Week 2

Exercise 1

Consider the following formation control problem for two vehicles. Denote by $x_i(t)$ the position of the i-th vehicle. The stationary dynamics of the system is the following

$$\begin{cases} x_1(t+1) = x_1(t) + u_1(t) + w_1(t) \\ x_2(t+1) = x_2(t) + u_2(t) + w_2(t) \end{cases}$$

where $u_i(t), i = 1,2$ is the control at each vehicle, and $w_i(t), i = 1,2$ are disturbances. Assume that the stochastic process $\{w(k)\}_{k \geq 0}$, $w(k) = \begin{bmatrix} w_1(t) & w_2(t) \end{bmatrix}^T$, is white Gaussian with

$$\mathbb{E} w(k) w(k)^T = I_2$$

The goal is to design a stabilizing controller so that $J = \mathbb{E} x_1^2 + (x_1 - x_2)^2 + u_1^2 + u_2^2$ is minimized. This corresponds to a scenario in which a) the first vehicle has a notion of absolute position and wants to stabilize its position, and b) the two vehicles need to rendez-vous.

- Centralized control with no information on the disturbances: Assume $u = Lx$. Compute the optimal L and the corresponding cost;
- Centralized control with information on the disturbances: Assume $u = Lx + Mw$. Compute the optimal L, M and the corresponding cost;
- Deentralized control with no information on the disturbances: Assume $u = Lx + Mw$ and $M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$, namely u_1 is in feedback with the state and in feedforward with w_1 only, while u_2 is in feedback with the state and in feedforward with w_2 only. Is this system partially nested? Compute the optimal L, M and the corresponding cost.

Exercise 2

Consider the system in Figure 1. Give conditions on $P_{yu}(s)$ such that the set of lower triangular matrices is quadratically invariant with respect to P_{yu}.

Consider now the following irrigation model

$$\begin{cases} \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} B_1 u_1(t) \\ B_2 u_2(t) \\ B_3 u_3(t) \\ B_4 u_4(t) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \end{bmatrix} \\ y_i(t) = x_i(t), i = 1, \dots, 4 \end{cases}$$

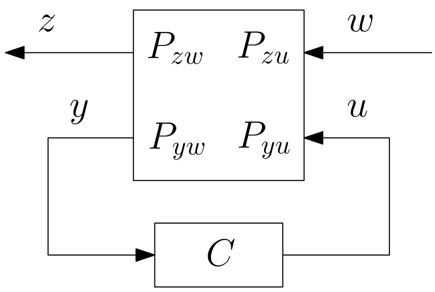

Figure 1: LF representation

Which of the following control structures are quadratically invariant?

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} = \begin{bmatrix} K_{11} & 0 & 0 & 0 \\ K_{12} & K_{22} & 0 & 0 \\ 0 & K_{32} & K_{33} & 0 \\ 0 & 0 & K_{43} & K_{44} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix}$$

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} = \begin{bmatrix} K_{11} & 0 & 0 & 0 \\ K_{12} & K_{22} & 0 & 0 \\ K_{31} & K_{32} & K_{33} & 0 \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix}$$

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ 0 & K_{22} & K_{23} & K_{24} \\ 0 & 0 & K_{33} & K_{34} \\ 0 & 0 & 0 & K_{44} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix}$$

Exercise 3

We consider a master-slave system. The master's equations are

$$\begin{cases} \tau_M \dot{q}_M = -q_M + u_M + K_M w_M \\ y_M = q_M + \eta_M \end{cases}$$

where q_M is the position of the master, w_M the operator command, η_M measurement noise and u_M the master device control.

The slave's equations are

$$\begin{cases} m_S \ddot{q}_S = -\beta_S \dot{q}_S + H_S q_S + u_S + w_S \\ y_S = q_S + \eta_S \end{cases}$$

where q_S is the position of the slave, w_S is the external disturbance, η_S measurement noise and u_S the slave device control.

The regulated output z is $z = \begin{bmatrix} q_M & q_S - q_M & u_M & u_S \end{bmatrix}^T$. This choice corresponds to a) control master's position and b) control the slave's position to the master's, while c) saving energy.

The control structure is constrained as follows

$$\begin{cases} U_M(s) = K_{11}(s) Y_M(s) + K_{12}(s) e^{-hs} Y_S(s) \\ U_S(s) = K_{21}(s) e^{-hs} Y_M(s) + K_{22}(s) Y_S(s) \end{cases}$$

a) Check that the system is quadratically invariant with respect to the proposed control structure

b) Compute the optimal $\mathcal{H}_2$ controller following the paper *On the decentralized H^2 optimal control of bilateral teleoperation systems with time delays.* Use the following numerical values

$$\tau_M = 1, \quad K_M = 1, \quad m_S = 5, \quad \beta_S = 10, \quad H_S = 2, \quad K_S = 5, \quad h = .1$$

Hint: in the computation of π_h, it is enough to approximate e^{-sh} via the Padè approximation of order 3*. Once Π_1 and Π_2 have been obtained, use* `minreal` *with a certain tolerance to perform zero-pole cancellations.*