A Course on Distributed Control

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Schedule

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Mo April  8 at 1315-1430 lecture
Mo April 15 at 1315-1600 lecture and exercises
Fr April 26 at 0815-1100 lecture and exercises
We   May  8 at 0915-1200 lecture and exercises
We   May 15 at 0915-1200 lecture and exercises
Mo   May 20 at 1315-1600 lecture and exercises
Mo   May 27 at 1315-1500 exercises
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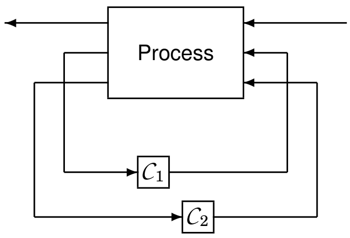
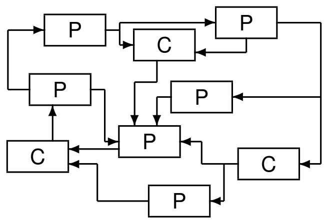
Control with Information Constraints



Can we stabilize the system? Are the optimal controllers linear? Can they be computed efficiently?

These questions will be adressed during the first two lectures.

Control Synthesis from a Decentralized Perspective



Can local controllers be designed without knowledge of the entire system?
What level of performance can be achieved this way?

This will be the main topic in of lecture 3-4.

A Course of Six Lectures

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1. Introduction
   Fixed modes, Team theory, Witsenhausen's counterexample

2. Partial nestedness and quadratic invariance
   Control with information delays
   Example: Tele-operation

3. Dual decomposition
   The saddle algorithm
   Example: The Internet protocol

4. Distributed MPC
   Example: Water Supply Network

5. Distributed control of positive systems. Consensus algorithms

6. Spatially invariant systems.
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Outline of Lecture 3

- **Dual decomposition and the saddle algorithm**
  [Arrow/Hurwicz/Uzawa 1958]
- Example: The TCP protocol [Low/Paganini/Doyle 2002]

50 year old idea: Dual decomposition

$$\min_{z_i}[V_1(z_1,z_2)+V_2(z_2)+V_3(z_3,z_2)]$$
$$=\max_{p_i}\min_{z_i,v_i}\Big[V_1(z_1,v_1)+V_2(z_2)+V_3(z_3,v_3)+p_1(z_2-v_1)+p_3(z_2-v_3)\Big]$$

The optimum is a Nash equilibrium of the following game:

The three computers try to minimize their respecive costs

Computer 1: $\min_{z_1,v_1}\left[V_1(z_1,v_1)-p_1v_1\right]$

Computer 2: $\min_{z_2}\left[V_2(z_2)+(p_1+p_3)z_2\right]$

Computer 3: $\min_{z_3,v_3}\left[V_3(z_3,v_3)-p_3v_3\right]$

while the "market makers" try to maximize their payoffs

Between computer 1 and 2: $\max_{p_1}\left[p_1(z_2-v_1)\right]$

Between computer 2 and 3: $\max_{p_3}\left[p_3(z_2-v_3)\right]$

Decentralized Bounds on Suboptimality

Given any $p_1, p_3, \bar{z}_1, \bar{z}_2, \bar{z}_3$, the distributed test

$$V_1(\bar{z}_1,\bar{z}_2)-p_1\bar{z}_2 \le \alpha \min_{z_1,v_1}\left[V_1(z_1,v_1)-p_1v_1\right]$$
$$V_2(\bar{z}_2)+(p_1+p_3)\bar{z}_2 \le \alpha \min_{z_2}\left[V_2(z_2)+(p_1+p_3)z_2\right]$$
$$V_3(\bar{z}_3,\bar{z}_2)-p_3\bar{z}_2 \le \alpha \min_{z_3,v_3}\left[V_3(z_3,v_3)-p_3v_3\right]$$

implies that the globally optimal cost J^* is bounded as

$$J^* \le V_1(\bar{z}_1,\bar{z}_2)+V_2(\bar{z}_2)+V_3(\bar{z}_3,\bar{z}_2) \le \alpha J^*$$

Proof: Add both sides up!

The saddle point algorithm

Update in gradient direction:

Computer 1: $\begin{cases} \dot z_1 &= -\partial V_1/\partial z_1 \\ \dot v_1 &= -\partial V_1/\partial z_2 + p_1 \end{cases}$

Computer 1 and 2: $\dot p_1 = z_2 - v_1$

Computer 2: $\dot z_2 = -\partial V_2/\partial z_2 - p_1 - p_3$

Computer 2 and 3: $\dot p_3 = z_2 - v_3$

Computer 3: $\begin{cases} \dot z_3 &= -\partial V_3/\partial z_3 \\ \dot v_3 &= -\partial V_3/\partial z_2 + p_3 \end{cases}$

Globally convergent if V_i are convex!
[Arrow, Hurwicz, Usawa 1958]

Global stability of saddle algorithm

$$\min_{Rx=0} V(x) = \max_p \min_x [V(x) + p^T Rx]$$

$$\begin{cases} \dot x = -G\left[(\partial V/\partial x)^T + R^T p\right] \\ \dot p = HRx \end{cases}$$

$G, H > 0$ adjustment rates

$$\begin{bmatrix} \ddot x \\ \ddot p \end{bmatrix} = \begin{bmatrix} -G(\partial^2 V/\partial x^2) & -GR^T \\ HR & 0 \end{bmatrix} \begin{bmatrix} \dot x \\ \dot p \end{bmatrix}$$

$$\mathbf{V} = |\dot x|^2_{G^{-1}} + |\dot p|^2_{H^{-1}}$$

$$\begin{aligned} \frac{d}{dt}\mathbf{V} &= \dot x^T G^{-1} \ddot x + \dot p^T H^{-1} \ddot p \\ &= -\dot x^T \left[(\partial^2 V/\partial x^2)\dot x + R^T \dot p\right] + \dot p^T (R\dot x) \\ &= -\dot x^T (\partial^2 V/\partial x^2)\dot x \le 0 \end{aligned}$$

Example: Three Trading Units (The Beer Game)

Consumer utility	$U_1(w_1 + u_{11}) - p_1 u_{11}$
Retailer utility	$U_2(w_2 - u_{21} + u_{22}) + p_1 u_{21} - p_2 u_{22}$
Factory utility	$U_3(w_3 - u_{32}) + p_2 u_{32}$

Consumer demand: $\dot u_{11} = -U_1'(w_1 + u_{11}) - p_1$

Consumer market: $\dot p_1 = u_{11} - u_{21}$

Retailer supply and demand: $\begin{cases} \dot u_{21} = U_2'(w_2 - u_{21} + u_{22}) + p_1 \\ \dot u_{22} = -U_2'(w_2 - u_{21} + u_{22}) - p_2 \end{cases}$

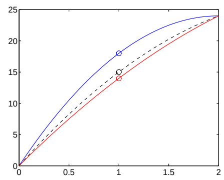
Factory market: $\dot p_2 = u_{22} - u_{32}$

Factory supply rate: $\dot u_{32} = -U_3'(w_3 - u_{32}) + p_2$

Example: Three Trading Units

Three utility functions plotted together with possible equilibrium point.

$$\begin{aligned} U_1(x_1) &= 24 - 6(x_1 - 2)^2 \\ U_2(x_2) &= 27 - 3(x_2 - 3)^2 \\ U_3(x_2) &= 32 - 2(x_3 - 4)^2 \end{aligned}$$

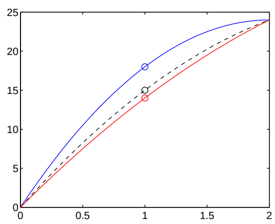


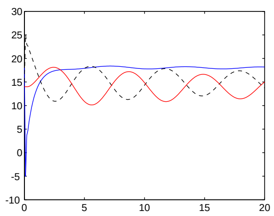
When prices and quantities have settled, there is no trade incentive. The equilibrium is a global optimum (social welfare):

$$\max_{u_1,u_2} [U_1(w_1 + u_1) + U_2(w_2 - u_1 + u_2) + U_3(w_3 - u_2)]$$

This is a Nash equilibrium for the game with five players, three agents and two markets.

Gradient dynamics tend to be oscillative





Network congestion control

Maximize $U_i(x)$ subject to $\sum_i R_{li} x_i \le c_l$. Introduce link prices p_l:

$$\begin{aligned} \max_{x_i \ge 0} \sum_i U_i(x_i) &= \min_{p_l \ge 0} \max_{x_i \ge 0} \sum_i \left[U_i(x_i) - \sum_l p_l \left(R_{li} x_i - c_l\right) \right] \\ &= \min_{p_l \ge 0} \max_{x_i \ge 0} \sum_i \left[U_i(x_i) - x_i \sum_l p_l R_{li} \right] + \sum_l p_l c_l \end{aligned}$$

To update the send rate x_i, we need to know $\sum_l p_l R_{li}$. To update the price p_l, we need $R_{li} x_i - c_l$. Are these quantities locally known?

Global stability of discrete saddle algorithm

$$\min_{Rx=0} U(x) = \max_p \min_x [U(x) + p^T Rx]$$

The discrete time saddle algorithm

$$\begin{cases} x^+ = x - G\left[(\partial U/\partial x)^T + R^T p\right] \\ p^+ = p + HRx \end{cases}$$

is stable for convex U provided that $G, H > 0$ and

$$3R^T HR < -(\partial^2 U/\partial x^2) < \frac{1}{3} G^{-1}$$

Exercise: Prove this using the Lyapunov function
$V = |x - x^*|^2_{G^{-1}} + |p - p^*|^2_{H^{-1}} - 2(p - p^*)^T R(x - x^*)$

What did we achieve?

- Optimality test inherits structure of original problem
- Prices show the relative importance of different terms
- Suboptimality bounds indicate *where* things went wrong
- Sparsity structure useful for efficient computations