Discussion session 5

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Compute NE in finite games - Introduction

- Computation of mixed Nash equilibria for finite games
- Two player games
- Express two player games as a LP in order to find mixed Nash equilibria
- Player 1 n actions, player 2 m actions
- A and B n x m payoff matrices
- X and Y mixed strategies of player 1 and 2 as
  $X \equiv \{x : \sum_{i=1}^{n} x_i = 1, , x_i \geq 0\}$ and
  $Y \equiv \{y : \sum_{i=1}^{m} y_i = 1, , y_i \geq 0\}$
- Payoffs:

$$u_1(x, y) = x^T Ay$$
$$u_2(x, y) = x^T By$$

Zero-Sum finite games

- A mixed strategy profile (x^*, y^*) is a mixed strategy Nash equilibrium iff

$$(x^*)^T A y^* \geq x^T A y^* \qquad \forall x \in X$$
$$(x^*)^T B y^* \geq (x^*)^T B y \qquad \forall y \in Y$$

- for a zero sum game B = -A;
- Is possible to show that finding the mixed strategy Nash equilibrium strategies reduces to solve a pair of linear optimization problems:

$$\min_{y \in Y} \max_{x \in X} x^T AY = \min_{y \in Y} \max\{Ay_i\} \qquad i = \{1, \ldots, n\}$$
$$\max_{x \in X} \min_{y \in Y} x^T AY = \max_{x \in X} \min\{Ax_j\} \qquad j = \{1, \ldots, m\}$$

Nonzero-Sum finite games

- Generally to solve a bimatrix game it is needed to transform it into a bilinear programming problem.
- A mixed strategy profile (x^*, y^*) is a mixed Nash equilibrium of the bimatrix game (A, B) if and only if there exists a pair (p^*, q^*) such that (x^*, y^*, p^*, q^*) is a solution to the following bilinear programming problem:

$$\max\{x^T Ay + x^T By - p - q\}$$

$$s.t$$

$$\begin{aligned} Ay &\leq p\mathbf{1}_n & By &\leq q\mathbf{1}_m \\ \sum_i x_i &= 1 & \textstyle\sum_j y_j &= 1 \\ x &\geq 0 & y &\geq 0 \end{aligned}$$

Nonzero-Sum finite games

- We now introduce the Linear Complementarity Problem Formulation
- We add the slack variables $r_i \in \mathcal{R}^n$, $r_i \geq 0$, $i \in \{1, 2\}$ and $v_i \in \mathcal{R}$, $i \in \{1, 2\}$;
- We drop the normalization constraints (i.e. the sum equal to one , practically we allow the extraneous solution $z = [0, 0]^T$) and define:

$$
\begin{aligned}
z &= [x, y]^T \\
r &= [r_1, r_2]^T \quad q = [\mathbf{1}_n, \mathbf{1}_m]^T \\
U &= \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix}
\end{aligned}
$$

- we can write:

$$
\begin{aligned}
Uz + r &= q \\
z \geq 0 &\quad r \geq 0 \\
z^T r &= 0
\end{aligned}
$$

Computing approximate Nash Equilibria

- Given some scalar $\epsilon > 0$, a mixed strategy profile $(\bar{x}, \bar{y})$ is an ϵ equilibrium if

$$x^T A\bar{y} \leq \bar{x}^T A\bar{y} + \epsilon \quad \forall x \in X$$
$$\bar{x}^T By \leq \bar{x}^T B\bar{y} + \epsilon \quad \forall y \in Y$$

- A mixed strategy is called k-uniform if it is the uniform distribution on a multiset S of pure strategies, with $|S| = k$;
- Consider a 2-player game with n pure strategies. Assume that all the entries of the matrices A and B are between 0 and 1. Let (x^*, y^*) be a mixed Nash equilibrium and let $\epsilon > 0$. For all $k \geq \frac{32 \log n}{\epsilon^2}$, there exists a pair of k-uniform strategies $(\bar{x}, \bar{y})$ such that:
  - $(\bar{x}, \bar{y})$ is an ϵ equilibrium
  - $|\bar{x}^T A\bar{y} - (x^*)^T Ay^*| < \epsilon$
  - $|\bar{x}^T B\bar{y} - (x^*)^T By^*| < \epsilon$
- Consider a 2-player game with n pure strategies for each player. There exists an algorithm that is quasi-polynomial in n for computing an ϵ - Nash equilibrium.

Evolution and Learning in Games - Game of life

- Each organism is born programmed to play a particular strategy.
- Payoffs given as fitness (i.e., expected number of offsprings). If the organism is successful, it has greater fitness and more offspring, also programmed to play in the same way. If it is unsuccessful, it likely dies without offspring.
- Mutations imply that some of these offspring will randomly play any one of the feasible strategies.
- At each instant, each agent is randomly matched with one other agent, and they play a symmetric strategic form game, each agent is programmed (committed to) to playing a given strategy.
- Strategies with higher payoffs expand and those with lower payoffs contract.

Evolutionarily Stable Strategies

- Consider a two player, symmetric strategic form game, so we write it simply as $\langle S, u \rangle$. A (possibly mixed) strategy is $\sigma \in \Sigma$
- A strategy $\sigma^* \in \Sigma$ is evolutionarily stable if there exists $\bar{\epsilon} > 0$ such that for any $\sigma \neq \sigma^*$and for any $\epsilon < \bar{\epsilon}$, we have $u(\sigma^*, \epsilon\sigma + (1-\epsilon)\sigma^*) > u(\sigma, \epsilon\sigma + (1-\epsilon)\sigma^*)$
- equivalently, A strategy $\sigma^* \in \Sigma$ is evolutionarily stable if for any $\sigma \neq \sigma^*$, we have $u(\sigma^*, \sigma^*) \geq u(\sigma, \sigma^*)$.
- if, for some $\sigma \in \Sigma$, $u(\sigma^*, \sigma^*) = u(\sigma, \sigma^*)$, then $u(\sigma^*, \sigma) > u(\sigma, \sigma)$.
- A strict (symmetric) Nash equilibrium of a symmetric game is an evolutionarily stable strategy.
- An evolutionarily stable strategy is a Nash equilibrium.
- The converses of the two preceding results are not true in general.

Monomorphic and Polymorphic Evolutionarily Stability

- We could require an evolutionarily stable strategy (ESS) to be monomorphic—that is, all agents to use the same (pure) strategy.
- The alternative is polymorphic, where different strategies coexist, mimicking a mixed strategy equilibrium.

Replicator dynamics

- Strategies are enumerated by s= 1,2,..,K. Denote the fraction of the population playing strategy s by x_s.
- Discrete "ticks"

$$x_s(t+\tau) = x_s(t)\frac{\tau[u(s,\sigma(t)) - \bar{u}(\sigma(t))]}{\bar{u}(\sigma(t))}$$

$$\bar{u}(\sigma(t)) = \sum_{s=1}^{K} x_s(t)u(s,\sigma(t))$$

- continuous time:

$$\dot{x_s}(t) = \frac{d}{dt}x_s(t) = x_s(t)\frac{[u(s,\sigma(t)) - \bar{u}(\sigma(t))]}{\bar{u}(\sigma(t))}$$

Replicator Dynamics: Implications

- Distribution x^* could be:
  - Stationary state: $\dot{x_s}(t) = 0$ for all s
  - asymptotically stable state: starting from any x_0 in this neighborhood, dynamics induced by (Continuous replicator) approach x^*.
- If x^* is a Nash equilibrium, then it is a stationary state.
- If x^* is asymptotically stable, then it is a Nash equilibrium.
- If x^* is evolutionarily stable, then it is asymptotically stable.

Replicator Dynamics: Implications

- Random matching is not always the best model, so is possible to incorporate a graph like structure with more focused interactions (e.g. only with agents neighborhood).
- Imitation is another tool i.e. individuals imitate the strategies of others in proportion to how much they outperform the average in the population, both with global or local knowledge of the agent.
- fictitious play is one of the earliest learning rule involving imitation. The basic idea of fictitious play is that each player assumes that his opponent is using a stationary mixed strategy, and updates his beliefs about this stationary mixed strategies at each step.
- Players choose actions in each period (or stage) to maximize that period's expected payoff given their prediction of the distribution of opponent's actions. Essentially agents try to forecast the behaviour of the others.

References

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