A Course on Distributed Control

Anders Rantzer and Enrico Lovisari

Schedule

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Mo April  8 at 1315-1430 lecture
Mo April 15 at 1315-1600 lecture and exercises
Fr April 26 at 0815-1100 lecture and exercises
We   May  8 at 0915-1200 lecture and exercises
We   May 15 at 0915-1200 lecture and exercises
We   May 22 at 0915-1200 lecture and exercises
We   May 29 at 0915-1200 exercises
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A Course of Six Lectures

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1. Introduction
   Fixed modes, Team theory, Witsenhausen's counterexample

2. Partial nestedness and quadratic invariance
   Control with information delays
   Example: Tele-operation

3. Dual decomposition
   The saddle algorithm
   Example: The Internet protocol

4. Distributed MPC
   Example: Water Supply Network

5. Distributed control of positive systems. Consensus algorithms

6. Spatially invariant systems.
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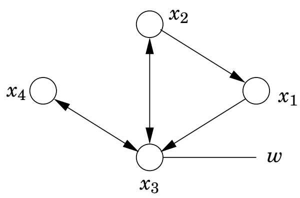
Lecture 5

- **Examples**
- Positive Systems
- Distributed Verification and Synthesis
- Positively Dominated Systems
- Consensus Algorithms

Example 1: Transportation Networks



$$\begin{bmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\\ \dot x_4\end{bmatrix} = \begin{bmatrix} -1-\ell_{31} & \ell_{12} & 0 & 0\\ 0 & 2-\ell_{12}-\ell_{32} & \ell_{23} & 0\\ \ell_{31} & \ell_{32} & 3-\ell_{23}-\ell_{43} & \ell_{34}\\ 0 & 0 & \ell_{43} & -4-\ell_{34}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ w\\ 0\end{bmatrix}$$

How do we select ℓ_{ij} to minimize the gain from w to $\sum_i x_i$?

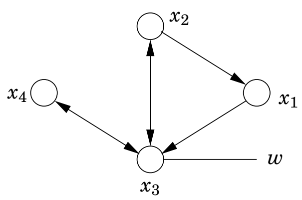
Transportation Networks in Practice

Application projects in Lund:

- Cloud computing / server farms
- Heating and ventilation in buildings
- Traffic flow dynamics
- Production planning and logistics

Example 2: A vehicle formation

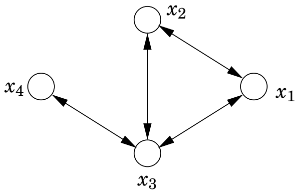
Example 2: Vehicle Formations



$$\begin{cases} \dot x_1 = -x_1 + \ell_{13}(x_3 - x_1) \\ \dot x_2 = \ell_{21}(x_1 - x_2) + \ell_{23}(x_3 - x_2) \\ \dot x_3 = \ell_{32}(x_2 - x_3) + \ell_{34}(x_4 - x_3) + w \\ \dot x_4 = -4x_4 + \ell_{43}(x_3 - x_4) \end{cases}$$

How do we select ℓ_{ij} to minimize the gain from w to $\sum_i x_i$?

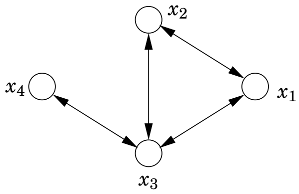
Example 3: Mass-spring system



$$\ddot{x}_i + d_i\dot{x} + k_i x_i = \sum_j \ell_{ij}(x_j - x_i) + w_i \qquad i = 1, \dots, N$$

Given masses m_i and local spring constants k_i, select the ℓ_{ij} to minimize the gain from w to x?

Example 4: Consensus Dynamics



$$x_i(t+1) = x_i(t) + \sum_j \ell_{ij}[x_j(t) - x_i(t)] \qquad i = 1, \dots, N$$

What parameters ℓ_{ij} guarantee convergence to consensus?
Can we maximize the speed of convergence?

Outline

- Examples
- **Positive Systems**
- Distributed Verification and Synthesis
- Positively Dominated Systems
- Consensus Algorithms

Positive systems have nonnegative impulse response

If the matrices A, B and C have nonnegative coefficients except possibly for the diagonal of A, then the system

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$

has non-negative impulse response $Ce^{At}B$.

Examples:

- Probabilistic model with x_k the probability of state k.
- Economic system with x_k the quantity of commodity k.
- Chemical reaction with x_k the concentration of reactant k.
- Ecological system with x_k the population of species k.

Positive Systems and Nonnegative Matrices

Classics:

- Perron (1907) and Frobenius (1912)
- Leontief (1936)
- Hirsch (1985)

Books:

- Gantmacher (1959)
- Berman and Plemmons (1979)
- Luenberger (1979)

Recent control related work:

- Angeli and Sontag (2003)
- Moreau (2004)
- Tanaka and Langbort (2010)

Stability of Positive systems

Suppose the matrix A has nonnegative off-diagonal elements. Then the following conditions are equivalent:

(i) The system $\frac{dx}{dt} = Ax$ is exponentially stable.

(ii) There exits a vector $\xi > 0$ such that $A\xi < 0$. (The vector inequalities are elementwise.)

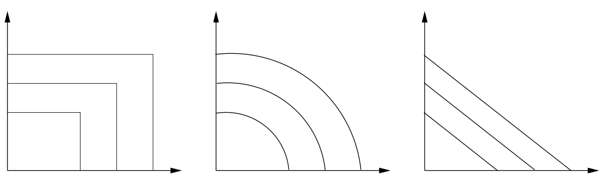
(iii) There exits a vector $z > 0$ such that $A^T z < 0$.

(iv) There is a *diagonal* matrix $P \succ 0$ such that $A^T P + PA \prec 0$

Lyapunov Functions of Positive systems

Solving the three alternative inequalities gives three different Lyapunov functions:

$A\xi < 0$ $\qquad$ $A^T P + PA \prec 0$ $\qquad$ $A^T z < 0$



$V(x) = \max_k (x_k/\xi_k)$ $\qquad$ $V(x) = x^T P x$ $\qquad$ $V(x) = z^T x$

Performance of Positive systems

Suppose that $\mathbf{G}(s) = C(sI - A)^{-1}B + D$ where $A \in \mathbb{R}^{n\times n}$ is Metzler, while $B \in \mathbb{R}_+^{n\times 1}$, $C \in \mathbb{R}_+^{1\times n}$ and $D \in \mathbb{R}_+$. Define $\|\mathbf{G}\|_\infty = \sup_\omega |G(i\omega)|$. Then the following are equivalent:

(i) The matrix A is Hurwitz and $\|\mathbf{G}\|_\infty < \gamma$.

(ii) The matrix $\begin{bmatrix} A & B \\ C & D - \gamma \end{bmatrix}$ is Hurwitz.

(iii) There is diagonal $P \succ 0$ such that $\dot{x} = Ax + Bw$ gives

$$\frac{d}{dt} x(t)^T P x(t) + |Cx(t) + Dw(t)|^2 \le \gamma^2 |w(t)|^2$$

(iv) There is $0 < p \in \mathbb{R}^n$ such that $\dot{x} = Ax + Bw$ gives

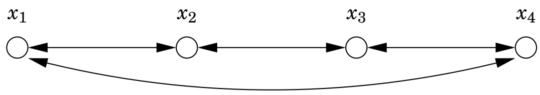
$$\frac{d}{dt}\left(p^T |x(t)|\right) + |Cx(t) + Dw(t)| \le \gamma |w(t)|$$

Moreover, if A is Hurwitz, then $\|\mathbf{G}\|_\infty = \mathbf{G}(0)$.

Outline

- Introduction
- Positive Systems
- **Distributed Verification and Synthesis**
- Positively Dominated Systems
- Consensus Algorithms

A Distributed Stability Test



Stability of $\dot{x} = Ax$ follows from existence of $\xi_k > 0$ such that

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix}}_{A} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The first node verifies the inequality of the first row.

The second node verifies the inequality of the second row.

...

Verification is scalable!

A Distributed Search for Stabilizing Gains

Suppose $\begin{bmatrix} a_{11}-\ell_1 & a_{12} & 0 & a_{14} \\ a_{21}+\ell_1 & a_{22}-\ell_2 & a_{23} & 0 \\ 0 & a_{32}+\ell_2 & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix} \geq 0$ for $\ell_1, \ell_2 \in [0, 1]$.

For stabilizing gains ℓ_1, ℓ_2, find $0 < \mu_k < \xi_k$ such that

$$\begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and set $\ell_1 = \mu_1/\xi_1$ and $\ell_2 = \mu_2/\xi_2$. Every row gives a local test.

Distributed synthesis by linear programming (gradient search).

Optimizing H_∞ Performance

Let $\mathcal{D}$ be the set of diagonal matrices with entries in $[0, 1]$.
Suppose $B, C, D \geq 0$ and $A + ELF$ is Metzler for all $L \in \mathcal{D}$.

If $F \geq 0$, then the following are equivalent:

(*i*) There exists $L \in \mathcal{D}$ such that $A + ELF$ is Hurwitz and $\|C[sI - (A + ELF)]^{-1}B + D\|_\infty < \gamma$.

(*ii*) There exist $\xi \in \mathbb{R}^n_+$, $\mu \in \mathbb{R}^m_+$ with

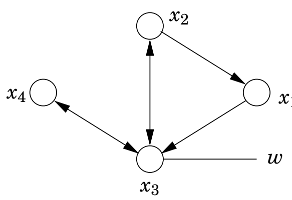
$$A\xi + E\mu + B < 0 \qquad C\xi + D < \gamma \qquad \mu \leq F\xi$$

Alternatively, if $E \geq 0$, then (i) is equivalent to

(*iii*) There exist $p \in \mathbb{R}^n_+$, $q \in \mathbb{R}^m_+$ with

$$A^T p + F^T q + C^T < 0 \quad B^T p + D < \gamma \quad q \leq E^T p$$

Example 1: Transportation Networks



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -1-\ell_{31} & \ell_{12} & 0 & 0 \\ 0 & 2-\ell_{12}-\ell_{32} & \ell_{23} & 0 \\ \ell_{31} & \ell_{32} & 3-\ell_{23}-\ell_{43} & \ell_{34} \\ 0 & 0 & \ell_{43} & -4-\ell_{34} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ w \\ 0 \end{bmatrix}$$

How do we select $\ell_{ij} \in [0, \bar{\ell}]$ to minimize the gain from w to $\sum_i x_i$?

Example 1: Transportation Networks

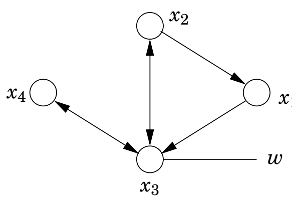
$$A = \mathrm{diag}\{-1, 2, 3, -4\} \qquad K = 0$$

$$L = \mathrm{diag}\{\ell_{31}, \ell_{12}, \ell_{32}, \ell_{23}, \ell_{43}, \ell_{34}\}/\bar{\ell}$$

$$E = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \qquad F = \begin{pmatrix} \bar{\ell} & 0 & 0 & 0 \\ 0 & \bar{\ell} & 0 & 0 \\ 0 & \bar{\ell} & 0 & 0 \\ 0 & 0 & \bar{\ell} & 0 \\ 0 & 0 & \bar{\ell} & 0 \\ 0 & 0 & 0 & \bar{\ell} \end{pmatrix}$$

The closed loop matrix is $A + ELF$.

Example 1: Transportation Networks



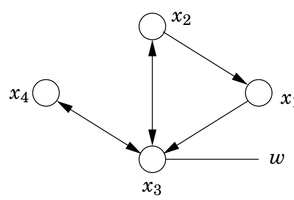
Minimize $\sum_i \xi_i$ subject to

$$\begin{aligned} 0 &\geq -\xi_1 - \mu_{13} + \mu_{21} \\ 0 &\geq 2\xi_2 - \mu_{21} - \mu_{23} + \mu_{32} \\ 0 &\geq 3\xi_3 + \mu_{13} + \mu_{23} - \mu_{32} - \mu_{34} + \mu_{43} + 1 \\ 0 &\geq -4\xi_4 + \mu_{34} - \mu_{43} \end{aligned}$$

and $0 \leq \mu_{ij} \leq \bar{\ell}\xi_j$. Then select ℓ_{ij} to get $\mu_{ij} = \ell_{ij}\xi_j$.

Optimal solution has $\ell_{13} = \ell_{23} = \ell_{43} = 0, \quad \ell_{21} = \ell_{32} = \ell_{34} = \bar{\ell}$.

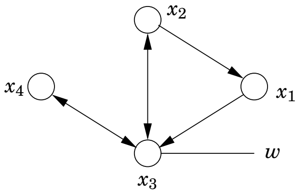
Example 2: Vehicle Formations



$$\begin{cases} \dot{x}_1 = -x_1 + \ell_{13}(x_3 - x_1) \\ \dot{x}_2 = \ell_{21}(x_1 - x_2) + \ell_{23}(x_3 - x_2) \\ \dot{x}_3 = \ell_{32}(x_2 - x_3) + \ell_{34}(x_4 - x_3) + w \\ \dot{x}_4 = -4x_4 + \ell_{43}(x_3 - x_4) \end{cases}$$

Select $\ell_{ij} \in [0, \bar{\ell}]$ to minimize the gain from w to $\sum_i x_i$?

Example 2: Vehicle Formations



Minimize p_3 subject to

$$
\begin{aligned}
0 &\geq -p_1 - q_{13} + q_{21} + 1 \\
0 &\geq -q_{21} - q_{23} + q_{32} + 1 \\
0 &\geq q_{13} + q_{23} - q_{32} - q_{34} + q_{43} + 1 \\
0 &\geq -4p_4 + q_{34} - q_{43} + 1
\end{aligned}
$$

and $0 \leq q_{ij} \leq \bar{\ell} p_j$. Then select ℓ_{ij} to get $q_{ij} = \ell_{ij} p_j$.
Optimality: $\ell_{13} = \ell_{23} = \ell_{43} = 0, \quad \ell_{32} = \ell_{34} = \bar{\ell}, \quad 2 < \ell_{21} \leq \bar{\ell}$.

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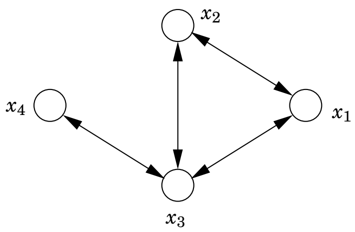
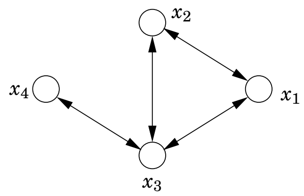
Example 3: Mass-spring system



$$
\ddot{x}_i + d_i \dot{x} + k_i x_i = \sum_j \ell_{ij}(x_j - x_i) + w_i \qquad i = 1, \ldots, N
$$

Given masses m_i and local spring constants k_i, select $\ell_{ij} \in [0, \bar{\ell}]$ to minimize the gain from w_1 to x_1?

Example 3: Mass-spring system



$$
\ddot{x}_i + d_i \dot{x} + k_i x_i = \sum_j \ell_{ij}(x_j - x_i) + w_i \qquad d_i \geq k_i
$$

In frequency domain:

$$
X_i(s) = \frac{1}{s^2 + d_i s + k_i} \left[\sum_j \ell_{ij}(X_j(s) - X_i(s)) + W_i(s) \right]
$$

Externally Positive Systems

$\mathbf{G} \in \mathbb{RH}_\infty^{m \times n}$ is called *externally positive* if if the corresponding impulse response $g(t)$ is nonnegative for all t. The set of all such matrices is denoted $\mathbb{PH}_\infty^{m \times n}$.

Suppose $\mathbf{G}, \mathbf{H} \in \mathbb{PH}_\infty^{n \times n}$. Then

- $\mathbf{GH} \in \mathbb{PH}_\infty^{n \times n}$
- $a\mathbf{G} + b\mathbf{H} \in \mathbb{PH}_\infty^{n \times n}$ when $a, b \in \mathbb{R}_+$.
- $\|\mathbf{G}\|_\infty = \|\mathbf{G}(0)\|$.
- $(I - \mathbf{G})^{-1} \in \mathbb{PH}_\infty^{n \times n}$ if and only if $\mathbf{G}(0)$ is Schur.

Positively Dominated Systems

$\mathbf{G} \in \mathbb{RH}_\infty^{m \times n}$ is called *positively dominated* if $|\mathbf{G}_{jk}(i\omega)| \leq \mathbf{G}_{jk}(0)$ for $\omega \in \mathbb{R}$. The set of all such matrices is denoted $\mathbb{DH}_\infty^{m \times n}$.

Suppose $\mathbf{G}, \mathbf{H} \in \mathbb{DH}_\infty^{n \times n}$. Then

- $\mathbf{GH} \in \mathbb{DH}_\infty^{n \times n}$
- $a\mathbf{G} + b\mathbf{H} \in \mathbb{DH}_\infty^{n \times n}$ when $a, b \in \mathbb{R}_+$.
- $\|\mathbf{G}\|_\infty = \|\mathbf{G}(0)\|$.
- $(I - \mathbf{G})^{-1} \in \mathbb{DH}_\infty^{n \times n}$ if and only if $\mathbf{G}(0)$ is Schur.

Example 3: Mass-spring system

$$
\ddot{x}_i + d_i \dot{x} + k_i x_i = \sum_j \ell_{ij}(x_j - x_i) + w_i
$$

$$
\left(s^2 + d_i s + k_i + \sum_j \bar{\ell}_{ij} \right) X_i(s) = \sum_j \left(\ell_{ij} X_j(s) + (\bar{\ell}_{ij} - \ell_{ij}) X_i(s) \right) + W_i(s)
$$

$$
X = (\mathbf{A} + \mathbf{E}L\mathbf{F})X + \mathbf{B}W
$$

The transfer matrices $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{A} + \mathbf{E}L\mathbf{F}$ are positively dominated for all $L \in \mathcal{D}$ provided that $d_i \geq k_i + \sum_j \bar{\ell}_{ij}$.

Exercises:

How do you compute a stabilizing gain matrix $L \in \mathcal{D}$?

How do you compute $L \in \mathcal{D}$ to minimize gain from w to x?

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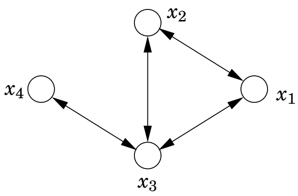
Example 4: Consensus Dynamics

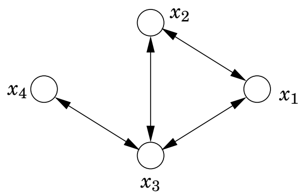


$$x_i(t+1) = x_i(t) + \sum_j \ell_{ij}[x_j(t) - x_i(t)] \qquad i = 1, \ldots, N$$

What parameters ℓ_{ij} guarantee convergence to consensus?
Can we maximize the speed of convergence?

Can the previous theory be used?

Example 4: Consensus Dynamics



$$x_i(t+1) = x_i(t) + \sum_j \ell_{ij}[x_j(t) - x_i(t)] \qquad i = 1, \ldots, N$$

What parameters ℓ_{ij} guarantee convergence to consensus?
Can we maximize the speed of convergence?

Can the previous theory be used? No, system not stabilizable!

Example 4: Consensus Dynamics

$$x_i(t+1) = x_i(t) + \sum_j \ell_{ij}[x_j(t) - x_i(t)] \qquad i = 1, \ldots, N$$

Positive system if $\ell_{ij} \geq 0$ and $\sum_j \ell_{ij} \leq 1$. The total system is

$$x(t+1) = Wx(t)$$

where $W \geq 0$ and $W\mathbf{1} = \mathbf{1}$, i.e. W is a *stochastic matrix*.
The state $x(t)$ converges to the average if and only if

$$\lim_{t \to \infty} W^t = \frac{\mathbf{1}\mathbf{1}^T}{n}$$

and the speed of convergence is given by the spectral radius of

$$W - \frac{\mathbf{1}\mathbf{1}^T}{n}$$

If W is symmetric, this equals $\|W - \frac{\mathbf{1}\mathbf{1}^T}{n}\|$ which is convex in W!

Open problems

Let

$$x_i(t+1) = x_i(t) + \sum_j \ell_{ij}[x_j(t-1) - x_i(t)] + w_i(t)$$

where w_i are white noise sequences.

Problems:

1. Find ℓ_{ij} to minimize variance of x
2. Do the same thing with ℓ_{ij} transfer functions

Summary

Classical hard problems solvable for positive systems:

- Static output feedback
- H_∞/L_1 optimal decentralized controllers
- No need for global information
- Verification and synthesis *scale linearly* !
- Consensus theory is different — not stabilizable!

Further reading:

Rantzer, Distributed control of positive systems, 2012
Xiao/Boyd, Fast linear iterations for distributed averaging, 2004