Homework assignment 3

Exercises 3, 4, and 5 are Hand-in exercises.

1. Assume that $T : \mathbb{R}^n \to \mathbb{R}^n$ is $\frac{1}{\beta}$-cocoercive and that $\gamma \in (0, \frac{2}{\beta})$.

a. Motivate graphically that $\mathrm{Id} - \gamma T$ is $\frac{\gamma\beta}{2}$-averaged.

b. Show that $\mathrm{Id} - \gamma T$ is $\frac{\gamma\beta}{2}$-averaged.

2. Assume that $T : \mathbb{R}^n \to \mathbb{R}^n$ is $\frac{1}{\beta}$-cocoercive with $\beta > 0$.

a. Using graphical arguments, estimate a tight Lipschitz constant to $2T - \beta\mathrm{Id}$

b. Prove that the estimated Lipschitz constant holds.

c. Show that the estimated Lipschitz constant is tight. That is, provide a $\frac{1}{\beta}$-cocoercive operator such that the Lipschitz inequality holds with equality.

3. Assume that T is α-averaged with $\alpha \in (0, \frac{1}{2})$. Let $R = 2T - \mathrm{Id}$.

a. Using graphical arguments, estimate an averagedness parameter for R.

b. Show that the averagedness parameter provided above holds.

4. Assume that $\alpha \in (0, 1)$ and recall that an operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is α-averaged if $T = (1 - \alpha)\mathrm{Id} + \alpha R$ for some nonexpansive operator R. Show that the following are equivalent

(i) T is α-averaged

(ii) $(1 - \alpha^{-1})\mathrm{Id} + \alpha^{-1} T$ is nonexpansive

(iii) the following holds for all $x, y \in \mathbb{R}^n$

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \tfrac{1-\alpha}{\alpha}\|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2$$

Hint: You may use that for any $u, v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$:

$$\|\lambda u + (1 - \lambda)v\|^2 + \lambda(1 - \lambda)\|u - v\|^2 = \lambda\|u\|^2 + (1 - \lambda)\|v\|^2.$$

5. Assume that T is $\frac{1}{\beta}$-cocoercive with $\beta \in (0, 1)$ and let $R = 2T - \mathrm{Id}$. Then R is α-negatively averaged.

a. Using graphical arguments, estimate α.

b. Show that R is negatively averaged with the above estimate.

6. Assume that T is β-negatively averaged for $\beta \in (0, 1)$. Let $S = (1 - \alpha)\mathrm{Id} + \alpha T$ with $\alpha \in (0, 1)$. Show that S is contractive with the above estimated contraction factor.

7. Suppose that $T + \alpha\mathrm{Id}$ is $\frac{1}{\alpha+\beta}$-cocoercive with $\alpha + \beta > 0$. Then T is Lipschitz continuous.

a. Using graphical arguments, estimate a Lipschitz constant to T.

b. Show that the above provided Lipschitz constant holds.

8. Assume that f is proper closed and σ-strongly convex and let $h = f + \frac{1}{2}\|\cdot\|^2$. Provide a smoothness parameter to h^* and cocoercivity parameter to ∇h^*.

9. Assume that f is proper closed σ-strongly convex and β-smooth and let $h = f + \frac{1}{2}\|\cdot\|^2$.

a. Provide a smoothness parameter to $h^* - \frac{1}{2(1+\beta)}\|\cdot\|^2$.

b. For $\beta > \sigma$, provide a cocoercivity parameter of $\nabla h^* - \frac{1}{1+\beta}\mathrm{Id}$.

c. For $\beta = \sigma$, provide a Lipschitz constant to $\nabla h^* - \frac{1}{1+\beta}\mathrm{Id}$.