Convex Sets

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Today's lecture

- convex sets
- convex, affine, conical hulls
- closure, interior, relative interior, boundary, relative boundary
- separating and supporting hyperplane theorems
- tangent and normal cones

Euclidean setting

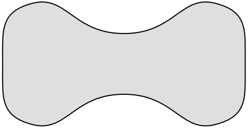
- in this course, we will consider Euclidean spaces $\mathbb{R}^n$ (although most results hold for general real Hilbert spaces)
- examples of Euclidean spaces
  - "standard":
$$\langle x, y \rangle = x^T y \qquad \|x\| = \sqrt{x^T x}$$
  - square matrices:
$$\langle X, Y \rangle = \mathrm{tr}(X^T Y) \qquad \|X\| = \sqrt{\mathrm{tr}(X^T X)} = \|X\|_F$$
  - skewed Euclidean (H positive definite):
$$\langle x, y \rangle = x^T H y \qquad \|x\| = \sqrt{x^T H x}$$

Convex sets

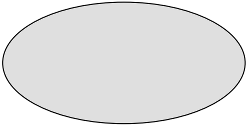
- a set S is convex if for every $x, y \in S$ and $\theta \in [0,1]$:

$$\theta x + (1-\theta)y \in S$$

- "every line segment that connect any two points in S is in S"

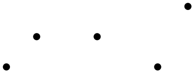


A nonconvex set



A convex set

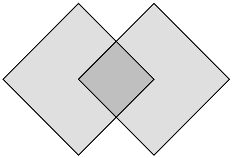
A nonconvex set



A nonconvex set

Intersection and union

- the intersection $C_1 \cap C_2$ of two convex sets C_1, C_2 is convex
- the union $C_1 \cup C_2$ of two convex sets C_1, C_2 need not be convex



(intersection: darker gray, union: lighter gray)

Set sum and set difference

- the set sum is also called the *Minkowski sum*
- the set sum of C_1 and C_2 is denoted $C_1 + C_2$ and is defined as

$$C_1 + C_2 := \{x \mid x = x_1 + x_2, \text{ with } x_1 \in C_1, x_2 \in C_2\}$$

- set sum of two convex sets is convex
- the set difference is denoted $C_1 - C_2$ and is defined as

$$C_1 - C_2 := \{x \mid x = x_1 - x_2, \text{ with } x_1 \in C_1, x_2 \in C_2\}$$

- set difference of two convex sets is convex

Image and inverse image of set

let

- $L : \mathbb{R}^n \to \mathbb{R}^m$ be an affine mapping, i.e. $Lx = L_0 x + y_0$
- $C \subseteq \mathbb{R}^n$ be a convex set
- $D \subseteq \mathbb{R}^m$ be a convex set

then

- the image set $L(C)$

  $$L(C) := \{y \in \mathbb{R}^m \mid y = Lx, x \in C\}$$

  is a convex set in $\mathbb{R}^m$
- the inverse image set

  $$L^{-1}(D) := \{x \in \mathbb{R}^n \mid Lx = y, y \in D\}$$

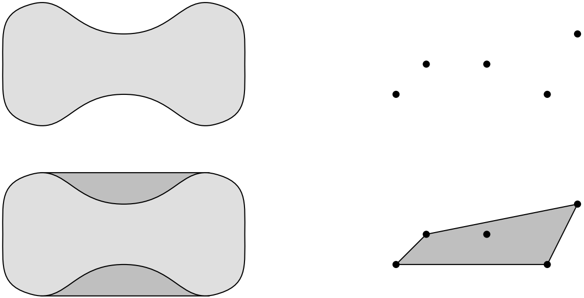
  is a convex set in $\mathbb{R}^n$

Convex combination and convex hull

- convex combination: of $x_1, \ldots, x_k$ is any points x on the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k$$

  where $\theta_1 + \ldots + \theta_k = 1$ and $\theta_i \geq 0$
- convex hull conv S: set of all convex combinations of points in S
- what are convex hulls of?



Affine sets

- an affine set V contains the entire (affine) line

$$\{\alpha x + (1-\alpha) y \mid \alpha \in \mathbb{R}\}$$

  whenever $x, y \in V$

- also called affine subspace or affine manifold
- which of the following are affine sets? if affine, what dimension?
  (a) point: $\{x\}$
  (b) line: $\{x \mid x = \alpha x_1 + (1-\alpha) x_2, x_1 \neq x_2, \alpha \in [0,1]\}$
  (c) line: $\{x \mid x = \alpha x_1 + (1-\alpha) x_2, x_1 \neq x_2, \alpha \in \mathbb{R}\}$
- (a) and (c) are affine, dimension 0 and 1 respectively

Affine hyperplanes

- an important affine set is the *affine hyperplane* $H_{s,r}$, defined as

$$H_{s,r} := \{x \in \mathbb{R}^n \mid \langle s, x \rangle = r\}$$

- the vector s is called *normal vector* to the hyperplane
- if $s \neq 0$, what is dimension is affine hyperplane in $\mathbb{R}^n$? $n-1$
- any affine set of dimension $n-1$ can be represented by hyperplane

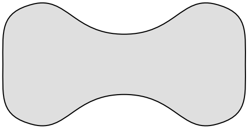
Affine combination and affine hull

- affine combination: of $x_1, \ldots, x_k$ is any points x on the form

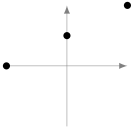
$$x = \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k$$

  where $\theta_1 + \ldots + \theta_k = 1$

- (affine combination lacks $\theta_i \geq 0$ compared to convex combination)
- affine hull aff(S): set of all affine combinations of points in S
- what is affine hull of the following sets (in $\mathbb{R}^2$)?



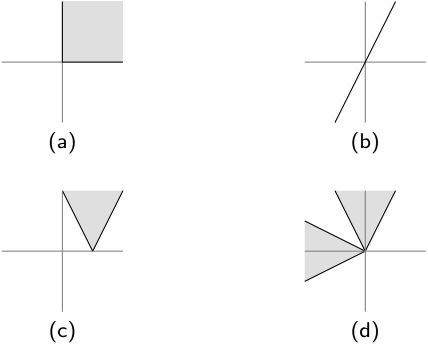
aff(S)=$\mathbb{R}^2$



aff(S)=$\{x \in \mathbb{R}^2 \mid x_1 - 2x_2 = -1\}$

Convex cones

- a cone K contains the half-line $\{\alpha x \mid \alpha > 0\}$ if $x \in K$
- which of the following figures are cones?



- (a), (b), (d) are cones
- a convex cone is a cone that is convex (which are convex cones?)
- (a), (b) convex cones

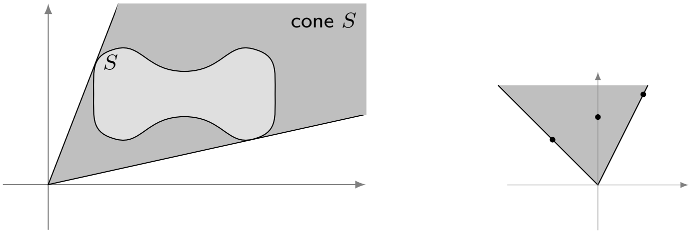
Conical combinations and conical hull

- conical combination: of $x_1, \ldots, x_k$ is any points x on the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k$$

  where $\theta_1, \ldots, \theta_k \geq 0$

- conical hull cone S: set of all conical combinations of points in S



- note: cone $S = \mathbb{R}^n$ if $0 \in \text{int } S$
- we have cone $S = \mathbb{R}_+(\text{conv } S)$ (see right figure)

Closure

- *closure* of a set if denoted by cl S
- $x \in \text{cl } S$ if for all $\epsilon > 0$ there exists $y_\epsilon \in B_\epsilon(x)$ with $y_\epsilon \in S$, where

$$B_\epsilon(x) = \{y \in \mathbb{R}^n \mid \|y - x\| < \epsilon\}$$

- (the point y_ϵ may be x itself $\Rightarrow$ cl $S \supseteq S$)
- the closure of $S \subseteq \mathbb{R}^n$ is the set of such x:

$$\text{cl } S = \{x \in \mathbb{R}^n \mid \forall \epsilon > 0, \exists y_\epsilon \in B_\epsilon(x) \text{ such that } y_\epsilon \in S\}$$

- a set S is *closed* iff cl $S = S$

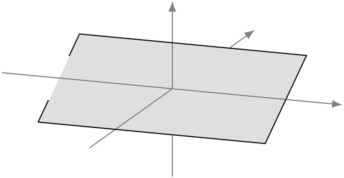
Closure – Examples

- example in $\mathbb{R}^2$:

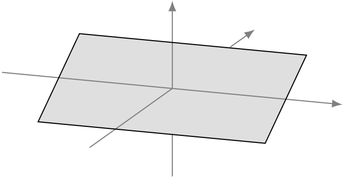
S

cl S

- example in $\mathbb{R}^3$, what is closure?



S



cl S

- embedding in higher dimensional spaces does not affect closure

Interior

- interior of a set $S \subseteq \mathbb{R}^n$ is denoted int S
- $x \in$ int S if there is $\epsilon > 0$ such that $B_\epsilon(x) \subseteq S$, where

$$B_\epsilon(x) = \{y \in \mathbb{R}^n \mid \|y - x\| < \epsilon\}$$

- the interior is the set of such x:

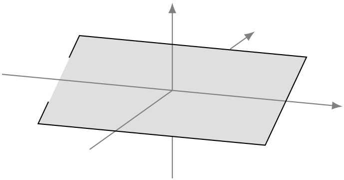
$$\text{int } S = \{x \in S \mid B_\epsilon(x) \subseteq S\}$$

Interior – Examples

- example in $\mathbb{R}^2$:

S int S

- example in $\mathbb{R}^3$, what is interior?



- int $S = \emptyset$, reason: no 3D ball fits in S since 2D
- need something to take care of this

Relative interior

- relative interior of a set S if denoted relint S or ri S
- $x \in \text{ri } S$ if there is $\epsilon > 0$ such that $B_\epsilon(x) \cap \text{aff } S \subseteq S$
- interior with respect to the affine hull where S lies
- the relative interior is the set of such x:

$$\text{ri } S = \{x \in S \mid B_\epsilon(x) \cap \text{aff } S \subseteq S\}$$

- note:
    - ri $S \subseteq S$
    - if S nonempty and convex, then ri $S \neq \emptyset$
    - if aff $S = \mathbb{R}^n$, then ri $S =$ int S
- concept of relative interior important for convex analysis!

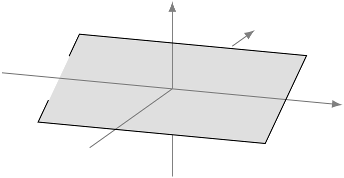
Relative interior – Examples

- example in $\mathbb{R}^2$:

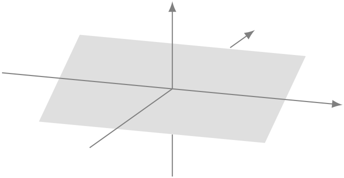
S

ri S

- embed in $\mathbb{R}^3$:



S



ri S

relative interior nonempty, but interior empty

- what is interior and relative interior of the singleton $\{x\} \subset \mathbb{R}^n$?
(ri $\{x\} = \{x\}$ and int $\{x\} = \emptyset$, since aff $\{x\} = \{x\}$)

Intersection results

let S_1 and S_2 be convex and satisfy ri $S_1 \cap$ ri $S_2 \neq \emptyset$, then

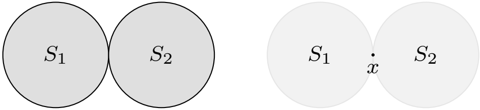
- the relative interiors satisfy

$$\text{ri } (S_1 \cap S_2) = \text{ri } S_1 \cap \text{ri } S_2$$

- the closures satisfy

$$\text{cl } (S_1 \cap S_2) = \text{cl } S_1 \cap \text{cl } S_2$$

- can you construct a counter-example for relative interior:



$\text{ri}S_1 \cap \text{ri}S_2 = \emptyset$ $\text{ri}(S_1 \cap S_2) = \{x\}$

- (qualification ri $S_1 \cap$ ri $S_2 \neq \emptyset$ will be very important later)

Product spaces

for $i = 1, \ldots, k$, let $C_i \in \mathbb{R}^{n_i}$ be convex sets, then

- the relative interiors satisfy

$$\text{ri } (C_1 \times \cdots \times C_k) = (\text{ri } C_1) \times \cdots \times (\text{ri } C_k)$$

- the closures satisfy

$$\text{cl } (C_1 \times \cdots \times C_k) = (\text{cl } C_1) \times \cdots \times (\text{cl } C_k)$$

Image and inverse image of set

let

- $L : \mathbb{R}^n \to \mathbb{R}^m$ be an affine mapping, i.e. $Lx = L_0 x + y_0$
- $C \subseteq \mathbb{R}^n$ be a convex set
- $D \subseteq \mathbb{R}^m$ be a convex set

then

- for the image $L(C)$, we have

$$\text{ri}[L(C)] = L(\text{ri } C), \qquad \text{cl}[L(C)] = L(\text{cl } C)$$

- for the inverse image $L^{-1}(D)$, we have

$$\text{ri}[L^{-1}(D)] = L^{-1}(\text{ri } D), \qquad \text{cl}[L^{-1}(D)] = L^{-1}(\text{cl } D)$$

Boundary and relative boundary

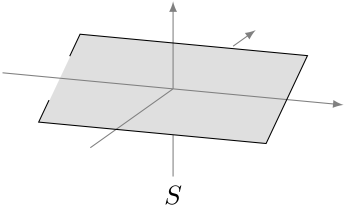
- the boundary of S is denoted by bd S and is defined as:

$$\text{bd } S := \text{cl } S \backslash \text{int } S$$

- since interior often empty, we also define relative boundary:

$$\text{rbd } S = \text{cl } S \backslash \text{ri } S$$

- what is boundary and relative boundary in figure?



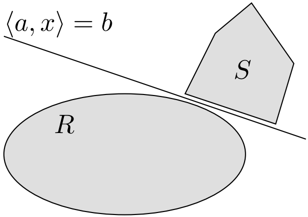
(boundary: cl S, relative boundary: full empty rectangle)

Separating hyperplane theorem

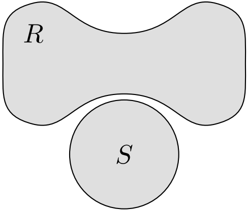
- suppose that R and S are two non-intersecting convex sets
- then there exists $a \neq 0$ and b such that

$$\langle a, x\rangle \leq b \qquad \text{for all } x \in R$$
$$\langle a, x\rangle \geq b \qquad \text{for all } x \in S$$

- the hyperplane $\{x \mid \langle a, x\rangle = b\}$ is called separating hyperplane



Example

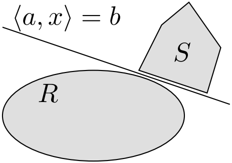


Counter-example
R nonconvex

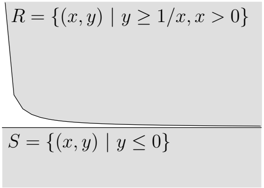
A strictly separating hyperplane theorem

- suppose that R and S are non-intersecting closed and convex sets and that one of them is compact (closed and bounded)
- then there exists $a \neq 0$ and b such that

$$\begin{aligned} \langle a, x \rangle &< b && \text{for all } x \in R \\ \langle a, x \rangle &> b && \text{for all } x \in S \end{aligned}$$



Example



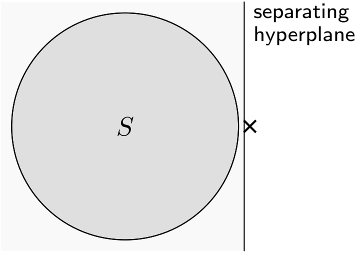
Counter example
S and R not bounded

Consequence

a closed convex set S is the intersection of all halfspaces that contain it

proof:

- let H be the intersection of all halfspaces containing S
- $\Rightarrow$: obviously $x \in S \Rightarrow x \in H$
- $\Leftarrow$: assume $x \notin S$, since S closed and convex and x compact (a point), there exists a strictly separating hyperplane, i.e., $x \notin H$ (see figure)

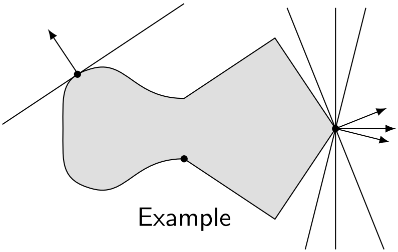


Supporting hyperplanes

- the hyperplane $H_{s,r} = \{y \mid \langle s, y\rangle = r\}$ supports S at $x \in \text{bd } S$ if

$$\langle s, y\rangle \leq r \text{ for all } y \in S \quad \text{and} \quad \langle s, x\rangle = r$$

  i.e., if S is in a halfspace delimited by $H_{s,r}$ that passes through x
- such hyperplanes are referred to as *supporting hyperplanes*
- (note: we only define supporting hyperplanes for boundary points)

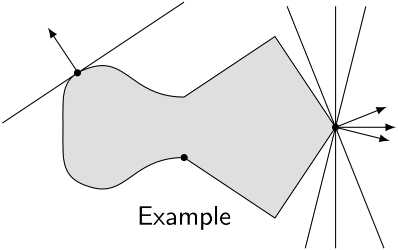


Example

Supporting hyperplane theorem

Let S be a nonempty convex set and let $x \in \mathrm{bd}(S)$. Then there exists a supporting hyperplane to S at x.

- proof
  - $\mathrm{int}(S) \neq \emptyset$: apply separating hyperplane theorem to the sets $\{x\}$ and $\mathrm{int}(S)$
  - $\mathrm{int}(S) = \emptyset$: then bd$S = S$ and S in affine subspace with dim aff $S \leq n - 1$, all affine subspaces of dim $n - 1$ are hyperplanes, therefore there exist a hyperplane $H_{s,r}$ such that $S =$ bd $S \subseteq H_{s,r}$ and hence in half-space defined by hyperplane
- can define for points on rbd S instead, degenerate case disappears
- does not hold for nonconvex sets



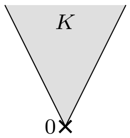
Example

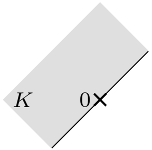
Polar cones

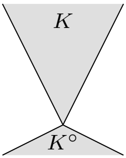
- the polar cone K° to the convex cone K is defined as:

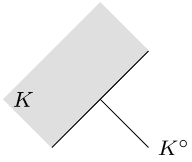
$$K^\circ := \{s \in \mathbb{R}^n \mid \langle s, x\rangle \le 0 \text{ for all } x \in K\}$$

- it is the set of normal vectors to supporting hyperplanes to K at 0
- the bipolar cone satisfies $K^{\circ\circ} := (K^\circ)^\circ = \text{cl } K$
- what is polar cone of





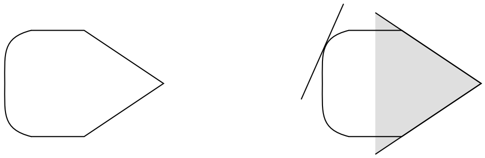




Canonical approximations of sets

- smooth sets can locally be approximated by affine manifold

- for nonsmooth sets, we can approximate with a cone



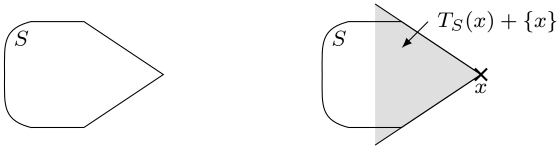
(reduces to affine manifold in smooth case)

Tangent cone operator

- the cone approximation to set S is called a *tangent cone*
- for *closed and convex* sets, the tangent cone $T_S(x)$ is defined as

$$T_S(x) = \overline{\mathsf{cone}}\,(S - \{x\}) = \mathsf{cl}\ \mathbb{R}_+(S - \{x\})$$

i.e., shift current point to origin, and form conical hull



- $T_S(x)$ is often visualized by $T_S(x) + \{x\}$ (i.e., shifted to x)

Normal cone operator

- the normal cone operator to a (maybe nonconvex) set S is

$$N_S(x) = \begin{cases} \{s \mid \langle s, y - x \rangle \leq 0 \text{ for all } y \in S\} & \text{if } x \in S \\ \emptyset & \text{else} \end{cases}$$

  i.e., vectors that form obtuse angle between s and all $y - x$, $y \in S$

- if $x \in \text{int } S$, what is $N_S(x)$? $N_S(x) = 0$

Relation to supporting hyperplanes

- since $N_S(\text{int } S) = 0$, the normal cone can be written as

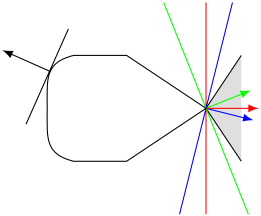
$$N_S(x) = \begin{cases} \{s \mid \langle s, y\rangle \le \langle s, x\rangle \text{ for all } y \in S\} & \text{if } x \in S \cap \text{bd } S \\ 0 & \text{if } x \in \text{int } S \\ \emptyset & \text{else} \end{cases}$$

- on boundary: $N_S(x)$ is set of normals to supporting hyperplanes
- if S convex, we know that $N_S(x) \neq \emptyset$ for all $x \in S \cap \text{bd } S$ (supporting hyperplane theorem)

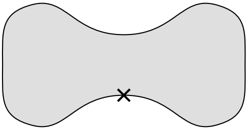
Examples

we consider boundary points only

- a convex example:



- a nonconvex example



($N_S(x) = \emptyset$ at marker since no supporting hyperplane)

Relation between tangent and normal cones

- suppose that S is nonempty closed and convex
- what is the relation between $T_S(x)$ and $N_S(x)$ for $x \in S$?
- they are polar to each other, $N_S(x) = (T_S(x))^\circ$:

$$
\begin{aligned}
N_S(x) &= \{s \mid \langle s, y - x \rangle \le 0 \text{ for all } y \in S\} \\
&= \{s \mid \langle s, d \rangle \le 0 \text{ for all } d \in (S - \{x\})\} \\
&= \{s \mid \langle s, d \rangle \le 0 \text{ for all } d \in \text{cone}\,(S - \{x\})\} \\
&= \{s \mid \langle s, d \rangle \le 0 \text{ for all } d \in \overline{\text{cone}}\,(S - \{x\})\} \\
&= \{s \mid \langle s, d \rangle \le 0 \text{ for all } d \in T_S(x))\} \\
&= (T_S(x))^\circ
\end{aligned}
$$

- proof $T_S(x) = (N_S(x))^\circ$: since $T_S(x)$ is closed by definition:

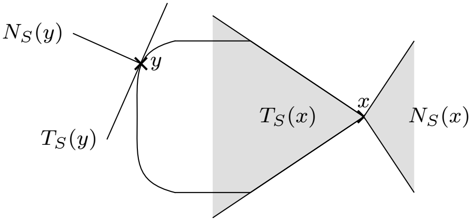
$$T_s(x) = (T_s(x)^\circ)^\circ = N_S(x)^\circ$$

- therefore, for convex sets, the tangent cone can be defined as

$$T_S(x) = \{d \mid \langle s, d \rangle \le 0 \text{ for all } s \in N_S(x)\}$$

Graphical representation

- example with normal cones and tangent cones for a convex set S
- cones shifted corresponding points x and y
- we see that the cones are polar



A calculus rule

- for $x \in S_1 \cap S_2$ with S_1, S_2 closed and convex, there holds:

$$T_{S_1 \cap S_2} \subseteq T_{S_1}(x) \cap T_{S_2}(x), \qquad N_{S_1 \cap S_2} \supseteq N_{S_1}(x) + N_{S_2}(x)$$

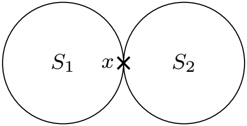
- under the additional constraint qualification that

$$(\text{ri } S_1) \cap (\text{ri } S_2) \neq \emptyset$$

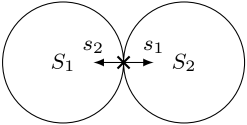
we have equality (this will be shown later!)

Example constraint qualification

- example that indicates constraint qualification is needed:



- what is $N_{S_1 \cap S_2}(x)$ and $N_{S_1}(x) + N_{S_2}(x)$?
  - $N_{S_1 \cap S_2}(x) = N_{\{x\}}(x) = \mathbb{R}^2$
  - $N_{S_1}(x) = \mathbb{R}_+\{s_1\}$
  - $N_{S_2}(x) = \mathbb{R}_+\{s_2\}$
  - $N_{S_1}(x) + N_{S_2}(x) = \mathbb{R}\{s_1\} = \mathbb{R}\{s_2\}$



- constraint qualification important for many results in convex analysis