Homework assignment 2

Exercises 4 and 7 are Hand-in exercises.

1. Compute the conjugate of the following functions (with the standard inner-product $\langle x,y\rangle = x^T y = \sum_{i=1}^n x_i y_i$).

a. Compute the conjugate of $f(x) = \|x\|$.

b. Compute the conjugate of $f(x) = \|x\|_1$.

c. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear operator, let $b \in \mathbb{R}^m$, and let $V_b = \{x \mid Lx = b\}$ be nonempty. Compute the conjugate of $\iota_{V_b}(x)$.

d. Let $C = \{x \mid x \le 0\}$. Compute the conjugate of $f(x) = \iota_C(x)$.

2. Compute the subdifferential of the following functions using (consequences of) Fenchel-Young's equality.

a. $f(x) = \|x\|$

b. $f(x) = \|x\|_1$

c. $f(x) = \iota_{V_b}(x)$, where $V_b = \{x \mid Lx = b\} \neq \emptyset$, $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator and $b \in \mathbb{R}^m$.

3. Suppose that $\gamma \in (0,\infty)$ and f is proper. Show that

$$(\gamma f)^* = \gamma(f^* \circ \gamma^{-1}\mathrm{Id}).$$

4. Suppose that f is proper, that $q = \frac{1}{2}\|\cdot\|^2$, and that $\gamma \in (0,\infty)$.

a. Show that

$$\begin{aligned}(f+\gamma q)^*(y) - \gamma^{-1}q(y) &= -\inf_z\{f(z) + \tfrac{\gamma}{2}\|\gamma^{-1}y - z\|^2\} \\ &=: -((f \,\Box\, \gamma q) \circ \gamma^{-1}\mathrm{Id})(y)\end{aligned}$$

Hint: Start by explicitly stating the definition of $(f+\gamma q)^*$.

b. Assume that g is proper closed and convex. Use **a.** to show that g^* is $\frac{1}{\gamma}$-smooth if g is γ-strongly convex.
Hint: g is γ-strongly convex if there exists convex f such that $g = f + \frac{\gamma}{2}\|\cdot\|^2 = f + \gamma q$ and (since g^* is convex) g^* is γ^{-1}-smooth if there exists a convex function h such that $g^* = \gamma^{-1}q - h$.

c. Show that

$$(\gamma q - f^*)^* = \gamma^{-1}q + \gamma^{-1}(\gamma f - q)^*$$

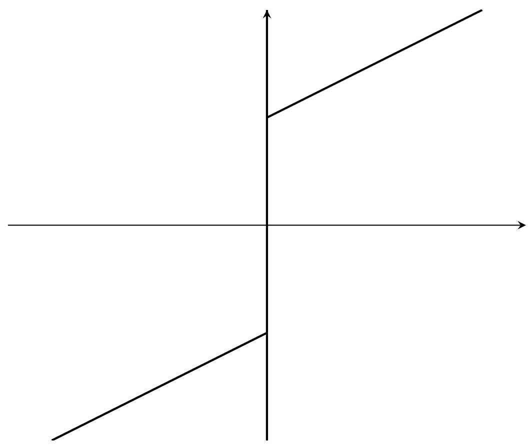
d. Suppose that g is proper closed and convex. Use **c.** to show that g^* is γ^{-1}-strongly convex if g is convex and γ-smooth.
Hint: as in **b.** and that a smooth function g satisfies $g = \gamma q - h = \gamma q - (h^*)^*$ for some proper closed and convex h.

e. Suppose that f is proper closed and convex. Show that f is γ-strongly convex if and only if f^* is γ^{-1}-smooth, and that f is γ^{-1}-smooth if and only if f^* is γ-strongly convex.

5. Suppose that f is proper closed and convex. Let ∂f be given by

$$\partial f(x) = \begin{cases} \frac{1}{2}(x-1) & \text{if } x \leq 0 \\ [-\frac{1}{2}, \frac{1}{2}] & \text{if } x = 0 \\ \frac{1}{2}(x+1) & \text{if } x \geq 0 \end{cases}$$

i.e., $\partial f(x)$ is given by:



Draw f, ∂f^* and f^*.

6. Suppose that f is proper and has an affine minorizer.

a. Show that the infimal convolution between f^{**} and the standard 2-norm $\|\cdot\|$ satisfies

$$(f^{**} \,\Box\, \|\cdot\|)(x) := \inf_y\{f^{**}(y) + \|y - x\|\} = \sup_{\|s\|\leq 1}\{\langle s, x\rangle - f^*(s)\}.$$

Hint: $(f^{**})^* = f^*$.

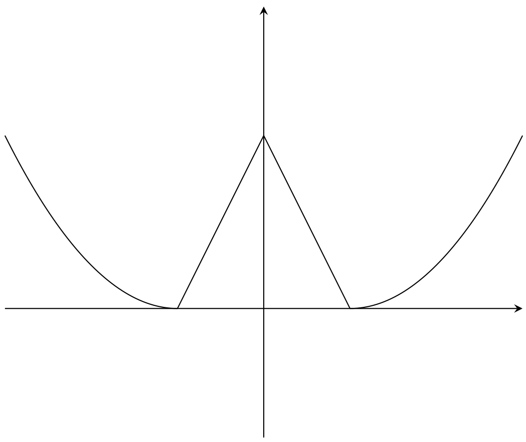
b. Use **a.** to show

$$(f^{**} \,\Box\, \|\cdot\|)(x) = \sup_{\|s\|\leq 1, r}\{\langle s, x\rangle - r \mid \langle s, z\rangle - r \leq f(z) \text{ for all } z \in \mathbb{R}^n\}$$

i.e, it is the supremum of all affine minorizors with slope $\|s\| \leq 1$.

c. Draw f^{**} and $f^{**} \,\Box\, \|\cdot\|$ for

$$f(x) = \begin{cases} \frac{1}{2}(x+1)^2 & \text{if } x \leq -1 \\ 2x + 1 & \text{if } -1 \leq x \leq 0 \\ -2x + 1 & \text{if } \leq x \leq 1 \\ \frac{1}{2}(x-1)^2 & \text{if } x \geq 1 \end{cases}$$



7. Consider an optimization problem of the form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \leq 0 \\ & Lx = b \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g = (g_1, \ldots, g_k) : \mathbb{R}^n \to \mathbb{R}^k$ are closed and convex, and $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping. Further assume that Slater's constraint qualification holds, i.e., that there exists $\bar{x}$ such that $g(\bar{x}) < 0$ and $L\bar{x} = b$. Show that the optimality conditions for this problem can be written as

$$\begin{aligned} &0 \in \partial f(x) + \sum_{i=1}^{k} \mu_i \partial g_i(x) + L^* \lambda \\ &0 = Lx - b \\ &0 \geq g(x) \\ &0 \leq \mu \\ &0 = \mu_i g_i(x) \text{ for all } i = 1, \ldots, k \end{aligned}$$

These are called KKT (Karush-Kuhn-Tucker) conditions that are usually stated for differentiable functions f, g.

Hint: You may use all results in Exercise 1 and the result from Exercise 1.2 that the normal cone operator to $\iota_{Lx=b}(x)$ is given by

$$N_{\iota_{Lx=b}}(x) = \begin{cases} L^* \lambda & \text{if } Lx = b \\ \emptyset & \text{else} \end{cases}$$

for any $\lambda \in \mathbb{R}^m$.

8. Let $f(x) = \frac{1}{2} x^T H x + q^T x$ where H is symmetric positive semi-definite. Further, let $L \in \mathbb{R}^n \to \mathbb{R}^m$ be a matrix. Show that

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Lx = b \end{array}$$

can be solved by solving a linear system of equations.