Homework assignment 1

Exercises 2, 7, and 8 are Hand-in exercises.

1. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear operator. Show that all points $x \in \mathbb{R}^n$ in the affine set $V = \{x \in \mathbb{R}^n \mid Lx = 0\}$ satisfies $x \in \text{ri } V$. That is, show that $V = \text{ri } V$.

2. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a nontrivial linear operator, i.e., Lx is not zero for all $x \in \mathbb{R}^n$. Further, let $V = \{x \in \mathbb{R}^n \mid Lx = 0\}$.

a. Compute the normal cone to the affine set V at $x = 0$. Involve the adjoint operator L^* in the expression. Note: The adjoint operator to the linear operator $L : \mathbb{R}^n \to \mathbb{R}^m$ is defined as the unique operator $L^* : \mathbb{R}^m \to \mathbb{R}^n$ that satisfies

$$\langle Lx, y\rangle = \langle x, L^* y\rangle$$

for all $x \in \mathbb{R}^n$ and all $y \in \mathbb{R}^m$.

b. Compute the normal cone operator to V for any $x \in \mathbb{R}^n$

c. Compute the normal cone operator to $V_b := \{x \in \mathbb{R}^n \mid Lx = b\}$ (assume $V_b \neq \emptyset$) for any $x \in \mathbb{R}^n$.

3. Construct and example of two nonempty closed convex sets, where the set sum is not closed.

4. The strictly separating hyperplane theorem assumes that the two sets S and R are closed and convex, and that one of them is compact. Provide an example where S is closed convex and bounded, and R is convex and bounded for which no strictly separating hyperplane exists.

5. Show that $\text{dom } (g \circ L) = L^{-1}(\text{dom } g)$.

6. Assume that $C = \{x \mid g(x) \leq 0\}$ where $g : \mathbb{R}^n \to \mathbb{R}$ is convex. Slater's condition is that there exists $\bar{x}$ such that $g(\bar{x}) < 0$.

a. Show that this is implies that $\text{int } C \neq \emptyset$.

b. Construct a function g such that no $\bar{x}$ exists such that $g(\bar{x}) < 0$, but where $C = \{x \mid g(x) \leq 0\}$ has nonempty interior.

7. Suppose that $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, k$ are convex (and finite-valued). Let $g(x) = (g_1(x), \ldots, g_k(x)) : \mathbb{R}^n \to \mathbb{R}^k$ and consider the set $C = \{x \mid g(x) \leq 0\}$. Further, assume that there exists $\bar{x} \in \mathbb{R}^n$ such that $g(\bar{x}) < 0$, (vector-wise comparison). Show that the normal cone to C for any $x \in \mathbb{R}^n$ can be written as

$$N_C(x) = \begin{cases} \sum_{i=1}^{k} \mu_i \partial g_i(x) & \text{if } g(x) \leq 0 \\ \emptyset & \text{else} \end{cases}$$

with the additional constraints that $\mu_i g_i(x) = 0$ and $\mu_i \geq 0$ for all $i = 1, \ldots, k$.
Hint: The assumption implies that $\text{int } C_i = \{x \mid g_i(x) < 0\}$, that $\text{bd } C_i = \{x \mid g_i(x) = 0\}$ (which you can use without proving it).

8. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and suppose that f is proper closed and convex and that $\text{ri dom } f \neq \emptyset$. Further suppose, if nothing else is stated, that f is σ-strongly convex with $\sigma \in (0, \infty)$.

a. Show that the nonempty level-sets of f are bounded.
Hint: At any $x \in \text{ri dom } f$ there exists a subgradient to $f - \frac{\sigma}{2}\|\cdot\|^2$. Use this to show that $f(y) \to \infty$ as $\|y\| \to \infty$.

b. Show that the infimum of $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is attained, i.e., show that $\text{argmin}_x f(x)$ exists.

c. Show by a counter-example that $\text{argmin}_x f(x)$ need not exist if f is merely strictly convex.

9. Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is finite-valued and convex. Show that the directional derivative

$$d \mapsto f'(x, d) := \lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t}$$

is convex in d for fixed x.

10. Compute subdifferentials of the following functions.

a. Assume that C is a nonempty set. Show that $\partial \iota_C(x) = N_C(x)$, where

$$\iota_C(x) = \begin{cases} 0 & \text{if } x \in C \\ \infty & \text{else} \end{cases}$$

b. Compute the subdifferential of $f(x) = \frac{1}{2}\|x\|^2$.

c. Compute the subdifferential of $f(x) = \|x\| = \sqrt{\sum_i x_i^2}$.

d. Compute the subdifferential of $f(x) = \|x\|_1 = \sum_i |x_i|$.

e. Compute the subdifferential of $f(x) = \langle c, x \rangle$.

11. In relation to the result that a closed function is convex if and only if $\text{dom } f$ is convex and $\text{dom } \partial f \supseteq \text{ri dom } f$, provide counter-examples if some of the assumptions do not hold.

a. Construct a closed nonconvex function f with $\text{dom } \partial f \supseteq \text{ri dom } f$ but $\text{dom } f$ is not convex.

b. Construct a closed nonconvex function f with $\text{dom } \partial f \subset \text{ri dom } f$ with $\text{dom } f$ convex.

c. Construct a nonconvex (not closed) function f with $\text{dom } \partial f \supseteq \text{ri dom } f$ and $\text{dom } f$ convex.