Week 5

Exercise 1

Consider a network of 5 agents trying to achieve formation control. The position of agent i is denoted by $x_i \in \mathbb{R}$. Agent 1 has a GPS and can measure and modify its position according to

$$\dot{x}_1(t) = -x_1(t) + B_1 w(t)$$

The other agents update their positions based on distance measurements:

$$\begin{cases} \dot{x}_2(t) = \ell_{21}(x_1(t) - x_2(t)) + \ell_{23}(x_3(t) - x_2(t)) + B_2 w(t) \\ \dot{x}_3(t) = \ell_{32}(x_2(t) - x_3(t)) + \ell_{34}(x_4(t) - x_3(t)) + B_3 w(t) \\ \dot{x}_4(t) = \ell_{43}(x_3(t) - x_4(t)) + \ell_{45}(x_5(t) - x_4(t)) + B_4 w(t) \\ \dot{x}_5(t) = \ell_{51}(x_1(t) - x_5(t)) + \ell_{52}(x_2(t) - x_5(t)) + \ell_{54}(x_4(t) - x_5(t)) + B_5 w(t) \end{cases}$$

Here $w(t)$ is an external disturbance, and we assume that the weights ℓ_{ij} satisfy $\ell_{ij} \in [0; 1]$. We want to minimize the gain of the system from w to $\sum_i x_i$.

- Write the system in terms of the matrices A, E, F and L following the paper *Distributed Control of Positive Systems.*
- Compute the optimal set of weights in the two cases
  - $B_1 = 0$, $B_2 = 10$, $B_3 = B_4 = B_5 = 1$;
  - $B_1 = 0$, $B_2 = B_3 = B_4 = 1$, $B_5 = 10$.
- Interpret the results.

Exercise 2

Rate of convergence towards the consensus is the most common but by no means the only possible performance index for a consensus protocol. Consider the following system corresponding to a noisy consensus scenario

$$x(t+1) = Wx(t) + w(t), \quad x(t) \in \mathbb{R}^N$$

where $W \in \mathbb{R}_+^{N \times N}$ is a *symmetric* matrix such as $W\mathbf{1} = \mathbf{1}$ and $W_{ij} \geq 0, \forall i, j$, the communication graph is connected (so all the eigenvalues of W apart from that in 1 are stable), and w is white Gaussian with zero mean and variance $\mathbb{E}ww^T = I$. Assume $x(0)$ Gaussian with zero mean and uncorrelated with the noise process.

- Simulate the system. Notice that there is no consensus value, and that the states form a stochastic cloud.

- Call $J(W) = \lim_{t\to\infty} \frac{1}{N}\mathbb{E}||x(t) - \frac{1}{N}\sum_i x_i(t)\mathbf{1}||^2$ the average variance of the state around the (time-varying) consensus value at stationarity. Show that

$$J(W) = \frac{1}{N}\sum_{t\geq 0} ||W^t(I - \frac{1}{N}\mathbf{1}\mathbf{1}^T)||_F^2 = \frac{1}{N}\sum_{\lambda\neq 1}\frac{1}{1-\lambda^2}$$

where λ are the (stable) eigenvalues of W, $||A||_F = \sqrt{\text{trace}\{A^T A\}}$ is the Frobenius norm of a matrix, and $\text{trace}\{A\} = \sum_i A_{ii}$ is the sum of the diagonal values of a matrix.

Hints (all of them are easy to check):

- *let* $\Omega = I - \frac{1}{N}\mathbf{1}\mathbf{1}^T$. *You need to use several times that* $\Omega^2 = \Omega$, *that* $W\Omega = \Omega W$, *and that* $(W\Omega)^t = W^t\Omega$ *for* $t \geq 1$
- *for any square matrix* A, $\text{trace}\{A\}$ *is the sum of the eigenvalues of* A
- *cyclic property of the trace:* $\text{trace}\{AB\} = \text{trace}\{BA\}$
- *if* W *is a positive symmetric matrix with* $W\mathbf{1} = \mathbf{1}$ *and* v *is an eigenvector of* W *with associated eigenvalue* $\lambda < 1$, *then* $\mathbf{1}^T v = 0$

Exercise 3 - Optional

In this exercise we study information patterns in the consensus algorithm.

- We know that the rate of convergence of a consensus algorithm with matrix W is given by ρ, its spectral radius, namely the largest eigenvalue (in absolute value) smaller than 1. Show that for symmetric matrices the problem[1]

$$\min_{W\sim\mathcal{G}} \rho(W)$$

is a convex problem. Deduce that if we allow for more edges to be used, the corresponding ρ cannot decrease.

- De Brujin's graphs are a particular class of directed fast mixing graphs. A De Brujin's graph with parameters k and n and $N = k^n$ nodes has the following adjacency matrix[2]

$$A_{(n,k)} = \mathbf{1}_k \otimes I_{k^{n-1}} \otimes \mathbf{1}_k^T$$

where $\otimes$ denotes the Kronecker product (`kron(A, B)` in MatLab). Take $n = 3$ and $k = 2$. Try to draw the graph. Build the consensus matrix

$$W_{(3,2)} = \frac{1}{2}\mathbf{1}_2 \otimes I_4 \otimes \mathbf{1}_2^T$$

Show that using $W_{(3,2)}$ you can achieve *average* consensus in finite time (this holds for any (n,k)). How many consensus iterations do you need to converge? How is it related to the geometry of the graph? Can you give an intuitive explanation on the basis of how information is exchanged among the agents?

[1] Given $\mathcal{G} = (V, \mathcal{E}, P \sim \mathcal{G}$ means that $W_{ij} > 0$ only if $(j,i) \in \mathcal{E}$

[2] The adjacency matrix of a graph $\mathcal{G} = (V,\mathcal{E})$, with nodes $V = \{1,\ldots,N\}$ and edges $(i,j) \in \mathcal{E}$, is the matrix $A \in \{\pm 1, 0\}^{N\times N}$ such that $A_{ij} = 1$ if $(j,i) \in \mathcal{E}$, and $A_{ij} = 0$ otherwise.

- De Brujin's graphs are part of a class of graphs called expander graphs. The corresponding consensus matrices can be built so that $\rho \leq 1-\delta, \delta > 0$ for any value of N, the number of agents in the graph. This makes this class of graphs appealing as usually $\rho \to 1$ as the number of nodes $N \to +\infty$. What is the spectral radius of $W_{(3,2)}$ (no calculation needed)?

- Try adding an edge to a De Brujin's graph and use a random weight. What happens to the spectral radius? Can you decrease it? Is this coherent with the first point of this exercise?

- Consider a generic undirected connected graph $\mathcal{G}$. Find a matrix W_f which achieves agreement in the least number of steps (*Hint: use a tree*). Consider a symmetric matrix $W_s \sim \mathcal{G}$ built using the so called Metropolis weights:

  $$[W_s]_{ij} = \begin{cases} \frac{1}{\max\{|\mathcal{N}|_i+1, |\mathcal{N}|_j+1\}}, & (i,j) \in \mathcal{E}, i \neq j \\ 1 - \sum_k [W_s]_{ik}, & i = j \end{cases}$$

  Simulate the system using W_f and W_s. Compare the results. Which is fastest? What is the advantage in using the symmetric matrix? What is the advantage using a consensus matrix built on a De Brujin's graph?

Hint: a powerful variational characterization of the eigenvalues of a symmetric matrix holds. In our case, it reads as follows: the largest eigenvalue of W which is not 1 is given by

$$\lambda_2(W) = \sup\{\frac{x^T W x}{x^T x}, \mathbf{1}^T x = 0, x \neq 0\}$$

and its smallest eigenvalue is given by

$$\lambda_n(W) = \inf\{\frac{x^T W x}{x^T x}, x \neq 0\}$$