Exercise session 6

LMI approach to H_∞ control.

Reading Assignment

[Dullerud & Paganini] Chapter 7

Exercises

E6.1 Consider the state-space system

$$\dot{x} = Ax + Bw, \quad x(0) = x_0,$$
$$z = Cx,$$

and suppose that $\left\| \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \right\|_\infty < 1$. Let X be the stabilizing solution of the Riccati equation

$$A^*X + XA + C^*C + XBB^*X = 0.$$

(a) Show that the trajectories of the system satisfy the identity

$$|z(t)|^2 - |w(t)|^2 = -|w(t) - B^*Xx(t)|^2 - \frac{d}{dt}\left(x(t)^*Xx(t)\right).$$

(b) Show that

$$\sup_{w \in L_2[0,\infty)} \left(\|z\|^2 - \|w\|^2\right) = x_0^* X x_0,$$

and find the signal $w(t)$ which achieves that optimum.

E6.2 Show that in a *state feedback* H_∞ synthesis problem (i.e. when $y = x$ is the measurement), the controller can be taken to be static without any loss of performance.

E6.3 Connections to Riccati solutions for the H_∞ problem. Let

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right]$$

satisfy the normalization conditions

$$D_{12}^* \left[C_1 \quad D_{12} \right] = \left[0 \quad I \right] \quad \text{and} \quad D_{21} \left[B_1^* \quad D_{21}^* \right] = \left[0 \quad I \right].$$

(a) Show that the H_∞ synthesis is equivalent to the feasibility of the LMIs $X > 0$, $Y > 0$ and

$$\begin{bmatrix} A^*X + XA + C_1^*C_1 - C_2^*C_2 & XB_1 \\ B_1^*X & -I \end{bmatrix} < 0,$$
$$\begin{bmatrix} AY + YA^* + B_1B_1^* - B_2B_2^* & YC_1^* \\ C_1Y & -I \end{bmatrix} < 0,$$
$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0.$$

(b) Now denote $P = Y^{-1}$, $Q = X^{-1}$. Convert the above conditions to the following:

$$A^*P + PA + C_1^*C_1 + P(B_1B_1^* - B_2B_2^*)P < 0,$$
$$AQ + QA^* + B_1B_1^* + Q(C_1^*C_1 - C_2^*C_2)Q < 0,$$
$$\rho(PQ) \leq 1.$$

These are two *Riccati inequalities* plus a spectral radius coupling condition. Formally analogous conditions involving the corresponding Riccati *equations* can be obtained when the plant satisfies some additional technical assumptions. For details consult the references in [Dullerud & Paganini] Chapter 7.

Hand-In problems:

H6.1 Consider the plant

$$P = \left[\begin{array}{cc|cc} -4 & 25 & 0.8 & -1 \\ -10 & 29 & 0.9 & -1 \\ \hline 10 & -25 & 0 & 1 \\ 13 & 25 & 1 & 0 \end{array}\right].$$

Apply [Dullerud & Paganini, Theorem 7.9] to verify if there exists a K such that the lower linear fractional transformation

$$\|T_{zw}\|_\infty = \|P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\|_\infty < \gamma,$$

where $\gamma = 1$. If so, construct a K using the method described in [Dullerud & Paganini, Section 7.3].

H6.2 Repeat the problem above for $\gamma = 1.5$.