Week 6

Exercise 1

Consider an infinite line of agents indexed by $i \in \mathbb{Z}$. Denote by ψ_i the position of the i-th agent, which is updated according to

$$\frac{d}{dt}\psi_i(t) = \frac{1}{2}(\psi_{i-1}(t) + \psi_{i+1}(t)) - \psi_i(t) + u_i(t) + w_i(t)$$

where $u_i(t)$ and $w_i(t)$ denote the control and the disturbance at the i-th agent, respectively. The control u_i is to be designed in such a way that the $\mathcal{H}_2$ gain $w \to (\psi, u)$ is minimized.

- Consider the system
$$\frac{d}{dt}x(t) = \alpha x(t) + u(t) + w(t)$$
Solve the $\mathcal{H}_2$ control problem finding u in such a way that the gain $w \to (x, u)$ is minimized. In particular
  - Solve the Riccati Equation
$$2p\alpha - p^2 + 1 = 0$$
  - Observe that the control is given by $u(t) = -px(t)$
- Solve the control problem for the spatially invariant system. Notice that the dual group of $\mathbb{Z}_N$ is $\mathbb{D} = \{\lambda \in \mathbb{C} | \, |\lambda| = 1\}$. Compute the operator $\hat{p}(e^{j\omega})$, $\omega \in [-\pi, \pi)$, and (numerically) its inverse Fourier transform p_i, $i \in \mathbb{Z}$.
- Estimate the exponential decay of the convolution operator p_i by estimating in which region of $\mathbb{C}$ one can extend analytically $\hat{p}(e^{j\omega})$ to $\hat{p}_e(\sigma)$. Obtain a truncation of p_i.
- Simulate the system and compare the results.

Exercise 2

Consider the following circulant matrix

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

and the following system on $\mathbb{Z}_2$

$$\frac{d}{dt}\psi(t) = P\psi(t) + u(t) + w(t)$$

- Diagonalize the system using the following Fourier matrix

$$F = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- Solve the $\mathcal{H}_2$ control problem to minimize the gain from w to (ψ, u)
- Optional: in general, a circulant matrix on $\mathbb{Z}_N$ has the form

$$P = \sum_{j=0}^{N-1} g_j \Pi^j$$

where Π is the circulant matrix associated with the first upper diagonal:

$$\Pi = \begin{bmatrix} 0_{N-1\times 1} & I_{N-1} \\ 1 & 0_{1\times N-1} \end{bmatrix}$$

For example, for $N = 3$,

$$\Pi = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

  - Show that $P_{jk} = g(k - j)$ (difference in $\mathbb{Z}_N$). The function $\{g(j)\}_{j=0,\dots,N-1}$ is called the generator of the matrix. Can you see why?
  - The set $\{\Pi^i\}_{i=0,\dots,N-1}$ with the standard multiplication is a group. Check it.
  - The Fourier matrix of dimension N is such that its (j,k)-th entry is

$$\mathcal{F}_{jk} = \frac{1}{\sqrt{N}} e^{i2\pi\frac{jk}{N}}, \quad j = 0, \dots, N-1, k = 0, \dots, N-1$$

    Use $\mathcal{F}$ to diagonalize the matrix P. Show that the eigenvalues of P are

$$\lambda_k = \sum_{j=0}^{N-1} g_j e^{i2\pi jk/N}, k = 0, \dots, N-1$$