Game Theory 2014

Exta Lecture 1 (BoB)

- Pursuit Evasion Games
- Isaacs' Equation
- Singular Surfaces
- The Lady in the Lake
- The Homicidal Taxi Driver
- The Exam

Baser/Olsder, Ch. 8, except 8.2.2

To understand

- *Isaacs equation and Minimum principle for Pursuit evasion games*
- *definition of semi-permeable surface and barriers*

Material

- Copies from Baser/Olsder

Pursuit-Evasion Games

The start of differential games in the 1950s-60s

Special case of two player zero-sum game treated before

$$
\begin{aligned}
\dot{x} &= f(t, x(t), u^1(t), u^2(t))), \quad x(0) = x_0 \\
T &= \inf\{t \in R : (x(t), t) \in \Lambda\} \\
L(u^1, u^2) &= \int_0^T g(t, x(t), u^1(t), u^2(t))\, dt + q(T, x(T))
\end{aligned}
$$

u^1, minimizer, pursuer, P
u^2, maximizer, evader, E

Saddle Point

Feedback-strategies $u^i(t) = \gamma^i(t, x(t))$

Solutions are often first obtained in open-loop strategies and then synthesized to feedback strategies, provided that they both exist.

$$J(\gamma^{1*}, \gamma^2) \leq J(\gamma^{1*}, \gamma^{2*}) \leq J(\gamma^1, \gamma^{2*})$$

Upper and Lower Isaacs Equations

$$\overline{V}(t,x) = \min_{\gamma_1} \max_{\gamma_2} \left\{ \int_t^T g(s, x(s), \gamma^1(s, x(s)), \gamma^2(s, x(s))\, ds + q(T, x(T)) \right\}$$

$$-\frac{\partial \overline{V}}{\partial t} = \min_{u^1} \max_{u_2} \left[\frac{\partial \overline{V}}{\partial x} f(t, x, u^1, u^2) + g(t, x, u^1, u^2) \right]$$

$$\underline{V}(t,x) = \max_{\gamma_2} \min_{\gamma_1} \left\{ \int_t^T g(s, x(s), \gamma^1(s, x(s)), \gamma^2(s, x(s))\, ds + q(T, x(T)) \right\}$$

$$-\frac{\partial \underline{V}}{\partial t} = \max_{u^2} \min_{u_1} \left[\frac{\partial \underline{V}}{\partial x} f(t, x, u^1, u^2) + g(t, x, u^1, u^2) \right]$$

(Provided $\overline{V}$ and $\underline{V}$ exist and are differentiable)

Geometrical Interpretation

Assume $g = 0$ (can always transform to this case).

See Figure 8.1 on p. 426. The Value function should satisfy

$$V(t,x) \;=\; \min_{\gamma^1}\max_{\gamma^2} q(T,x(T)) = \max_{\gamma^2}\min_{\gamma^1} q(T,x(T))$$

Minimizer chooses γ^1 to make the inner product between $\begin{pmatrix} V_x \\ V_t \end{pmatrix}$ and $\begin{pmatrix} f \\ 1 \end{pmatrix}$ minimal, and symmetricly for the maximizer.

Hence u^{1*} and u^{2*} are chosen as the arguments of

$$\min_{u^1}\max_{u^2}(V_x f + V_t) = \max_{u^2}\min_{u^1}(V_x f + V_t)$$

Semipermeable Surfaces

Along the equilibrium trajectory, V is constant and

$$\min_{u^1} \max_{u^2} (V_x f + V_t) = 0$$

If evader E plays optimally, $u^2 = u^{2*}$ then

$$V_x f(t, x, u^1, u^{2*}) + V_t \geq 0 \qquad \forall u^1$$

This means that maximizer E can assure that V never decreases. He can make sure the surface $V(t, x) = c$ is only traversed towards increasing V, i.e. V is made a semi-permeable surface.

Similarly minimizer P can assure V never increases by playing u^{1*}, i.e. he can make $V = c$ a semi-permeable surface in the other direction.

When both play optimally the state stays on the level set $V = c$

The Isaacs Condition

The interchangeability of min and max is called the Isaacs condition

Interchangeability certainly holds if for all p it holds that

$$\min_{u^1} \max_{u^2} (p' f + g) = \max_{u^2} \min_{u^1} (p' f + g)$$

A special case of this is when

$$\begin{aligned} f(t, x, u^1, u^2) &= f_1(t, x, u^1) + f_2(t, x, u^2) \\ g(t, x, u^1, u^2) &= g_1(t, x, u^1) + g_2(t, x, u^2) \end{aligned}$$

Note though that the problem with existence of a smooth V does not follow from this.

Theorem 8.1

If there exists a smooth V such that

- Isaacs equation holds
- $V(T,x) = q(T,x)$ when $l(T,x(T)) = 0$.
- Either $u^1 = \gamma^{1*}$ or $u^2 = \gamma^{2*}$ assures that the target set is reached in finite time

then V is the value function and γ^{1*} and γ^{2*} constitutes a saddle point

Example, smoothness is required

$$\begin{aligned} \dot{x} &= u_1 + u_2, \quad |u_1| \leq 1, \ \ |u_2| \leq 2, \quad x(0) = 0 \\ l(t,x) &= x^2 - 1, \quad q(T, x(T)) = |x(T)| - T \end{aligned}$$

Separable, so min max = max min

$$-V_t = \min_{|u_1| \leq 1} \max_{|u_2| \leq 2} (V_x(u_1 + u_2)) = |V_x|$$

One solution given by $V(x,t) = |x| - t$.

There are also spurious solutions, for instance $V(x,t) = 2 - t - |x|$.

Candidates for V can often be found via the minimum principle, Theorem 8.2

Example

Read Example 8.1 where the value function $V = x_1 + x_2$ guarantees a saddle point and the semi-permeable surfaces are illustrated.

Skip 8.2.2

Capturability

When can the minimizer/pursuer P, force the game to terminate?

Consider the cost functional

$$J = \begin{cases} -1 & \text{if } (x(t), t) \in \Lambda \text{ for some } t < \infty \\ 1 & \text{else} \end{cases}$$

See Figure 8.3, p. 433

A point is said to be on the usable part (UP) of $\partial\overline{\Lambda}$ if (where ν is the outward pointing normal of Λ at x)

$$\max_{u^2} \min_{u^1} \nu' f(x, u^1, u^2) \leq 0$$

This means that the state penetrates $\overline{\Lambda}$. So the Pursuer, u^1 can force the game to terminate, whatever u^2 does.

Barriers

The (barrier) surface S separates terminating states from non-terminating states. If $p(x)$ is a normal to S then

$$\min_{u^1} \max_{u^2} p'(x) f(x, u^1, u^2) = 0$$

So for $u^{i*} = \gamma^{i*}(x)$ this leads to

$$p'(x) f(x, \gamma^{1*}, \gamma^{2*}(x)) = 0$$

Differentiation leads to the equation

$$\frac{dp}{dt} = -\left(\frac{\partial f}{\partial x}\right)' p, \quad p(T) = \nu$$

where ν is the outward normal of $\overline{\Lambda}$ at S.

Example – The Homicidal Taxi Driver

The pursuer is driving a circular car with radius β and constant velocity $v_1 = 1$. The car has a minimal turning radius ω_1. He is trying to run over the evader, a pedestrian running with speed v_2. The pedestrian can change direction momentarily.

Use taxi-centric coordinate system. x_2 axis is along the velocity vector of the taxi.

$$\begin{aligned} \dot{x}_1 &= -u^1 x_2 + v_2 \sin u^2 \\ \dot{x}_2 &= -1 + u^1 x_1 + v_2 \cos u^2, \qquad |u^1| \leq 1 \end{aligned}$$

Capture if $x_1^2 + x_2^2 \leq \beta^2$

Usable Part and Capture Barriers

Usable part of $x_1^2 + x_2^2 = \beta^2$:

$$\max_{u^2}\min_{u^1} \nu' f(x, u^1, u^2) \le 0$$

$$\max_{u^2}\min_{u^1} x_1(-u^1 x_2 + v_2 \sin u^2) + x_2(-1 + u^1 x_1 + v_2 \cos u^2)$$

$$= -x_2 + v_2\beta \le 0$$

See Figure 8.4 p 437. With the normal $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ we get

$$\min_{u^1}\max_{u^2} p_1(-u^1 x_2 + v_2 \sin u^2) + p_2(-1 + u^1 x^1 + v_2 \cos u^2) \equiv 0$$

which gives

$$u^{1*} = \mathrm{sgn}\ (p_1 x_2 - p_2 x_1)$$

$$\sin u^{2*} = p_1/(p_1^2 + p_2^2), \quad \cos u^{2*} = p_2/(p_1^2 + p_2^2)$$

The Barriers

Complicated to solve for all possible cases

$$\dot{p}_1 = -p_2 u^1, \qquad \dot{p}_2 = -p_1 u^1$$

with $p_1(T) = \cos\alpha$, $p_2(T) = \sin\alpha$. For t close to T it can be shown that $u^1 = \text{sign } x_1$, hence

$$p_1(t) = \cos(t - T + \alpha), \quad p_2(t) = \sin(t - T + \alpha)$$

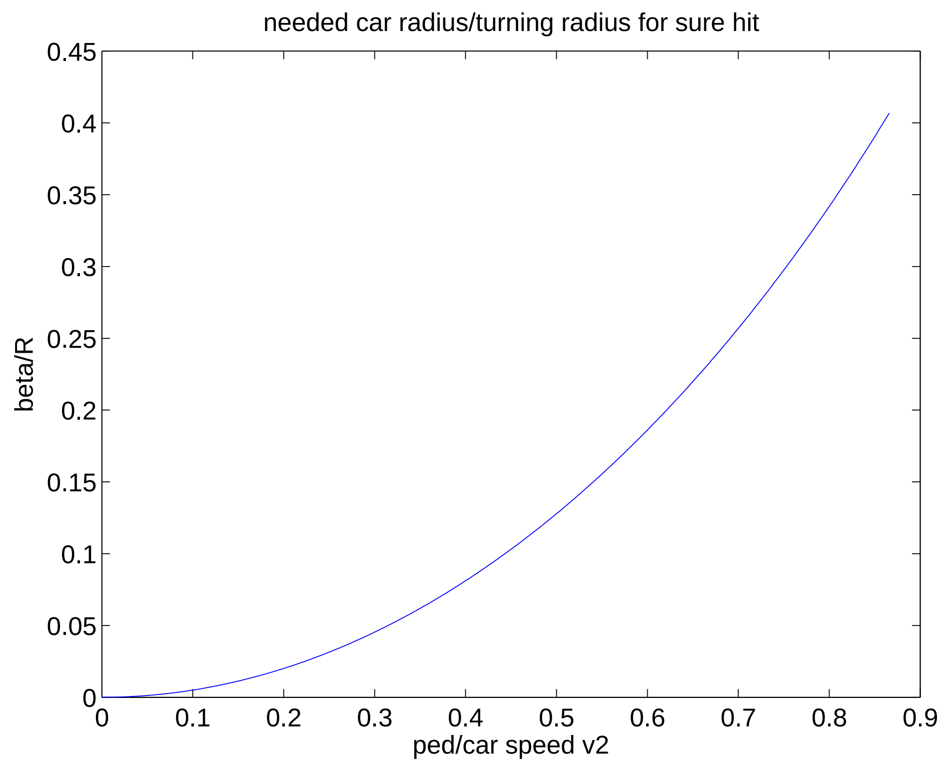
which gives

$$\begin{aligned} x_1(t) &= (\beta + (t-T)v_2)\cos(t - T + \alpha) + 1 - \cos(t - T), \\ x_2(t) &= (\beta + (t-T)v_2)\sin(t - T + \alpha) - \sin(t - T) \end{aligned}$$

Result

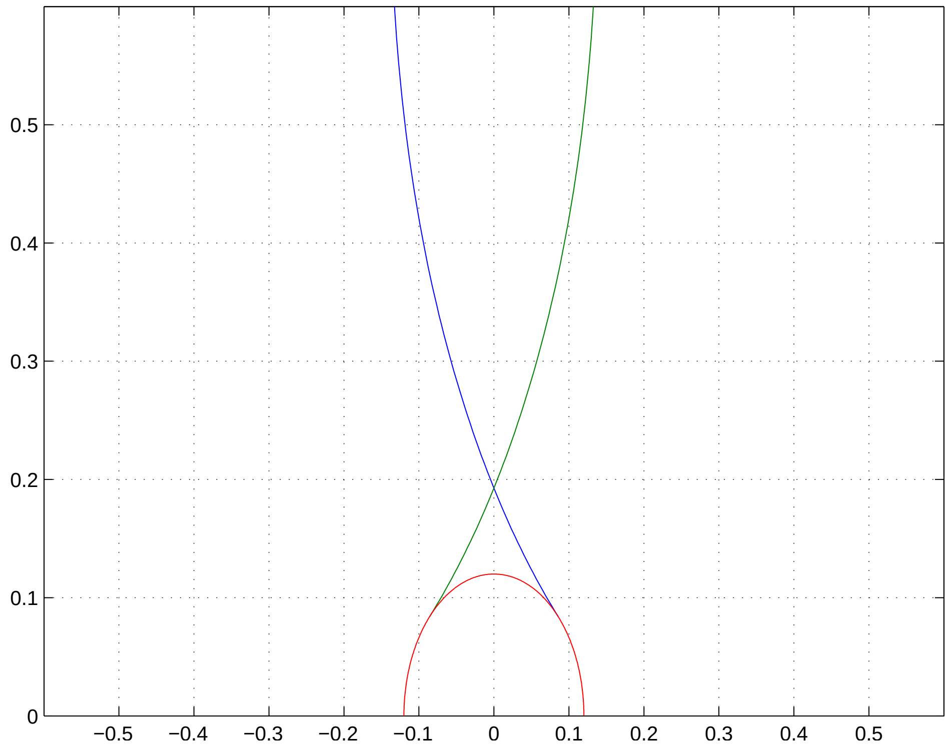
If $\beta^2 + v_2^2 < 1$ (the other cases are more complicated) then pedestrian survives if

$$\beta < v_2 \arcsin(v_2) + \sqrt{1 - v_2^2} - 1$$



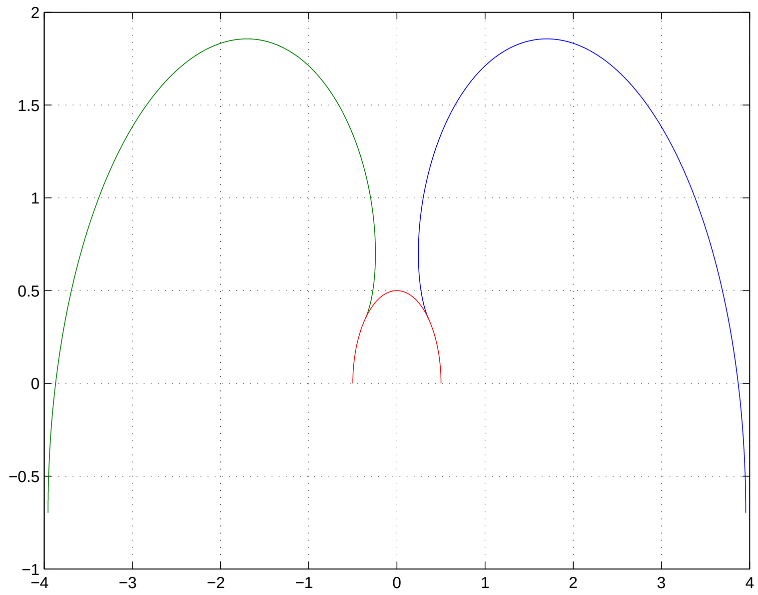
Barriers for $v_2 = 0.5, \beta = 0.12$

E escapes, except just in front of the car.



Barriers for $v_2 = 0.5$, $\beta = 0.5$

Sure hit. The Barrier ends, can go around. See figure 8.7, p.241



The Person in the Lake

The evader E is swimming with velocity v_2 in a circular pond of radius R . The Pursuer is running on the shore with velocity 1. P wants to intercept E when he/she reaches the shore. The goal function is $\theta(T)$, see figure 8.12.

In polar coordinates

$$\begin{aligned}\dot{\theta} &= \frac{v_2}{r}\sin(u^2) - \frac{u^1}{R}, \quad |u^1| \leq 1 \\ \dot{r} &= v_2 \cos(u^2)\end{aligned}$$

The Person in the Lake

Isaacs' equation (8.36) gives

$$
\begin{aligned}
u^{1*} &= \operatorname{sign}(\theta(T)) \\
\sin u^{2*} &= \frac{Rv_2}{r(t)}\operatorname{sign}(\theta(T)), \quad \text{if } r(t) > Rv_2
\end{aligned}
$$

When $r(t) < Rv_2$ the Isaacs equation gives $0 = 0$, hence no information. It is easy to see that in the middle of the pond, the evader can outmaneuver P, i.e. keep $\theta = \pi$

After E leaves the inner circle Rv_2 P runs in the same direction all of the time and E swims in a straight line tangent to the circle of radius Rv_2

Also study the nice interpretation in Fig 8.13b.

The Outcome

The outcome of the game is

$$|\theta(T)| = \pi + \arccos v_2 - \frac{1}{v_2}\sqrt{(1 - v_2^2)}$$

if $v_2 > 0.21723\ldots$ (otherwise the person can not escape).