Lecture 5

- LTV stability concepts
- Quadratic Lyapunov functions
- Feedback, Well-posedness, Internal Stability

Rugh Ch 6,7,12 (skip proofs of 12.6 and 12.7),14 (pp240-247) + (22,23,24,28)

Zhou, Doyle, Glover pp 117-124

Stability

For LTI systems $\dot{x} = Ax$ the stability concept was easy, we had the two concepts

i) Stability: $x(t)$ remains bounded
ii) Asymptotic stability: $x(t)$ goes to zero

Corresponding to eigenvalues of A

i) either in the open left half plane, or on imaginary axis (if all such Jordan blocks have size 1)
ii) in the open left half plane

For example $\dot{x} = 0$ is stable but not asymptotically stable

Stability is more subtle for LTV systems

And how is LTV stability reflected in the $\Phi(t, t_0)$-matrix?

Definition of Uniform Stability

The system $\dot{x}(t) = A(t)x(t)$ is called

uniformly stable if $\exists \gamma > 0$ such that (for all $t_0 \geq 0$ and $x(t_0)$)

$$|x(t)| \leq \gamma |x(t_0)|, \quad \forall t \geq t_0 \geq 0$$

uniformly asymptotically stable if it is uniformly stable and

$$\forall \delta > 0 : \ \exists T > 0 :$$

$$|x(t)| \leq \delta |x(t_0)|, \quad \forall t \geq t_0 + T, \ t_0 \geq 0$$

uniformly exponentially stable if $\exists \gamma, \lambda > 0$ such that

$$|x(t)| \leq \gamma |x(t_0)| e^{-\lambda(t-t_0)}, \quad t \geq t_0 \geq 0$$

Note: Rugh does not include the condition $t_0 \geq 0$.

Transition Matrix Conditions

From the relation $x(t) = \Phi(t, t_0)x(t_0)$ and the definition of matrix norm follows that the system $\dot{x}(t) = A(t)x(t)$ is

uniformly stable if $\exists \gamma > 0$

$$\|\Phi(t, t_0)\| \leq \gamma, \quad \forall t \geq t_0 \geq 0$$

uniformly asymptotically stable if it is uniformly stable and
$\forall \delta > 0 : \exists T > 0 :$

$$\|\Phi(t, t_0)\| \leq \delta, \quad \forall t \geq t_0 + T, \; t_0 \geq 0$$

uniformly exponentially stable if $\exists \gamma, \lambda > 0$ such that

$$\|\Phi(t, t_0)\| \leq \gamma e^{-\lambda(t-t_0)}, \quad \forall t \geq t_0 \geq 0$$

Comparisons

The first stability concept is the weakest.

The system $\dot{x} = 0$ is unif. stable but not unif. asymp. stable

The third condition at first looks stronger than the second, but surprisingly enough they are equivalent.

In fact we have the following result

Criterion for Exponential Stability

For the equation $\dot{x}(t) = A(t)x(t)$ with $\|A(t)\|$ bounded, the following three conditions are equivalent:

(i) The equation is uniformly exponentially stable.

(ii) The equation is uniformly asymptotically stable.

(iii) There exists a $\beta > 0$ such that

$$\int_\tau^t \|\Phi(t,\sigma)\| d\sigma \leq \beta \quad \forall t \geq \tau \geq 0$$

Proof

(i) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (ii) Let $\alpha = \sup_t \|A(t)\|$. Then asym. stab. follows from

$$\begin{aligned}
\|\Phi(t,\tau)\| &= \left\| I - \int_\tau^t \frac{\partial}{\partial\sigma} \Phi(t,\sigma) d\sigma \right\| \\
&= \left\| I + \int_\tau^t \Phi(t,\sigma) A(\sigma) d\sigma \right\| \\
&\leq 1 + \alpha \int_\tau^t \|\Phi(t,\sigma)\| d\sigma \ \leq \ 1 + \alpha\beta
\end{aligned}$$

$$\begin{aligned}
\|\Phi(t,\tau)\| &= \frac{1}{t-\tau} \int_\tau^t \|\Phi(t,\tau)\| d\sigma \\
&\leq \frac{1}{t-\tau} \int_\tau^t \|\Phi(t,\sigma)\| \cdot \|\Phi(\sigma,\tau)\| d\sigma \ \leq \ \frac{\beta}{t-\tau}(1+\alpha\beta)
\end{aligned}$$

Proof of (ii)$\Rightarrow$ (i)

Assume asymptotic stability. To prove exponential stability, select $\gamma, T > 0$ such that

$$\begin{aligned} \|\Phi(t,t_0)\| &\leq \gamma \quad \forall t \geq t_0 \\ \|\Phi(t_0+T,t_0)\| &\leq \frac{1}{2} \quad \forall t \geq t_0 + T \end{aligned}$$

Then

$$\begin{aligned} \|\Phi(t_0+kT,t_0)\| &\leq \|\Phi(t_0+kT,t_0+(k-1)T)\| \cdots \|\Phi(t_0+T,t_0)\| \\ &\leq \frac{1}{2^k}, \quad k = 1, 2, \dots \end{aligned}$$

$$\|\Phi(t,t_0)\| \leq \|\Phi(t,t_0+kT)\| \cdot \|\Phi(t_0+kT,t_0)\| \leq \frac{\gamma}{2^k},\ t \geq t_0$$

This proves exponential stability with $\lambda = \frac{\ln 2}{T}$.

An observation

The equation $\dot{x}(t) = A(t)x(t)$ is uniformly exponentially stable with rate λ, if and only if the equation

$$\dot{z}(t) \quad = \quad [A(t) - \alpha I]z(t)$$

is uniformly exponentially stable with rate $\lambda + \alpha$.

Proof. The lemma follows from the fact that $x(t)$ solves $\dot{x} = Ax$ if and only if $z(t) = e^{-\alpha t}x(t)$ solves $\dot{z} = [A - \alpha I]z$.

Warning: Stability Under Coordinate Change

Note that the scalar system

$$\dot{x} = x$$

is not stable, but the change of coordinates $z(t) = e^{-2t}x(t)$ gives the stable equation

$$\dot{z} = -z$$

This motivates some care when allowing for time varying coordinate changes

Lyapunov Transformation

An $n \times n$ continuously differentiable matrix function $T(t)$ is called a *Lyapunov transformation* if there exist $\rho > 0$ s.t.

$$\|T(t)\| \leq \rho, \quad \|T(t)^{-1}\| \leq \rho \quad \forall t$$

For such a transformation we have

$$\|\Phi_x(t, t_0)\| = \|T(t)\Phi_z(t, t_0)T(t_0)^{-1}\| \leq \rho^2 \|\Phi_z(t, t_0)\|$$

$$\|\Phi_z(t, t_0)\| = \|T(t)^{-1}\Phi_x(t, t_0)T(t_0)\| \leq \rho^2 \|\Phi_x(t, t_0)\|$$

Hence both uniform stability and uniform exponential stability are preserved under a coordinate transformation $x(t) = T(t)z(t)$ defined by a Lyapunov transformation.

Lyapunov Equation

The equation $\dot{x}(t) = A(t)x(t)$ is uniformly exponentially stable with rate λ, if there exists $Q > 0$ such that

$$A(t)^T Q + QA(t) \le -2\lambda Q$$

or $P > 0$ such that

$$PA(t)^T + A(t)P \le -2\lambda P$$

Note: For LTV systems, existence of such Q is a sufficient but not necessary condition for unif. exp. stability (for LTI systems it is both sufficient and necessary)

Proof

Given Q, we have

$$\frac{d}{dt}x^T Qx = x^T[A(t)^T Q + QA(t)]x \leq -2\lambda(x^T Qx)$$

so $x(t)^T Qx(t) \leq e^{-2\lambda t}x(0)^T Qx(0)$. From this follows that

$$\|x(t)\|^2 \leq e^{-2\lambda t}x(0)^T Qx(0)/\lambda_{min}(Q).$$

Given P, put $Q = P^{-1}$ and proceed as before

Linear Matrix Inequalities

An LMI is an expression of the form

$$A_0 + x_1 A_1 + \dots x_k A_k \geq 0$$

where $x_1, \dots, x_k$ are scalars and A_j given symetric matrices

Many control problems can be formulated as a search for x solving an LMI, and efficient SW exist for solving LMIs

Note that the scalars x_k can occur as elements in vectors or matrices. For example, the requirements

$$A^T Q + QA \leq 0, \qquad Q = Q^T$$

is an LMI (x being the elements of Q on or above the diagonal)

Matlab Software - CVX

After downloading CVX:

```
A1=[-5 -4;-1 -2];
A2=[-2 -1; 2 -2];
cvx_begin sdp
variable Q(2,2)
subject to
A1'*Q+Q*A1 < 0
A2'*Q+Q*A2 < 0
Q >= eye(2)
cvx_end
Q =
    4.1169   -0.8749
   -0.8749    6.8597
```

Feedback law from linear matrix inequality

If there exist $Y > 0$ and K such that

$$(AY + BK) + (AY + BK)^T \leq -2\lambda Y$$

then the LTI system

$$\dot{x} = (A + BL)x$$

with $L = KY^{-1}$, is uniformly exponentially stable with rate λ.

LTV Lyapunov Functions

By noting that

$$\frac{d}{dt}(x^T(t)Q(t)x(t)) = x^T(t)\left(A^T(t)Q(t) + Q(t)A(t) + \dot{Q}(t)\right)x(t)$$

it is easy to obtain several sufficient criteria for LTV stability (see Rugh Ch 7 for details)

Lyapunov Criteria for LTV

1. There exists $\eta > 0, \rho > 0, Q(t)$:

$$\eta I \leq Q(t) \leq \rho I, \quad A^T(t)Q(t) + Q(t)A(t) + \dot{Q}(t) \leq 0$$

$\Rightarrow |x|^2 \leq \rho/\eta |x(t_0)|^2 \Rightarrow$ uniform stability

2. There exists $\eta > 0, \rho > 0, \nu > 0, Q(t)$:

$$\eta I \leq Q(t) \leq \rho I, \quad A^T(t)Q(t) + Q(t)A(t) + \dot{Q}(t) \leq -\nu I$$

$\Rightarrow |x|^2 \leq \frac{\rho}{\eta} e^{-\frac{\nu}{\rho}(t-t_0)} |x(t_0)|^2 \Rightarrow$ uniform exponential stability

3. There exists $\rho > 0, \nu > 0, Q(t), t_0$:

$$\|Q(t)\| \leq \rho, \quad A^T(t)Q(t) + Q(t)A(t) + \dot{Q}(t) \leq -\nu I$$

$Q(t_0)$ not pos. semidef. $\Rightarrow$ not uniform stable

Constructing LTV Lyapunov Functions

The matrix differential equation

$$S(t) + A^T(t)Q(t) + Q(t)A(t) + \dot{Q}(t) = 0$$

has the solution

$$Q(t) = \int_t^\infty \Phi^T(\sigma,t)S(\sigma)\Phi(\sigma,t)d\sigma$$

if $A(t)$ is uniformly exponentially stable and $S(t)$ bounded.

Stability Margins

If the system is exponentially stable, stability will be maintained for small pertubations of the state equations

For instance one has the following result (Rugh exercise 8.12)

Let $\dot{x} = Ax$ be exponentially stable and let $g(x)$ satisfy

$$g(x) = o(\|x\|), \quad \text{as } x \to 0$$

then all solutions of

$$\dot{x} = Ax + g(x)$$

that start sufficiently close to the origin, converge to the origin

Uniform BIBO stability for LTV systems

The system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t)$$

is called bounded input bounded output stable if there is η such that the zero-state response (i.e. $x(t_0) = 0$)) satisfies

$$\sup_{t \geq t_0} \|y(t)\| \ \leq \ \eta \sup_{t \geq t_0} \|u(t)\|$$

for any t_0 and input $u(t)$.

Criteria for uniform BIBO-stability

1. Uniform BIBO stability $\Leftrightarrow$ exists ρ such that the impulse response satisfies

$$\int_{\tau}^{t} \|g(t,\sigma)\| d\sigma \leq \rho, \quad \forall \tau, t$$

2. Assume $A(t), B(t), C(t)$ are bounded and that controller and observer Gramians satisfy

$$\epsilon I \leq W(t-\delta, t)$$
$$\epsilon I \leq M(t, t+\delta)$$

for some positive ϵ, δ. Then
Uniform BIBO stability $\Leftrightarrow$ uniform exponential stability

Discrete Time

There are no big surprises going over to discrete time.

$\Phi(t, t_0)$ will change to $\Phi(k, k_0)$

$V(t) = x^T(t)Q(t)x(t)$ and $\dot{V}(t) < 0$ will change to
$V(k) = x^T(k)Q(k)x(k)$ and $V(k+1) - V(k) < 0$

Bounds of the form $\leq e^{-\lambda t},\ \ \lambda > 0$ will change to $\leq \lambda^k,\ \ \lambda < 1$

etc

Typical results are included on the next three frames

Internal Stability - Discrete Time

Definitions analog with continuous time.

With $\Phi(k, k_0) = A(k-1)\cdots A(k_0+1)A(k_0)$ we get e.g.

uniformly stable if $\exists \gamma > 0$

$$\|\Phi(k, k_0)\| \quad \leq \quad \gamma, \quad \forall k \geq k_0 \geq 0$$

uniformly exponentially stable if $\exists \gamma, \lambda < 1$ such that

$$\|\Phi(k, k_0)\| \leq \gamma \lambda^{k-k_0}, \quad \forall k \geq k_0 \geq 0$$

$$\Leftrightarrow \exists \beta : \qquad \sum_{i=k_0}^{k} \|\Phi(k, k_0)\| \leq \beta, \quad \forall k \geq k_0 \geq 0$$

Lyapunov criteria - Discrete Time

With Lyapunov function $V(k) = x^T(k)Q(k)x(k)$ we e.g. have

$$V(k+1) - V(k) = x^T(k)(A^T(k)Q(k+1)A(k) - Q(k))x(k)$$

Therefore discrete time results will look like:

If there exists positive η, ρ, ν and $Q(k)$ so that

$$\eta I \leq Q(k) \leq \rho I, \quad A^T(k)Q(k+1)A(k) - Q(k) \leq -\nu I$$

then the system is uniform exponentially stable.

Uniform BIBO stability - Discrete Time

Impulse response $g(k, k_0) = C(k)\Phi(k, k_0 + 1)B(k_0)$

1. Uniform BIBO stability $\Leftrightarrow$ exists ρ such that

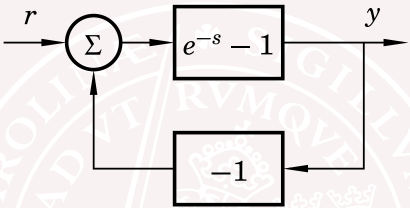
$$\sum_{i=k_0}^{k-1} \|g(k,i)\| \leq \rho, \quad \forall k \geq k_0 + 1$$

2. Assume $A(k), B(k), C(k)$ are bounded and that controller and observer Gramians satisfy

$$\epsilon I \leq W(k-l, k)$$
$$\epsilon I \leq M(k, k+l)$$

for some positive ϵ and integer l. Then
Uniform BIBO stability $\Leftrightarrow$ uniform exponential stability

Feedback - Well Posedness



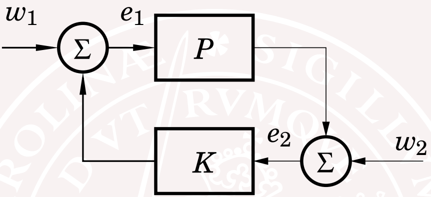
The transfer function from r to y is

$$\frac{e^{-s}-1}{1+e^{-s}-1} = 1 - e^{s}$$

This would give a way to implement the non-causal block e^{s}.

What is wrong?

Well Posedness



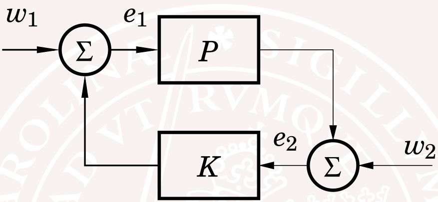
For rational functions P and K we say that the feedback system is well-posed if the transfer functions from $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ are all proper rational functions

Lemma 5.1 [ZDG] The feedback system is well-posed iff

$$I - D_P D_K \text{ is invertible}$$

(where D_P and D_K are the direct terms in P and K)

Internal Stability



Definition of Internal stability:
All states in P and K go to zero when $w = 0$.

Lemma 5.3 The (well-posed) feedback system in the figure is internally stable iff the "Gang of Four" transfer matrix

$$\begin{bmatrix} I & -K \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - KP)^{-1} & K(I - PK)^{-1} \\ P(I - KP)^{-1} & (I - PK)^{-1} \end{bmatrix}$$

from (w_1, w_2) to (e_1, e_2) is asymptotically stable.