Session 0

Math background

Reading Assignment

Get the book.

Exercise 0.1 Compute e^{At} for $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

Exercise 0.2 Prove the Courant-Fisher formula for symmetric A

$$\lambda_{max}(A) = \max_{\|x\|=1} x^T A x = \max_{x \neq 0} \frac{x^T A x}{x^T x},$$

for example using the decomposition $A = U \Lambda U^T$.

Exercise 0.3 Show that the spectral norm is given by

$$\|A\|_2 = \left(\max_{\|x\|=1} x^T A^T A x \right)^{1/2}.$$

Conclude that $\|A\|_2 = \left(\lambda_{max}(A^T A)\right)^{1/2} = \sigma_{max}(A)$.

Exercise 0.4 Show that for the spectral norm we have for invertible A

$$\|A^{-1}\|_2 \geq \|A\|_2^{-1}.$$

What holds for other induced matrix norms? For all matrix norms?

Exercise 0.5 Show that for any eigenvalue λ of A we have

$$|\lambda| \leq \|A\|_2$$

Is the same true for all induced matrix norms? For all matrix norms?

Exercise 0.6 Show that if A is symmetric with $0 < aI \leq A \leq bI$ then

$$0 < b^{-1} I \leq A^{-1} \leq a^{-1} I$$

Exercise 0.7 Show that $\|A\|_F^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_r^2$.

Exercise 0.8 Show that $\|AB\|_F \leq \|A\|_F \|B\|_F$, i.e. that the Frobenius-norm is submultiplicative.

Exercise 0.9 Show that $x \in N(A) \iff x \perp R(A^T)$.

Exercise 0.10 Show that $x(t) = e^{At} b \theta(t)$, is a solution to

$$\frac{dx}{dt} = Ax + bu, \qquad x(0-) = 0$$

where $u(t) = \delta(t)$.

Exercise 0.11 Consider $h(t) = ce^{At}b\theta(t)$ and $H(s) = c(sI - A)^{-1}b$ Show that

$$H(s) = c(I/s + A/s^2 + A^2/s^3 + \ldots)b = \sum_{k=1}^{\infty} h_k/s^k$$

where $h_k = h^{(k-1)}(0+) = cA^{k-1}b$ (the *Markov parameters*).

Exercise 0.12 Show that if A is asymptoically stable then Taylor-expansion of $H(s)$ around $s = 0$ gives

$$H(s) = \sum_{k=0}^{\infty} m_k s^k,$$

where $m_k = H^{(k)}(0)/k! = \int_0^\infty \frac{(-t)^k}{k!} h(t)dt = -cA^{-k-1}b$ (the *moments* of $h(t)$).

Hand in problems - to be handed in at exercise session

Do the two problems at the end of Lecture 0 and the problem below:

Handin 0.3: Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with the input

$$u(t) = u_0\delta(t) + u_1\delta^{(1)}(t) + \ldots + u_r\delta^{(r)}(t).$$

where the u_k are constants. Show that there is a solution of the form

$$x(t) = e^{At}v_0\theta(t) + v_1\delta(t) + \ldots + v_r\delta^{(r-1)}(t)$$

and determine the vectors v_k.