Lecture 6

Control Synthesis Using Linear Matrix Inequalities.

Text: Dullerud/Paganini, chapter 7.

- H_2 Optimal State Feedback Using LMIs
- The Kalman - Yakubovich - Lemma
- H_∞ Optimal State Feedback
- The Matrix Elimination Lemma
- H_∞ Optimal Output Feedback

H_2 Optimal State Feedback Using LMIs

Problem: Given $\dot{x} = Ax + B_1 w + B_2 u$ find a stabilizing control law $u = Kx$ that minimizes $E(|x|^2 + |u|^2)$.

Solution: The closed loop system is $\dot{x} = (A + B_2 K)x + B_1 w$, so $X := Exx^T$ satisfies $(A+B_2K)X + X(A+B_2K)^T + B_1B_1^T = 0$

This is a linear constraint on (X, KX)! $(Y = KX)$

$$\text{Minimize } \operatorname{trace} X + \operatorname{trace}(Y X^{-1} Y)$$
$$\text{with } X \succ 0, \ (AX + B_2Y) + (AX + B_2Y)^T + B_1B_1^T = 0$$

Then the optimal control law is $K = YX^{-1}$!

Stationary Stochastic Processes

If A is Hurwitz and w is white noise with intensity I, then the stationary solution to $\dot{x} = Ax + Bw$ has covariance $Exx^T = X$ satisfying

$$AX + XA^T + BB^T = 0$$

The KYP Lemma

Given A, B and $M = M^T$, with $\det(i\omega I - A) = 0$ for $\omega \in \mathbb{R}$, the following are equivalent:

(i) $$\begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0 \quad \text{for all } \omega.$$

(ii) There exists $P = P^T$ such that

$$M + \begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} \prec 0$$

The KYP is a classical result is sytems theory connecting frequency domain to time domain.

Proof Sketch for KYP Lemma

Suppose (ii) holds and (x,w) is a nonzero square integrable solution to $\dot{x} = Ax + Bw$.

$$0 > \int_{-\infty}^{\infty} \begin{bmatrix} x \\ w \end{bmatrix}^T \left(M + \begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ w \end{bmatrix} dt = \int_{-\infty}^{\infty} \left(\begin{bmatrix} x \\ w \end{bmatrix} M \begin{bmatrix} x \\ w \end{bmatrix} + x^T P \dot{x} \right) dt$$

$$= \int_{-\infty}^{\infty} \begin{bmatrix} x \\ w \end{bmatrix} M \begin{bmatrix} x \\ w \end{bmatrix} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{w}(i\omega)^* \begin{bmatrix} (i\omega - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} \ \end{bmatrix} \hat{w}(i\omega) d\omega$$

Hence (i) follows.

The implication (i) is more difficult, but can be done by a separating hyperplane argument.

H_∞ Optimal State Feedback

Problem: Given $\dot{x} = Ax + B_1 w + B_2 u$, $x(0) = 0$, find control law $u = Kx$ such that the closed loop satisfies

$$\int_0^\infty |x|^2 + |u|^2 dt \le \gamma^2 \int_0^\infty |w|^2 dt$$

Equivalent conditions (with notation $A_K = A + B_2 K$)

(1) $\int_{-\infty}^{\infty} |\hat{x}|^2 + |\hat{u}|^2 d\omega \le \gamma^2 \int_{-\infty}^{\infty} |\hat{w}|^2 d\omega$ when $\begin{cases} \hat{x} = (i\omega - A_K)^{-1} B_1 w \\ \hat{u} = K\hat{x} \end{cases}$

(2) $B_1^T (i\omega - A_K)^{-*} (i\omega - A_K)^{-1} B_1 + B_1^T (i\omega - A_K)^{-*} K^T K (i\omega - A_K)^{-1} B_1 \preceq \gamma^2 I$

More equivalent conditions:

(3) $\begin{bmatrix} (i\omega - A_K)^{-1} B_1 \\ I \end{bmatrix}^* M \begin{bmatrix} (i\omega - A_K)^{-1} B_1 \\ I \end{bmatrix} \preceq 0$ when $M = \begin{bmatrix} I + K^TK & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$

(4) $\exists P = P^T$: $\begin{bmatrix} A_K^T P + PA_K + I + K^TK & PB_1 \\ B_1^T P & -\gamma^2 I \end{bmatrix} \preceq 0$

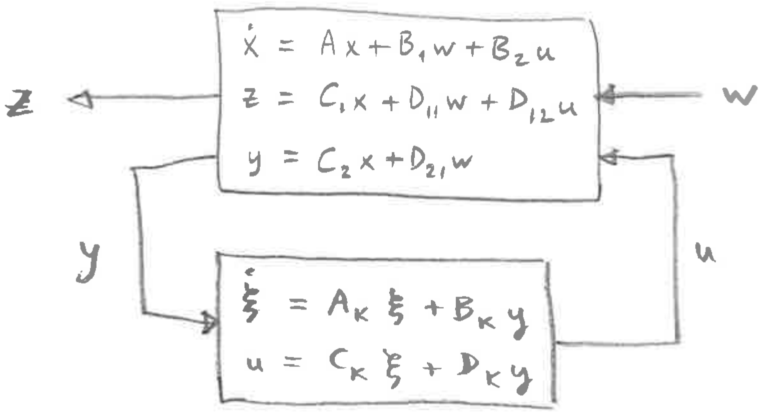
(5) $\exists P = P^T$: $\begin{bmatrix} (AP^{-1} + B_2 KP^{-1})^T + AP^{-1} + B_2 KP^{-1} + P^{-1}(I + K^TK)P^{-1} & B_1 \\ B_1^T & -\gamma^2 I \end{bmatrix} \preceq 0$

Solve by convex optimization over (P^{-1}, KP^{-1}) ∇

Comparison to Riccati Approach

- More expensive to solve LMI's than AREs.
- Fewer assumptions
- Extensions possible at the expense of conservatism
  - Diagonal P^{-1} and sparse KP^{-1} gives sparse controller K.
  - H_2 and H_∞ specifications can be merged.

Output Feedback Synthesis



Determine $J := \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$ to get $\int_0^\infty |z|^2 dt \leq \gamma^2 \int_0^\infty |w|^2 dt$.

Notation

$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \bar{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$ $\quad \bar{C} = [C_1 \ 0]$ $\quad \underline{C} = \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}$

$\underline{B} = \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix}$ $\quad \underline{D}_{21} = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}$ $\quad \underline{D}_{12} = [0 \ D_{21}]$

Then

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} + \begin{bmatrix} \underline{B} \\ \underline{D}_{12} \end{bmatrix} J \begin{bmatrix} \underline{C} & \underline{D}_{21} \end{bmatrix}$$

$$\hat{M}_{cl}(s) = C_{cl}(sI - A_{cl})^{-1} B_{cl} + D_{cl}$$

Closed Loop System

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \underbrace{\begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix}}_{A_{cl}} \begin{bmatrix} x \\ \xi \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 + B_2 D_K D_{21} \\ B_K D_{21} \end{bmatrix}}_{B_{cl}} w$$

$$z = \underbrace{\begin{bmatrix} C_1 + D_{12} D_K C_2 & D_{12} C_K \end{bmatrix}}_{C_{cl}} \begin{bmatrix} x \\ \xi \end{bmatrix} + \underbrace{(D_{11} + D_{12} D_K D_{21})}_{D_{cl}} w$$

LMI condition on the closed loop

It follows from the KYP lemma that the following two conditions are equivalent:

(i) A_{cl} is Hurwitz and $\|\hat{M}_{cl}\|_\infty < 1$

(ii) There exists a matrix $X_{cl} = X_{cl}^T$ such that

$$\begin{bmatrix} A_{cl}^T X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^T \\ B_{cl}^T X_{cl} & -I & D_{cl}^T \\ C_{cl} & D_{cl} & -I \end{bmatrix} < 0 \qquad (*)$$

Notice: The inequality is affine in X_{cl} and J individually, but not jointly.

The Matrix Elimination Lemma (Dullerud/Paganini 7.2)

Given matrices $P, Q, H = H^T$ let N_P and N_Q be full rank matrices with $\operatorname{Im} N_P = \operatorname{Ker} P$, $\operatorname{Im} N_Q = \operatorname{Ker} Q$. Then the following are equivalent:

(i) There exists J with $H + P^T J^T Q + Q^T J P^T < 0$

(ii) The inequalities $N_P^T H N_P < 0$, $N_Q^T H N_Q < 0$ hold.

Notice that $(*)$ can be written as

$$H_{X_{cl}} + Q^T J^T P_X + P_X^T J Q < 0$$

where

$$H_{X_{cl}} = \begin{bmatrix} \bar{A}^T X_{cl} + X_{cl} \bar{A} & X_{cl} \bar{B} & \bar{C}^T \\ \bar{B}^T X_{cl} & -I & D_{11}^T \\ \bar{C} & D_{11} & -I \end{bmatrix} \qquad \begin{aligned} P_X &= [\underline{B}^T X_{cl} \;\; 0 \;\; \underline{D}_{12}^T] \\ Q &= [\underline{C} \;\; \underline{D}_{21} \;\; 0] \end{aligned}$$

By the Matrix Elimination Lemma, J exists iff

$$N_{P_X}^T H_{X_{cl}} N_{P_X} < 0 \quad \text{and} \quad N_Q^T H_{X_{cl}} N_Q < 0$$

The inequality $N_{P_X}^T H_X N_{P_X} < 0$ can equivalently be written

$$N_P^T T_{X_{cl}} N_P < 0$$

where

$$T_{X_{cl}} = \begin{bmatrix} \bar{A} X_{cl}^{-1} + X_{cl}^{-1} \bar{A}^T & \bar{B} & X_{cl}^{-1} \bar{C}^T \\ \bar{B}^T & -I & D_{11}^T \\ \bar{C} X_{cl}^{-1} & D_{11} & -I \end{bmatrix}$$

Notice: $X_{cl} = \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}$ and $H_{X_{cl}}$ depends only on X!

$X_{cl}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}$ and $T_{X_{cl}}$ depends only on Y!

Lemma (Dullerud/Paganini 7.9)

Given $X \succ 0$ and $Y \succ 0$, the following are equivalent:

(i) There exist X_2, Y_2, $X_3 = X_3^T$, $Y_3 = Y_3^T \in \mathbb{R}^{n_K \times n_K}$ such that $0 \prec \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix} = \begin{bmatrix} Y & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}^{-1}$

(ii)

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \quad \text{and} \quad \operatorname{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + n_K$$

The rank condition goes away if $n_K = n$!

H_∞ Output Feedback Synthesis

A controller that gives $\|\hat{M}_{cl}\|_\infty < 1$ exists iff there exist symmetric matrices $X > 0$, $Y > 0$ such that

$$\begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^T X + X A & X B_1 & C_1^T \\ B_1^T X & -I & D_{11}^T \\ C_1 & D_{11} & -I \end{bmatrix} \begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix} < 0$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0, \quad \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AY + A^T Y & Y C_1^T & B_1 \\ C_1 Y & -I & D_{11} \\ B_1^T & D_{11}^T & -I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} < 0$$

where N_o, N_c are full rank matrices with $\operatorname{Im} N_o = \operatorname{Ker}[C_2 \; D_{21}]$, $\operatorname{Im} N_c = \operatorname{Ker}[B_2^T \; D_{12}^T]$.

Summary

- Both H_2 and H_∞ synthesis problems can be stated as convex optimization in terms of linear matrix inequalities.
- Convexity requires $n_K = n$.
- The H_∞ optimization involves coupling between estimation and control.