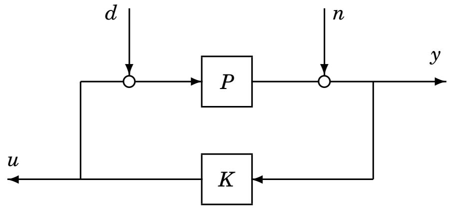
Lecture 7

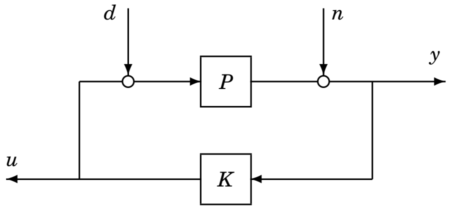
- An H_∞ Loop Shaping Procedure.
- Properties of the robustness margin $b_{P,K}$
- Justification of H_∞ Loop Shaping.
- The ν-gap Metric

What is Good Performance?



$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \begin{bmatrix} n \\ d \end{bmatrix}$$

What is Good Performance?



What is captured by the norm

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_\infty \quad ?$$

Remember: A controller should counteract disturbances, but be insensitive to measurement noise.

Loop-Shaping Design

Recall from Lecture 2 that a good performance controller design requires

- in the low frequency region:

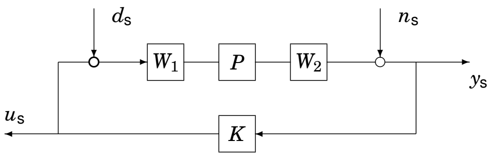
$$\underline{\sigma}(PK) >> 1, \quad \underline{\sigma}(KP) >> 1, \quad \underline{\sigma}(K) >> 1.$$

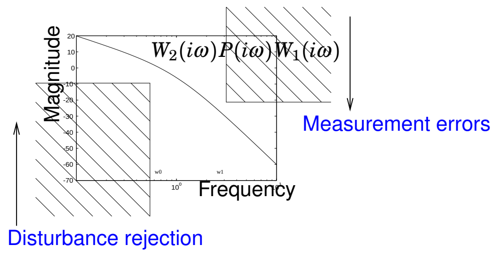
- in the high frequency region:

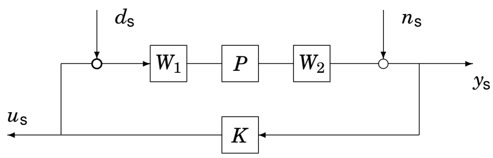
$$\overline{\sigma}(PK) << 1, \quad \overline{\sigma}(KP) << 1, \quad \overline{\sigma}(K) \leq M$$

where M is not too large.

Use weighting matrices!







1) Choose W_1 and W_2 and absorb them into the nominal plant P to get the shaped plant $P_s = W_2 P W_1$.
2) Design the controller K_∞ to minimize the H_∞ gain from (n_s, d_s) to (u_s, y_s). If the gain is large, the return to Step 1.
3) The final controller is $K = W_1 K_\infty W_2$.

(The H_∞ loop shaping design procedure was suggested by Glover and McFarlane, 1990.)

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A Notion of Loop Stability Margin

Introduce the quantity $b_{P,K}$

$$b_{P,K} = \begin{cases} \left\| \begin{bmatrix} I \\ K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_\infty^{-1} & \text{if } K \text{ stabilizes } P \\ 0 & \text{otherwise} \end{cases}$$

The larger $b_{P,K}$ is, the more robustly stable the closed loop system is.

Relation to Gain and Phase Margins

Theorem: Let P be a SISO plant and K be a stabilizing controller. Then

$$\text{gain margin} \geq \frac{1+b_{P,K}}{1-b_{P,K}},$$
$$\text{phase margin} \geq 2\arcsin(b_{P,K}).$$

Proof: For SISO system at every ω

$$b_{P,K} = \frac{1}{\|\dots\|_\infty} \leq \frac{|1+P(j\omega)K(j\omega)|}{\left\|\begin{bmatrix}1\\K\end{bmatrix}\begin{bmatrix}1 & P\end{bmatrix}\right\|} = \frac{|1+P(j\omega)K(j\omega)|}{\sqrt{1+|P(j\omega)|^2}\sqrt{1+|K(j\omega)|}}$$

So at frequencies where $k := -PK \in R^+$ we have

$$\begin{aligned} b_{P,K} &\leq \frac{|1-k|}{\sqrt{(1+|P|^2)(1+k^2/|P|^2)}} \leq \\ &\leq \frac{|1-k|}{\sqrt{\min_P\{(1+|P|^2)(1+k^2/|P|^2)\}}} = \frac{|1-k|}{|1+k|} \end{aligned}$$

from which the gain margin result follows.

Similarly at frequencies where $PK = -e^{i\theta}$

$$\begin{aligned} b_{P,K} &\leq \frac{|1-e^{i\theta}|}{\sqrt{(1+|P|^2)(1+1/|P|^2)}} \leq \\ &\leq \frac{|1-e^{i\theta}|}{\sqrt{\min_P\{(1+|P|^2)(1+1/|P|^2)\}}} = \frac{2|\sin(\theta/2)|}{2} \end{aligned}$$

which implies the phase margin result.

Robust Stabilization of Coprime Factors

Let $P = \tilde{M}^{-1}\tilde{N}$, where $\tilde{N}(i\omega)\tilde{N}(i\omega)^* + \tilde{M}(i\omega)\tilde{M}(i\omega)^* \equiv 1$. This is called *normalized coprime factorization*.

The process $P_\Delta = (\tilde{M}+\tilde{\Delta}_M)^{-1}(\tilde{N}+\tilde{\Delta}_N)$ in feedback with the controller K is stable for all $\Delta = [\tilde{\Delta}_N\ \tilde{\Delta}_M]$ with $\|\Delta\|_\infty \leq \epsilon$ iff

$$\left\|\begin{bmatrix}K\\I\end{bmatrix}(I+PK)^{-1}\begin{bmatrix}I & P\end{bmatrix}\right\|_\infty < \frac{1}{\epsilon} \qquad (1)$$

Finding K that achieves (1) is a problem of H_∞ optimization.

Proof

The interconnection of $P_\Delta = (\tilde{M}+\tilde{\Delta}_M)^{-1}(\tilde{N}+\tilde{\Delta}_N)$ and K can be rewritten as an interconnection of $\Delta = [\tilde{\Delta}_N\ \tilde{\Delta}_M]$ and

$$\begin{bmatrix}K\\I\end{bmatrix}(I+PK)^{-1}\tilde{M}^{-1}$$

The small gain theorem therefore gives the stability condition

$$\begin{aligned} \frac{1}{\epsilon} &> \left\|\begin{bmatrix}K\\I\end{bmatrix}(I+PK)^{-1}\tilde{M}^{-1}\right\|_\infty \\ &= \left\|\begin{bmatrix}K\\I\end{bmatrix}(I+PK)^{-1}\tilde{M}^{-1}\begin{bmatrix}\tilde{N} & \tilde{M}\end{bmatrix}\right\|_\infty \\ &= \left\|\begin{bmatrix}K\\I\end{bmatrix}(I+PK)^{-1}\begin{bmatrix}I & P\end{bmatrix}\right\|_\infty \end{aligned}$$

Computing Normalized Coprime Factors

Given $P(s) = C(sI-A)^{-1}B$, let Y be the stabilizing solution to

$$AY + YA^* - YC^*CY + BB^* = 0.$$

The matrix $A - YC^*C$ is stable, so we can put $L = -YC^*$.

Lemma: With $L = -YC^*$, a normalized factorization is given by

$$\begin{bmatrix}\tilde{N}(s) & \tilde{M}(s)\end{bmatrix} = \left(\begin{array}{c|cc} A+LC & B & L \\ \hline C & 0 & I\end{array}\right),$$

Proof: Denote $\mathcal{A}(s) = (sI - A + YC^*C)^{-1}$ and calculate

$$\begin{aligned} \tilde{N}\tilde{N}^* + \tilde{M}\tilde{M}^* &= I - C\mathcal{A}YC^* - CY\mathcal{A}^*C^* + C\mathcal{A}(B^*B+YC^*CY)\mathcal{A}^* \\ &= I + C\mathcal{A}(B^*B+YC^*CY-Y(\mathcal{A}^*)^{-1}-\mathcal{A}^{-1}Y)\mathcal{A}^*C^* \\ &= I + C\mathcal{A}(\underbrace{B^*B-YC^*CY+AY+YA^*}_{=0})\mathcal{A}^*C^* = I \end{aligned}$$

H_∞ Optimization of Normalized Coprime Factors

Theorem: Let $D = 0$ and $L = -YC^*$ where $Y \geq 0$ is the stabilizing solution to $AY + YA^* - YC^*CY + BB^* = 0$. Then $P = \tilde{M}^{-1}\tilde{N}$ is a normalized left coprime factorization and

$$\begin{aligned} \inf_{K-\text{stab}} \left\|\begin{bmatrix}K\\I\end{bmatrix}(I+PK)^{-1}\tilde{M}^{-1}\right\|_\infty &= \frac{1}{\sqrt{1-\lambda_{\max}(YQ)}} \\ &= \left(1 - \|\tilde{N}\ \tilde{M}\|_H^2\right)^{-1/2} \end{aligned}$$

where $Q(A-YC^*C) + (A-YC^*C)^*Q + C^*C = 0$.. Moreover, a controller achieving $\gamma > \gamma_{opt}$ is

$$\begin{aligned} K(s) &= \left(\begin{array}{c|c} A - BB^*X_\infty - YC^*C & -YC^* \\ \hline -B^*X_\infty & 0 \end{array}\right) \\ X_\infty &= \frac{\gamma^2}{\gamma^2-1}Q\left(I - \frac{\gamma^2}{\gamma^2-1}YQ\right)^{-1} \end{aligned}$$

Proof: Define

$$H_q = \begin{bmatrix} A - YC^*C & 0 \\ -C^*C & -(A-YC^*C)^* \end{bmatrix} \qquad T = \begin{bmatrix} I & -\frac{\gamma^2}{\gamma^2-1}Y \\ 0 & \frac{\gamma^2}{\gamma^2-1}I \end{bmatrix}$$

It is straightforward to see that $H_\infty = TH_qT^{-1}$. Since $Q = \text{Ric}(H_q)$ we have the stable invariant subspace for H_∞ as

$$T\begin{bmatrix}I\\Q\end{bmatrix} = \begin{bmatrix} I - \frac{\gamma^2}{\gamma^2-1}YQ \\ \frac{\gamma^2}{\gamma^2-1}Q \end{bmatrix}.$$

Finally $\exists X_\infty \geq 0$ iff

$$I - \frac{\gamma^2}{\gamma^2-1}YQ > 0 \quad \Leftrightarrow \quad \gamma^2 > \frac{1}{1-\lambda_{max}(YQ)}.$$

Note that Y and Q are controllability and observability Gramians for $[\tilde{N}\ \tilde{M}]$.

Right Coprime Factors

What if we have a normalized right coprime factorization $P = NM^{-1}$?

Theorem:

$$\left\|\begin{bmatrix}I\\K\end{bmatrix}(I+PK)^{-1}\begin{bmatrix}I & P\end{bmatrix}\right\| = \left\|\begin{bmatrix}I\\P\end{bmatrix}(I+KP)^{-1}\begin{bmatrix}I & K\end{bmatrix}\right\|.$$

Corollary: Let $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ be the normalized rcf and lcf, respectively. Then

$$\left\|\begin{bmatrix}K\\I\end{bmatrix}(I+PK)^{-1}\tilde{M}^{-1}\right\|_\infty = \left\|M^{-1}(I+KP)^{-1}\begin{bmatrix}I & K\end{bmatrix}\right\|_\infty.$$

Conclusion: It does not matter what kind of factorization we have. One can work with either left or right.

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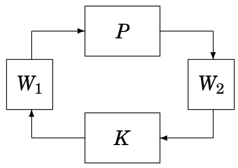
Conclusion: Performance depends strongly on open loop shape.



1) Choose W_1 and W_2 and absorb them into the nominal plant P to get the shaped plant $P_s = W_2 P W_1$.
2) Calculate $b_{opt}(P_s) = \sqrt{1 - \|\tilde{N}_s\ \tilde{M}_s\|_H^2}$. If it is small then return to Step 1 and adjust weights.
3) Select $\epsilon \leq b_{opt}(P_s)$ and design the controller K_∞ such that

$$\left\| \begin{bmatrix} I \\ K_\infty \end{bmatrix} (I + P_s K_\infty)^{-1} \tilde{M}_s^{-1} \right\|_\infty < \epsilon^{-1}.$$

4) The final controller is $K = W_1 K_\infty W_2$.

Remarks:

- In contrast to the classical loop shaping design we do not treat explicitly closed loop stability, phase and gain margins. Thus the procedure is simple.
- Observe that

$$\left\| \begin{bmatrix} I \\ K_\infty \end{bmatrix} (I + P_s K_\infty)^{-1} \tilde{M}_s^{-1} \right\|_\infty = \left\| \begin{bmatrix} W_2 \\ W_1^{-1} K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} W_2^{-1} & PW_1 \end{bmatrix} \right\|_\infty$$

so it has an interpretation of the standard H_∞ optimization problem with weights.
- BUT!!! The open loop under investigation on Step 1 is $K_\infty W_2 P W_1$ whereas the actual open loop is given by $W_1 K_\infty W_2 P$ and $P W_1 K_\infty W_2$. This is not really what we has shaped!

Thus the method needs validation.

Justification of H_∞ Loop Shaping

We show that the degradation in the loop shape caused by K_∞ is limited. Consider low-frequency region first.

$$\begin{aligned} \underline{\sigma}(PK) &= \underline{\sigma}(W_2^{-1} P_s K_\infty W_2) \geq \frac{\underline{\sigma}(P_s)\underline{\sigma}(K_\infty)}{\kappa(W_2)}, \\ \underline{\sigma}(KP) &= \underline{\sigma}(W_1 K_\infty P_s W_1^{-1}) \geq \frac{\underline{\sigma}(P_s)\underline{\sigma}(K_\infty)}{\kappa(W_1)} \end{aligned}$$

where κ denotes conditional number. Thus small $\underline{\sigma}(K_\infty)$ might cause problem even if P_s is large. Can this happen?

Theorem: Any K_∞ such that $b_{P_s,K_\infty} \geq 1/\gamma$ also satisfies

$$\underline{\sigma}(K_\infty) \geq \frac{\underline{\sigma}(P_s) - \sqrt{\gamma^2 - 1}}{\sqrt{\gamma^2 - 1}\underline{\sigma}(P_s) + 1} \quad \text{if } \underline{\sigma}(P_s) > \sqrt{\gamma^2 - 1}.$$

Corollary: If $\underline{\sigma}(P_s) >> \sqrt{\gamma^2 - 1}$ then $\underline{\sigma}(K_\infty) \geq 1/\sqrt{\gamma^2 - 1}$

Consider now high frequency region.

$$\begin{aligned} \overline{\sigma}(PK) &= \overline{\sigma}(W_2^{-1} P_s K_\infty W_2) \leq \overline{\sigma}(P_s)\overline{\sigma}(K_\infty)\kappa(W_2), \\ \overline{\sigma}(KP) &= \overline{\sigma}(W_1 K_\infty P_s W_1^{-1}) \leq \overline{\sigma}(P_s)\overline{\sigma}(K_\infty)\kappa(W_1). \end{aligned}$$

Can $\overline{\sigma}(K_\infty)$ be large if $\overline{\sigma}(P_s)$ is small?

Theorem: Any K_∞ such that $b_{P_s,K_\infty} \geq 1/\gamma$ also satisfies

$$\overline{\sigma}(K_\infty) \leq \frac{\sqrt{\gamma^2 - 1} + \overline{\sigma}(P_s)}{1 - \sqrt{\gamma^2 - 1}\overline{\sigma}(P_s)} \quad \text{if } \overline{\sigma}(P_s) < \frac{1}{\sqrt{\gamma^2 - 1}}.$$

Corollary: If $\overline{\sigma}(P_s) << 1/\sqrt{\gamma^2 - 1}$ then $\overline{\sigma}(K_\infty) \leq \sqrt{\gamma^2 - 1}$

One can get the idea of proof from SISO relation

$$b_{P,K} \leq \frac{|1 + P_s(j\omega)K_\infty(j\omega)|}{\sqrt{1 + |P_s(j\omega)|^2}\sqrt{1 + |K_\infty(j\omega)|^2}}.$$

Denote

$$\overline{\sigma}_i = \overline{\sigma}(W_i), \quad \underline{\sigma}_i = \underline{\sigma}(W_i), \quad \kappa_i = \kappa(W_i).$$

Theorem: Let P be the nominal plant and let $K = W_1 K_\infty W_2$ be the controller designed by loop shaping. If $b_{P_s,K_\infty} \geq 1/\gamma$ then

$$\begin{aligned} \overline{\sigma}(K(I + PK)^{-1}) &\leq \gamma\overline{\sigma}(\tilde{M}_s)\overline{\sigma}_1\overline{\sigma}_2, \\ \overline{\sigma}((I + PK)^{-1}) &\leq \min\{\gamma\overline{\sigma}(\tilde{M}_s)\kappa_2, 1{+}\gamma\overline{\sigma}(\tilde{N}_s)\kappa_2\}, \\ \overline{\sigma}(K(I + PK)^{-1}P) &\leq \min\{\gamma\overline{\sigma}(\tilde{N}_s)\kappa_1, 1{+}\gamma\overline{\sigma}(\tilde{M}_s)\kappa_1\}, \\ \overline{\sigma}((I + PK)^{-1}P) &\leq \frac{\gamma\overline{\sigma}(\tilde{N}_s)}{\underline{\sigma}_1\underline{\sigma}_2}, \\ \overline{\sigma}((I + KP)^{-1}) &\leq \min\{1{+}\gamma\overline{\sigma}(\tilde{N}_s)\kappa_1, \gamma\overline{\sigma}(\tilde{M}_s)\kappa_1\}, \\ \overline{\sigma}(P(I + KP)^{-1}K) &\leq \min\{1{+}\gamma\overline{\sigma}(\tilde{M}_s)\kappa_2, \gamma\overline{\sigma}(\tilde{N}_s)\kappa_2\} \end{aligned}$$

where

$$\overline{\sigma}(\tilde{N}_s) = \left(\frac{\overline{\sigma}^2(P_s)}{1 + \overline{\sigma}^2(P_s)}\right)^{1/2} \qquad \overline{\sigma}(\tilde{M}_s) = \left(\frac{1}{1 + \overline{\sigma}^2(P_s)}\right)^{1/2}$$

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ν-Gap Metric

$$\delta_\nu(P_1, P_2) = \begin{cases} \|(I+P_2P_2^*)^{-\frac{1}{2}}(P_1-P_2)(I+P_1^*P_1)^{-\frac{1}{2}}\|_\infty \\ \text{ if } \det(I+P_2^*P_1) \neq 0 \text{ on } jR \text{ and} \\ \text{ wno} \det(I+P_2^*P_1) + \eta(P_1) = \overline{\eta}(P_2), \\ 1 \quad \text{otherwise} \end{cases}$$

where $\overline{\eta}$ (η) is the number of closed (open) RHP poles and wno is winding number.

In scalar case it takes on the particularly simple form

$$\delta_\nu(P_1, P_2) = \sup_{\omega \in R} \frac{|P_2(j\omega) - P_1(j\omega)|}{\sqrt{1+|P_1(j\omega)|^2}\sqrt{1+|P_2(j\omega)|^2}}$$

whenever the winding number condition is satisfied.

Geometrical interpretation: Distance on the Riemann sphere

Example

Consider

$$P_1(s) = \frac{1}{s}, \quad P_2(s) = \frac{1}{s+0.1}.$$

We had $\|P_1 - P_2\|_\infty = +\infty$. However

$$\delta_\nu(P_1, P_2) \approx 0.09951$$

which means that the system are, in fact, very close.

Theorem

For any P_0, P and K

$$\arcsin b_{P,K} \geq \arcsin b_{P_0,K} - \arcsin \delta_\nu(P_0, P).$$

Corollary 1: If $b_{P_0,K} > \delta_\nu(P_0, P)$ then (P, K) is stable.

Corollary 2: For any P_0, P, K_0 and K

$$\arcsin b_{P,K} \geq \arcsin b_{P_0,K_0} - \arcsin \delta_\nu(P_0, P) - \arcsin \delta_\nu(K_0, K).$$

Proof: By Theorem we have

$$\begin{aligned} \arcsin b_{P,K_0} &\geq \arcsin b_{P_0,K_0} - \arcsin \delta_\nu(P_0, P) \\ \arcsin b_{P,K} &\geq \arcsin b_{P,K_0} - \arcsin \delta_\nu(K_0, K) \end{aligned}$$

What have we learned today?

- H_∞ optimization of normalized coprime factors.
- Left or right coprime factors - does not matter.
- Stability margin $b_{P,K}$. The larger the better. Relation to gain and phase margins.
- H_∞ loop shaping via pre- and postcompensations and optimization of $b_{P,K}$.
- Robustness in terms of δ_ν-gap