Week 1

Exercise 1

The stochastic processes $\{w_1(t)\}_{t\in\mathbb{Z}^+}$, $\{w_2(t)\}_{t\in\mathbb{Z}^+}$ and $\{w_3(t)\}_{t\in\mathbb{Z}^+}$ are white and Gaussian. For any $t \in \mathbb{Z}^+$,

$$\mathbb{E}\begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix}\begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix}^T = \begin{bmatrix} 2 & \varepsilon & 0 \\ \varepsilon & 2 & \gamma \\ 0 & \gamma & 4 \end{bmatrix}$$

where ε, γ are chosen so that this matrix is positive definite, i.e., $\gamma^2 + 2\varepsilon^2 < 8$.

Consider the following system

$$x(t+1) = \alpha x(t) + w_3(t) + u_1(t) + u_2(t)$$

where one decision maker chooses K_1 to produce $u_1(t) = K_1 w_1(t)$, while the second decision maker chooses K_2 to produce $u_2(t) = K_2 w_2(t)$.

a) Choose K_1 and K_2 to minimize

$$W(K_1, K_2) = \lim_{t\to\infty} \mathbb{E}x(t)^2$$

b) Discuss the cases $\varepsilon = 0$ and $\gamma = 0$

Exercise 2

Consider a couple of inverted pendula with mass $m = 1$ and length $l = 1$ and fixed pivots. There is no physical coupling among them, and the system is controlled by two torques applied at both pivots. The resulting system, linearized around the equilibrium point, admits the following state-space representation

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cu \end{cases}$$

where the state is $x = \begin{bmatrix} \theta_1 & \dot{\theta}_1 & \theta_2 & \dot{\theta}_2 \end{bmatrix}^T$, the input $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$ represents the two torques, and

$$A = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ g & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & g & 0 \end{array}\right] \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Each controller is only able to sense the angular position of the *other* pendulum, i.e.,

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

a) Show that there exists a stabilizing controller.

b) Find (numerically) a stabilizing controller. *Hint: write the system as a series plant - controller - plant - controller*

c) Compare the result with a centralized approach. Try to minimize

$$W = \int_{t\geq 0} ||x(t)||_2^2 + ||u(t)||_2^2 dt$$

Exercise 3

In this example we work out the Witsenhausen counterexample. Consider the 2-stages system in Figure 1. A variable x_0 is picked randomly in $\mathcal{N}(0, \sigma^2)$. A noise variable w is picked randomly in $\mathcal{N}(0, 1)$. The two are independent of each other.

1) First stage: The first controller observes $y_1 = x_0$ and decides $u_1 = \gamma_1(y_1)$. The variable is updated to $x_1 = x_0 + u_1$;

2) Second stage: The second controller observes $y_2 = x_1 + w$ and decides $u_2 = \gamma_2(y_2)$. The variable is updated to $x_2 = x_1 - u_2$.

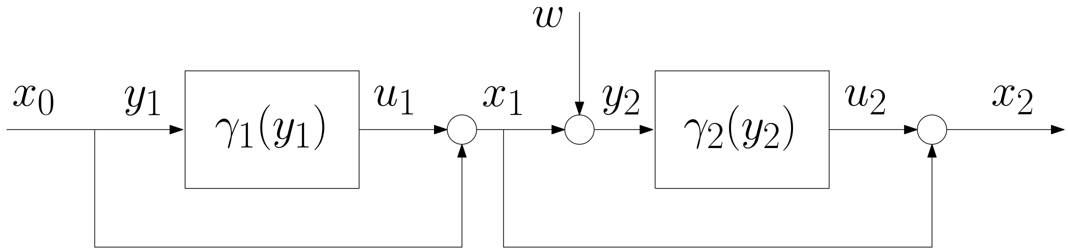

Figure 1: The 2-stages of the Witsenhausen counterexample.

In this problem, decision makers have to face a non-classical information pattern. In fact, the second decision maker cannot observe the whole information used by the first decision maker, or the actual control applied.

The cost function to be minimized is

$$W = \mathbb{E}[x_2^2 + k^2 u_1^2]$$

a) Assume classical information pattern, i.e., assume that $u_2 = \gamma_2(y_1, y_2, u_1)$. Which is the best choice for γ_1 and γ_2? Are they affine in the available information?

b) Consider the class of affine controllers $\gamma_1(y_1) = a_1 y_1 + b_1$ and $\gamma_2(y_2) = a_2 y_2 + b_2$. Give a formula for W as a function of a_1, b_1, a_2, b_2 and of σ^2 and k.

c) Obtain the parameters a_1, b_1, a_2, b_2 which minimize the cost. Assuming $k = 0.1$ and $\sigma = 10$, compute (numerically) the minimal cost.

d) Consider now a nonlinear controller of the type $\gamma_1(y_1) = -y_1 + \sigma \text{sgn}(y_1)$. Compute the best u_2. Notice that minimizing $\mathbb{E}x_2^2$ corresponds to finding the Bayesian estimate of x_1 given y_1.

e) Compute the cost with the obtained controllers. Evaluate the term $\mathbb{E}x_2^2$ numerically. Make a comparison with the affine case for the particular values $\sigma = 10$ and $k = 0.1$.