A Course on Distributed Control

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Schedule

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Mo April  8 at 1315-1430 lecture
Mo April 15 at 1315-1600 lecture and exercises
Fr April 26 at 0815-1100 lecture and exercises
We   May  8 at 0915-1200 lecture and exercises
We   May 15 at 0915-1200 lecture and exercises
We   May 22 at 0915-1000 lecture
We   May 29 at 0915-1200 exercises
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A Course of Six Lectures

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1. Introduction
   Fixed modes, Team theory, Witsenhausen's counterexample

2. Partial nestedness and quadratic invariance
   Control with information delays
   Example: Tele-operation

3. Dual decomposition
   The saddle algorithm
   Example: The Internet protocol

4. Distributed MPC
   Example: Water Supply Network

5. Distributed control of positive systems. Consensus algorithms

6. Spatially invariant systems.
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Lecture 6

- **Spatially invariant systems**
  - Separation of spatial frequencies
  - Approximation by spatial truncation

Spatially invariant systems

$$\frac{\partial}{\partial t}\psi(x,t) = [A\psi](x,t) + [B\psi](x,t)$$
$$y(x,t) = [C\psi](x,t) + [D\psi](x,t)$$

The variable $x = (x_1, \dots, x_d)$ is called the spatial variable. The components x_k could be integers, real numbers or, more generally, elements of a locally compact abelian group.

The operators A, B, C, D are assumed to be translation invariant, e.g. $AT_x = T_xA$ for every translation T_x.

Example 1 — String of vehicles

In an infinite string of vehicles, let p_i be the position of vehicle i relative to vehicle $i-1$. Let u_i and w_i be control force and disturbance acting on vehicle i. Then

$$\frac{d^2p_i}{dt^2}(t) = u_i(t) - u_{i-1}(t) + w_i(t) - w_{i-1}(t)$$
$$i = 0, \pm1, \pm2, \dots$$

Here the "spatial" variable i belongs to the set of integers.

Example 2 — Spring connected bodies

In an infinite string of bodies connected by springs, let p_i be the position of body i, which is subject to a control force u_i and a disturbance w_i. Then

$$\frac{d^2p_i}{dt^2}(t) = \frac{1}{2}[p_{i+1}(t) + p_{i-1}(t) - 2p_i(t)] + u_i(t) + w_i(t)$$
$$i = 0, \pm1, \pm2, \dots$$

Here the "spatial" variable i belongs to the set of integers.

Example 3

Consider the system

$$\begin{bmatrix}\dot\psi_1\\ \dot\psi_2\\ \dot\psi_3\\ \dot\psi_4\end{bmatrix} = \begin{bmatrix}\psi_2 + 2\psi_4\\ \psi_1 + 2\psi_3\\ \psi_4 + 2\psi_2\\ \psi_3 + 2\psi_1\end{bmatrix} = \begin{bmatrix}0&1&0&2\\1&0&2&0\\0&2&0&1\\2&0&1&0\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{bmatrix}$$

The system can also be written as

$$\dot\psi_{(i,j)} = \psi_{(i+1,j)} + 2\psi_{(i+1,j+1)}$$

where $i, j \in \{0, 1\}$ and addition is taken modulo 2.
Here the spatial variables are i and j.

Lecture 6

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- **Separation of spatial frequencies**
- Approximation by spatial truncation

Separation of spatial frequencies

Fourier transform in the spatial dimension gives

$$\begin{aligned}\frac{d}{dt}\widehat{\psi}(\lambda,t) &= \widehat{A}(\lambda)\widehat{\psi}(\lambda,t) + \widehat{B}(\lambda)\widehat{u}(\lambda,t)\\ y(\lambda,t) &= \widehat{C}(\lambda)\widehat{\psi}(\lambda,t) + \widehat{D}(\lambda)\widehat{u}(\lambda,t)\end{aligned}$$

Control synthesis can be done independently for different values of the "spatial frequency" λ. This gives

$$\widehat{u}(\lambda,t) = -\widehat{K}(\lambda)\widehat{\psi}(\lambda,t)$$

Transforming back from frequency domain gives

$$u(t) = -(K * \psi)(t)$$

where the convolution kernal K is obtained from $\widehat{K}$ by inverse Fourier transform.

Example 2 — Spring connected bodies

$$\begin{aligned}\frac{d^2p_i}{dt^2}(t) &= \frac{1}{2}[p_{i+1}(t) + p_{i-1}(t) - 2p_i(t)] + u_i(t) + w_i(t)\\ \frac{d^2\widehat{p}}{dt^2}(t) &= \underbrace{\frac{1}{2}(\lambda + \lambda^{-1} - 2)}_{\alpha_\lambda}\widehat{p}(\lambda,t) + \widehat{u}(\lambda,t) + \widehat{w}(\lambda,t)\end{aligned}$$

With $\psi = (p, \frac{dp}{dt})$ we get $\frac{d}{dt}\widehat{\psi} = \begin{bmatrix} 0 & 1 \\ \alpha_\lambda & 0 \end{bmatrix}\widehat{\psi} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}(\widehat{u} + \widehat{w})$.

The cost $\sum_i \int_0^\infty [p_i(t)^2 + (\frac{dp_i}{dt}(t))^2 + u_i(t)^2]dt$ is minimized by $\widehat{u}(t) = -(K * \psi)(t)$ where $\widehat{K}(\lambda) = [\widehat{P}_{12} \;\; \widehat{P}_{22}]$ and

$$\widehat{P}(\lambda) = \begin{bmatrix} \widehat{P}_{11} & \widehat{P}_{12} \\ \widehat{P}_{12} & \widehat{P}_{22} \end{bmatrix} \quad \text{where} \quad \begin{cases} \widehat{P}_{12}(\lambda) = \alpha_\lambda + \sqrt{\alpha_\lambda^2 + 1} \\ \widehat{P}_{22}(\lambda) = \sqrt{2\widehat{P}_{12} + 1} \\ \widehat{P}_{11}(\lambda) = (\widehat{P}_{12} - \alpha_\lambda)\widehat{P}_{22} \end{cases}$$

Discrete Fourier Transform

The discrete Fourier transform for N states is given by the state transformation matrix $[F]_{kl} = \frac{1}{\sqrt{N}}e^{i2\pi kl}$ where $k, l = 0, 1, \ldots, N-1$.

In particular, for $N = 2$

$$F = F^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Example 3

$$\begin{bmatrix} \dot\psi_1 \\ \dot\psi_2 \\ \dot\psi_3 \\ \dot\psi_4 \end{bmatrix} = \begin{bmatrix} \psi_2 + 2\psi_4 \\ \psi_1 + 2\psi_3 \\ \psi_4 + 2\psi_2 \\ \psi_3 + 2\psi_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}$$

The two spatial dimensions given by the permutations

$$\begin{aligned}(\psi_1, \psi_2, \psi_3, \psi_4) &\to (\psi_2, \psi_1, \psi_4, \psi_3) \to (\psi_1, \psi_2, \psi_3, \psi_4)\\ (\psi_1, \psi_2, \psi_3, \psi_4) &\to (\psi_3, \psi_4, \psi_1, \psi_2) \to (\psi_1, \psi_2, \psi_3, \psi_4)\end{aligned}$$

have corresponding state transformation matrices

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

which diagonalize $TSAS^{-1}T^{-1} = \text{diag}\{3, -3, -1, 1\}$. Design controllers for each state separately and transform back!

Lecture 6

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- Separation of spatial frequencies
- **Approximation by spatial truncation**

Localization of Controller

Control synthesis done independently for different λ gives

$$\widehat{u}(\lambda,t) = -\widehat{K}(\lambda)\widehat{\psi}(\lambda,t)$$

Transforming back from frequency domain gives

$$u(t) = -(K * \psi)(t)$$

Interestingly optimal controllers have a natural degree of decentralization, reflected in exponential decay of the convolution kernal. In particular, when the spatial coordinate is an integer, analytic extension of $\widehat{K}(\lambda)$ outside the unit circle to $R^{-1} \le |\lambda| \le R$ implies that $K(n)$ decays exponentially as $R^{-|n|}$.

Hence, a distributed controller can be obtained by truncation.

Summary

- Spatial invariance is inherited by optimal controllers
- Distributed controllers can be obtained by spatial truncation