Homework3 (Linear Algebra)

We have data values:

X	1	1.5	2	2.2	2.7	3.6	4.1	5.6	5.8	6
Y	15	43	125	185	450	1700	3137	13800	16300	19000

we want to find the interpolation polynomial of order 5, such that the summation of coefficients is 15.

Solve the constrained least squares problem by one of the explained methods in “Numerical Methods in Multibody Dynamics, E. Eich-Soellner, C. Führer”. (Those pages of the book have been attached)

If the condition is not much higher than the condition of M, the described method was successful.
The condition estimator is usually used in conjunction with a solver for linear equations, which is used here to compute $\widetilde{M}^{-1}\widetilde{D}$, and $\widetilde{M}^{-1}\widetilde{K}$ in order to write Eq. (2.1.12) in the explicit form $\dot{x} = Ax$ with

$$x := \begin{pmatrix} y \\ \dot{y} \end{pmatrix}; A := \begin{pmatrix} 0 & I \\ -\widetilde{M}^{-1}\widetilde{K} & -\widetilde{M}^{-1}\widetilde{D} \end{pmatrix}$$

Example 2.2.1 *To summarize this procedure, we will apply it to the constrained truck example (Ex. 1.3.2). This results in the following MATLAB statements*

```
[Q,R,P]=qr(G);   %  QR-Decomposition with Pivoting
R1=R(1:2,1:2);   %  Splitting up R into a triangular part R1
S=R(1:2,3:9);    %  and a rectangular S part
V=P*[-R1\S;eye(7)]; %
%
% transformation to state-space form
%
Mtilde=V'*M*V;  Ktilde=V'*K*V; Dtilde=V'*D*V;
A=[zeros(7),eye(7);-Mtilde\Ktilde,-Mtilde\Dtilde];
```

As result for the constrained truck, linearized around its nominal position we obtain

$$V = \begin{pmatrix} 1.0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.32 & -0.32 & 0.52 \\ 0 & 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.05 & 0.05 & -0.83 \\ 0 & 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.0 \end{pmatrix}$$

Thus, this procedure takes $p_1, p_2, p_4, p_5, p_6, p_8, p_9$ as state variables and p_3, p_7 as dependent ones. This can be verified by checking V or P.

Alternative methods can be found in Sec. 2.3.2.

2.3 Constrained Least Squares Problems

We will frequently encounter in the sequel constrained least squares problems of the form

$$\|Fx - a\|_2^2 = \min_x \tag{2.3.1a}$$

$$Gx - b = 0 \tag{2.3.1b}$$

with an $n_x \times n_x$ regular matrix F and a full rank $n_\mu \times n_x$ constraint matrix G. Typically, these systems occur when discretizing the linearized equations of motion of a constrained multibody system. In the nonlinear case, they occur inside an iteration process, cf. Sec. 5.3.
By introducing Lagrange multipliers μ, the problem reads equivalently

$$\begin{aligned} F^T F x - F^T a + G^T \mu &= 0 \qquad (2.3.2a) \\ Gx - b &= 0, \qquad (2.3.2b) \end{aligned}$$

or in matrix notation

$$\begin{pmatrix} F^T F & G^T \\ G & 0 \end{pmatrix} \begin{pmatrix} x \\ \mu \end{pmatrix} = \begin{pmatrix} F^T a \\ b \end{pmatrix}. \qquad (2.3.3)$$

The solution of this equation can be expressed in two different ways (see [GMW81]):

Range Space Formulation
From the first equation we obtain

$$x = (F^T F)^{-1} \left(F^T a - G^T \mu \right).$$

Inserting this into the second one gives

$$G(F^T F)^{-1}(F^T a - G^T \mu) - b = 0.$$

By assuming $G(F^T F)^{-1} G^T$ to be regular we finally get

$$\begin{aligned} x &= \left(I - (F^T F)^{-1} G^T \left(G(F^T F)^{-1} G^T \right)^{-1} G \right) (F^T F)^{-1} F^T a \\ &\quad + (F^T F)^{-1} G^T \left(G(F^T F)^{-1} G^T \right)^{-1} b \qquad (2.3.4a) \\ \mu &= (G(F^T F)^{-1} G^T)^{-1} (G(F^T F)^{-1} F^T a - b). \qquad (2.3.4b) \end{aligned}$$

Null Space Formulation
Defining V as a matrix spanning the null space of G and x_{p} as a particular solution of the second equation of (2.3.3), we get

$$x = Vy + x_{\mathrm{p}}.$$

Inserting this into the first equation of (2.3.3) and premultiplying by V^T results in

$$y = (V^T F^T F V)^{-1} V^T F^T (-F x_{\mathrm{p}} + a),$$

and finally we obtain

$$x = V(V^T F^T F V)^{-1} V^T F^T a + \left(I - V(V^T F^T F V)^{-1} V^T F^T F\right) x_{\mathrm{p}}. \tag{2.3.5}$$

x_{p} can be computed using the Moore–Penrose pseudo-inverse

$$x_{\mathrm{p}} = G^+ b \text{ with } G^+ = G^T (G G^T)^{-1}.$$

We get

$$x = \begin{pmatrix} F \\ G \end{pmatrix}^{\mathrm{CLSQ}+} \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.3.6}$$

with

$$\begin{pmatrix} F \\ G \end{pmatrix}^{\mathrm{CLSQ}+} := \begin{pmatrix} V(FV)^+ & (I - V(FV)^+ F)\, G^+ \end{pmatrix}. \tag{2.3.7}$$

Alternatively, a weighted generalized inverse $G^\dagger$ may be used instead of the Moore–Penrose pseudo-inverse:

$$x_{\mathrm{p}} = G^\dagger b \text{ with } G^\dagger = M^{-1} G^T (G M^{-1} G^T)^{-1} \tag{2.3.8}$$

with M being an invertible $n_x \times n_x$ matrix.
The particular steps in a numerical computation of the solution of a constrained least squares problem will be discussed in Sec. 2.3.2.
Both approaches can be related to each other by first setting $b = 0$ and equating (2.3.4a) with (2.3.5):

$$\left(I - (F^T F)^{-1} G^T (G (F^T F)^{-1} G^T)^{-1} G\right) (F^T F)^{-1} F^T = V (V^T F^T F V)^{-1} V^T F^T. \tag{2.3.9}$$

This relation will be used later.
On the other hand, by setting $a = 0$ we get

$$(I - V(V^T F^T F V)^{-1} V^T F^T F) G^\dagger = (F^T F)^{-1} G^T (G (F^T F)^{-1} G^T)^{-1}. \tag{2.3.10}$$

2.3.1 Pseudo-Inverses

In the special case of an underdetermined linear system we may look for the minimum norm least squares solution of this system. With $F = I$, $a = 0$ we obtain from Eq. (2.3.4)

$$x = G^T (G G^T)^{-1} b.$$