REPEATED GAMES

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Repeated Games(Perfect monitoring)

- Extensive form game: Allowing explicit representation of important aspects, lite the sequencing of players' possible moves, their choices at very decision point and their information (possible imperfect) about the other players.
- Subgame perfect equilibrium (SPE): Nash equilibrium in subgames.

- Repeated games (super games) can be seen as a stage game played at each date for some duration of T periods.
- Future payoff are discounted (money tomorrow is worse then money now).
- For example, a two-period game with stage could be

$$U = u^1 + \delta u^2$$

- The payoff to player i in the repeated game

$$u_i(a) = \sum_{t=0}^{T} \delta^t g_i(a_i^t, a_{-i}^t)$$

	Cooperate	Defect
Cooperate	$1, 1$	$-1, 2$
Defect	$2, -1$	$0, 0$

Figure: Prisoners Dilemma.

- What happens if $T < \infty$?
- SPE $\Leftrightarrow$ backward induction.
- Unique SPE: (D,D) at each date.

Theorem

Consider repeated game $G^T(\delta)$ for $T < \infty$. Suppose that the stage game G has a unique pure strategy equilibrium a^*. Then G^T gas a unique SPE, where $a^t = a^*, \forall t = 0, 1, ..., T$ regardless of history.

- Infinitely-Repeated game G^∞, repeat the stage game at $t = 0, 1, \ldots$
- The payoff to player i in the repeated game

$$u_i(a) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t g_i(a_i^t, a_{-i}^t)$$

- Threatens other players with a "worse" punishment (could be forever, non-forgiving).

$$a_i^t = \begin{cases} \bar{a}_i & \text{if } a^\tau = \bar{a}, \ \forall \tau < t \\ \underline{a}_i & \text{otherwise} \end{cases}$$

	Cooperate	Defect
Cooperate	$1, 1$	$-1, 2$
Defect	$2, -1$	$0, 0$

Figure: Prisoners Dilemma.

- Play C in every period unless someone has every played D in the past.
- Play D forever if someone has played D in the past.

- Payoff from C: $(1-\delta)[1+\delta+\delta^2 \ldots] = 1$
- Payoff from D: $(1-\delta)[2+0+0\ldots] = 2(1-\delta)$
- Cooperation better if $\delta \geq 1/2$
- Cooperation is best against cooperation and defect is best against defect.
- Both has to follow this strategy for this to be an SPE.

Folk Theorems

- People believed that you could support cooperation in repeated prisoners' dilemma and other type of similar games with sufficiently high discount factor.
- Feasible payoff: Convex hull

$$V = Conv\{v \in \mathbb{R}^{'} | \exists a \in A \quad \text{such that} \quad g(a) = v\}$$

Minmax Payoff

- Minmax payoff to player i: the lowest payoff that players i's opponent can hold him to:

$$\underline{v}_i = \min_{\alpha_{-i}} \left[\max_{\alpha_i} g_i(\alpha_i, \alpha_{-i}) \right]$$

$$m^i_{-i} = \arg\min_{\alpha_{-i}} \left[\max_{\alpha_i} g_i(\alpha_i, \alpha_{-i}) \right]$$

$$g_i(m^i_i, m^i_{-i}) = \underline{v}_i$$

Minmax payoff Lower Bounds

Theorem

Let α be a (possibly mixed) nash equilibrium of $G^{(\infty)}$ and $g_i(\alpha)$ be the payoff to player i in the equilibrium α. Then

$$g_i(\alpha) \geq \underline{v}_i$$

Folks Theorem

Theorem (Nash Folk Theorem)

If $(v_1, \ldots, v_I)$ is feasible and $v_i > \underline{v}_i$ for all v_i. Then there exists some $\underline{\delta} \leq 1$ such that for all $\delta > \underline{\delta}$ there is a unique Nash equilibrium of $G^\infty(\delta)$ with payoffs $(v_1, \ldots, v_I)$

Problems with Nash Folk Theorem

- Any payoff can be obtained as a Nash Equilibrium when players are patient enough.
- The strategies can be very costly to carry out for the punisher

	L (q)	R $(1-q)$
U	$6, 6$	$0, -100$
D	$7, 1$	$0, -100$

Figure:

- NE is (DL) minmax:

$$\underline{v}_1 = 0 \quad \underline{v}_2 = 1$$

	A	B	C
A	$3,3$	$0,4$	$-2,0$
B	$4,0$	$1,1$	$-2,0$
C	$0,-2$	$0,-2$	$-1,-1$

- When there is multiple equilibria in the state game, cooperation can be allowed in the finite game.
- NE (B,B) and (C,C) but most cooperative is (A,A).
- Support (A,A) in first period by threatening to switch to (C,C).

- Public signal: p
- Actions: (a_1, a_2), where $a_i \in (C, D)$
- Payoffs:

$$r_1(C,p) = 1 + p, \qquad r_1(D,p) = 4 + p,$$
$$r_2(C,p) = 1 + p, \qquad r_2(D,p) = 4 + p.$$

- Probability distribution for public signal p:

  $a_1 = a_2 = C \rightarrow p = X,$
  $a_1 \neq a_2 \qquad \rightarrow p = X - 2,$
  $a_1 = a_2 = D \rightarrow p = X - 4,$
  where X is a continuous random variable with cumulative distribution function $F(x)$ and $E[X] = 0$.
- The payoff matrix for this game takes the following form:

	C	D
C	$(1+X, 1+X)$	$(-1+X, 2+X)$
D	$(2+X, -1+X)$	(X, X)