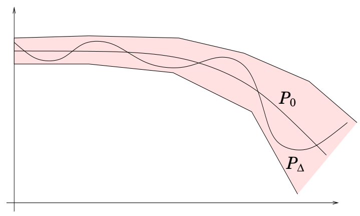
Lecture 3

- Unstructured Uncertainty Models
- Small Gain Theorem and Robust Stability
- Robust Performance
- Linear Fractional Transformations

Introduction

Recall that the purpose of robust control is that the closed loop performance should remain acceptable in spite of variations in the plant.

Methods to verify that a performance specification holds for all plants in a given set will be devoloped in this lecture and the next one.



Four kinds of specifications

Nominal stability
The closed loop is stable for the nominal plant P_0

Nominal performance
The closed loop specifications hold for the nominal plant P_0

Robust stability
The closed loop is stable for all plants in the given set P_Δ

Robust performance
The closed loop specifications hold for all plants in P_Δ

Basic Uncertainty Models

Let $\mathcal{D}$ be a set of all allowable Δ's.

Additive uncertainty model: $P_\Delta = P_0 + \Delta, \quad \Delta \in \mathcal{D}$.
Multiplicative uncertainty model: $P_\Delta = (I + \Delta)P_0, \quad \Delta \in \mathcal{D}$.
Feedback uncertainty model: $P_\Delta = P_0(I + \Delta P_0)^{-1}, \quad \Delta \in \mathcal{D}$.
Coprime factor uncertainty model:

Let $P_0 = NM^{-1}$, $M, N \in RH_\infty$ and

$$P_\Delta = (N + \Delta_N)(M + \Delta_M)^{-1}, \quad \begin{pmatrix} \Delta_N \\ \Delta_M \end{pmatrix} \in \mathcal{D}.$$

Miniproblem

Draw block diagrams for each of the previous uncertainty models!

- Very often
$$\mathcal{D} = \{W_1 \Delta W_2 \mid \|\Delta\|_\infty \leq 1\}$$
where W_1 and W_2 are given stable functions.
- The functions W_i provide the uncertainty profile. The main purpose of Δ is to account for phase uncertainty and to act as a scaling factor.
- Typically W is an increasing function of ω.
- The coprime factor uncertainty model is the most general form of all above.
- Construction of uncertainty models is a nontrivial task

Example

Let
$$P(s) = \frac{1}{s^2} e^{-\tau s}$$
where τ is known only to the extent that $\tau \in [0, 0.1]$.

Let the nominal plant be $P_0(s) = \frac{1}{s^2}$ and

$$P \in \mathcal{P}_\Delta = \{(1 + W\Delta)P_0 \mid \|\Delta\|_\infty \leq 1\}.$$

The weight should be chosen so that

$$\left| \frac{P(j\omega)}{P_0(j\omega)} - 1 \right| \leq |W(j\omega)|, \forall \omega \in R, \tau \in [0, 0.1].$$

So choose $|W(j\omega)| \geq |e^{-j\tau\omega} - 1|$ as tight as possible to reduce conservatism.

A suitable first order weight is $W(s) = 0.21s/(0.1s + 1)$

The Small Gain Theorem

Suppose $M \in RH_\infty^{m \times m}$. Then the closed loop system (M, Δ) is internally stable for all

$$\Delta \in \mathcal{B}RH_\infty := \{\Delta \in RH_\infty^{m \times m} \mid \|\Delta\|_\infty \leq 1\}$$

if and only if $\|M\|_\infty < 1$.

Proof: The internal stability of (M, Δ) is

$$\begin{pmatrix} I & -\Delta \\ -M & I \end{pmatrix}^{-1} \in RH_\infty.$$

Since $M, \Delta \in RH_\infty$ it is equivalent to $(I - M\Delta)^{-1} \in RH_\infty$ ([Zhou,Corollary 5.4]).

Thus we have to prove that $\|M\|_\infty < 1$ if and only if

$$(I - M\Delta)^{-1} \in RH_\infty, \quad \forall \Delta \in \mathcal{B}RH_\infty$$

Proof of Sufficiency

Let $\|M\|_\infty < 1$ and $\Delta \in \mathcal{B}RH_\infty$. Consider the Neumann series decomposition $(I - M\Delta)^{-1} = \sum_{n=0}^{+\infty} (M\Delta)^n$.

Then $(I - M\Delta)^{-1} \in RH_\infty$ since $M\Delta \in RH_\infty$ and

$$\begin{aligned} \|(I - M\Delta)^{-1}\|_\infty &\le \sum_{n=0}^{+\infty} \|M\Delta\|_\infty^n \\ &\le \sum_{n=0}^{+\infty} \|M\|_\infty^n \\ &= (1 - \|M\|_\infty)^{-1} < +\infty. \end{aligned}$$

Proof of Necessity

Assume that $\|M(j\omega)\| = \sigma \ge 1$ for some $\omega_0 \in [0, \infty]$. This means existence of singular vectors $\bar{u}, \bar{v} \in \mathbf{C}$ with $|\bar{u}| = |\bar{v}| = 1$ and $M(j\omega)\bar{v} = \sigma \bar{u}$. Define

$$\Delta(s) = \left[|\bar{v}_1| \frac{\alpha_1 - s}{\alpha_1 + s} \quad \dots \quad |\bar{v}_n| \frac{\alpha_m - s}{\alpha_m + s} \right]^T \left[|\bar{u}_1| \frac{\beta_1 - s}{\beta_1 + s} \quad \dots \quad |\bar{u}_n| \frac{\beta_m - s}{\beta_m + s} \right] \frac{1}{\sigma}$$

where α_j and β_k are chosen such that $\Delta(i\omega_0) = \bar{v}\bar{u}^* \frac{1}{\sigma}$. Then $\Delta \in RH_\infty^{m \times m}$, $\|\Delta\| = \sigma^{-1} \le 1$ and

$$\begin{aligned} \det[I - M(j\omega_0)\Delta(i\omega_0)] &= \det[I - M(j\omega_0)\bar{v}\bar{u}^*/\sigma] \\ &= 1 - \frac{\bar{u}^* M(j\omega_0)\bar{v}}{\sigma} = 0. \end{aligned}$$

Hence the closed loop system (M, Δ) is either not well-posed (if $\omega_0 = \infty$) or unstable (if $\omega_0 < \infty$).

Robust Stability under Unstructured Uncertainty

Theorem: Let $W_i \in RH_\infty$, $P_\Delta = P_0 + W_1 \Delta W_2$ for $\Delta \in RH_\infty$ and K be a stabilizing controller for P_0. Then K is robustly stabilizing for all $\Delta \in \mathcal{B}RH_\infty$ is and only if

$$\|W_2 K S_o W_1\|_\infty < 1.$$

Proof: Introduce

$$\begin{aligned} T_\Delta &= \begin{pmatrix} I & -K \\ -P_\Delta & I \end{pmatrix} = T_0 - \begin{pmatrix} 0 \\ W_1 \end{pmatrix} \Delta \begin{pmatrix} W_2 & 0 \end{pmatrix} \\ &= T_0 \left(I - T_0^{-1} \begin{pmatrix} 0 \\ W_1 \end{pmatrix} \Delta \begin{pmatrix} W_2 & 0 \end{pmatrix} \right) = T_0 \Phi \end{aligned}$$

Assuming nominal stability, i.e. $T_0^{-1} \in RH_\infty$, robust stability holds if and only if $\Phi^{-1} \in RH_\infty$ for all $\Delta \in \mathcal{B}RH_\infty$

Note that $\Phi \in RH_\infty$, so $\Phi^{-1} \in RH_\infty$ iff $\det \Phi$ has a stable inverse. The determinant identity in [Zhou,p. 14] yields

$$\det \Phi = \det \left(I - \begin{pmatrix} W_2 & 0 \end{pmatrix} T_0^{-1} \begin{pmatrix} 0 \\ W_1 \end{pmatrix} \Delta \right)$$

Hence robust stability is equivalent to the condition that

$$\left(I - \begin{pmatrix} W_2 & 0 \end{pmatrix} T_0^{-1} \begin{pmatrix} 0 \\ W_1 \end{pmatrix} \Delta \right)^{-1} \in RH_\infty$$

which in turn by small gain theorem is equivalent to

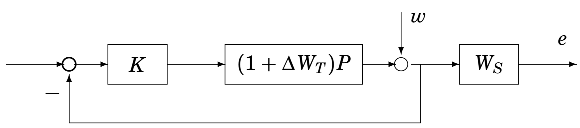
$$\left\| \begin{pmatrix} W_2 & 0 \end{pmatrix} T_0^{-1} \begin{pmatrix} 0 \\ W_1 \end{pmatrix} \right\|_\infty < 1$$

The desired condition follows as

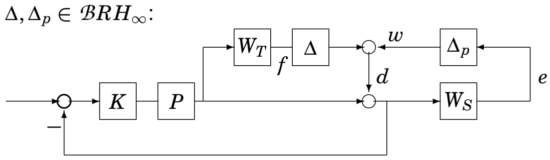
$$T_0^{-1} = \begin{pmatrix} S_i & KS_o \\ PS_i & S_o \end{pmatrix} \qquad \begin{pmatrix} W_2 & 0 \end{pmatrix} T_0^{-1} \begin{pmatrix} 0 \\ W_1 \end{pmatrix} = W_2 K S_o W_1$$

Uncertainty Model ($\|\Delta\| \le 1$)	Robust stability test
$(I + W_1 \Delta W_2)P$	$\|W_2 T_o W_1\|_\infty < 1$
$P(I + W_1 \Delta W_2)$	$\|W_2 T_i W_1\|_\infty < 1$
$(I + W_1 \Delta W_2)^{-1} P$	$\|W_2 S_o W_1\|_\infty < 1$
$P(I + W_1 \Delta W_2)^{-1}$	$\|W_2 S_i W_1\|_\infty < 1$
$P + W_1 \Delta W_2$	$\|W_2 K S_o W_1\|_\infty < 1$
$P(I + W_1 \Delta W_2 P)^{-1}$	$\|W_2 S_o P W_1\|_\infty < 1$
$(\tilde{M} + \tilde{\Delta}_M)^{-1}(\tilde{N} + \tilde{\Delta}_N)$ $\Delta = [\tilde{\Delta}_N \ \tilde{\Delta}_M]$	$\left\| \begin{pmatrix} K \\ I \end{pmatrix} S_o \tilde{M}^{-1} \right\|_\infty < 1$
$(N + \Delta_N)(M + \Delta_M)^{-1}$ $\Delta = [\Delta_N \ \Delta_M]$	$\left\| M^{-1} S_i \begin{pmatrix} K & I \end{pmatrix} \right\|_\infty < 1$

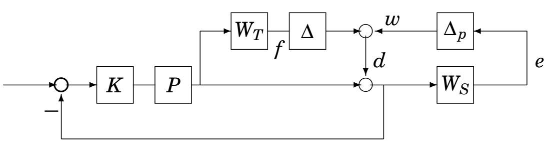
Robust Performance for Unstructured Uncertainty

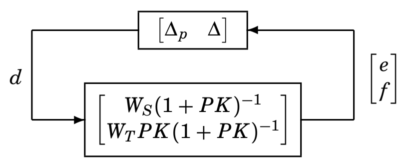


The closed loop map from w to e is $T_{ew} = W_S(1 + P_\Delta K)^{-1}$, where $P_\Delta = (1 + \Delta W_T)P$. Given robust stability, a robust performance specification is $\|T_{ew}\|_\infty < 1$ for all $\Delta \in \mathcal{B}RH_\infty$. This is equivalent to stability of the following diagram for $\Delta, \Delta_p \in \mathcal{B}RH_\infty$:

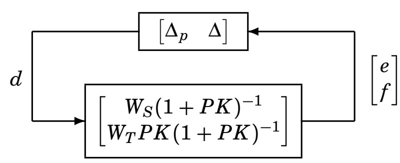


Equivalent Diagrams for Robust Stability





Condition for Robust Performance



Hence a small gain argument gives that the robust performance specification

$$\|T_{ew}\|_\infty < 1 \text{ for all } \Delta \in \mathcal{B}RH_\infty$$

is equivalent to the condition

$$\max_\omega \left[|W_S(1 + PK)^{-1}| + |W_T PK(1 + PK)^{-1}| \right] < 1$$

Linear Fraction Transformation

In complex analysis a linear fractional transformation (LFT) is a function in the form $F(s) = \frac{a+bs}{c+ds}$. If $c \neq 0$ then equivalently $F(s) = \alpha + \beta s(1-\gamma s)^{-1}$.

Definition: For a complex matrix $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ and other complex matrices Δ_l, Δ_u of appropriate size define a *lower* LFT with respect to Δ_l as

$$\mathcal{F}_l(M, \Delta_l) = M_{11} + M_{12}\Delta_l(I - M_{22}\Delta_l)^{-1}M_{21}$$

and an *upper* LFT with respect to Δ_u as

$$\mathcal{F}_u(M, \Delta_u) = M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12}$$

provided the inverse matrices exist.

Motivation

Consider closed loop systems

$$\begin{pmatrix} z_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ u_1 \end{pmatrix} \qquad u_1 = \Delta_l y_1$$

and

$$\begin{pmatrix} y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} u_2 \\ w_2 \end{pmatrix} \qquad u_2 = \Delta_u y_2$$

Then

$$T_{z_1 w_1} = \mathcal{F}_l(M, \Delta_l), \quad T_{z_2 w_2} = \mathcal{F}_u(M, \Delta_u).$$

Remark: In what follows we shall often use just LFT without distinguishing it to be lower or upper. It will be clear from context. Moreover $\mathcal{F}_u(N, \Delta) = \mathcal{F}_l(M, \Delta)$ where

$$N = \begin{pmatrix} M_{22} & M_{21} \\ M_{12} & M_{11} \end{pmatrix}.$$

Usage of LFT

- LFT is a useful way to standardize block diagram, that is to bring it to some canonical form.
- Systems with parametric uncertainties, i.e. with unknown coefficients in state space models can be represented as an LFT with respect to uncertain parameters (see examples in [Zhou]).
- Basic principle: use LFT to "pull out all uncertainties" which can appear in different points of a block diagram and to combine them in one uncertainty.

What have we learned today?

- Basic uncertainty models: additive, multiplicative, coprime factor etc.
- Robust stability — stability for all systems in a family closed by a single controller.
- Small Gain Theorem as a main tool for robust stability under unstructured uncertainty. Robust stability is equivalent to some H_∞ nominal performance.
- Conditions for robust performance are usually much harder to obtain explicitly.
- Linear Fractional Transformation as a standard way to represent an uncertain system combining all uncertainties in one.