A Course on Distributed Control

Anders Rantzer and Enrico Lovisari

Schedule

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Mo April  8 at 1315-1430 lecture
Mo April 15 at 1315-1600 lecture and exercises
Fr April 26 at 0915-1200 lecture and exercises
Tu   May  7 at 1315-1600 lecture and exercises
Mo   May 13 at 1315-1600 lecture and exercises
Mo   May 20 at 1315-1600 lecture and exercises
Mo   May 27 at 1315-1500 exercises
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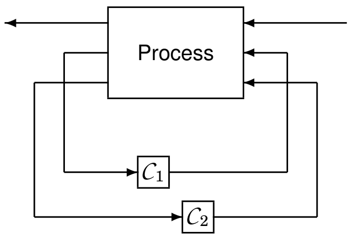
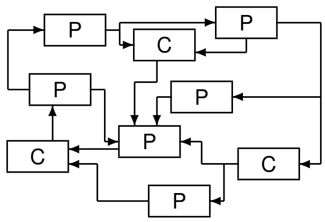
Control with Information Constraints



Can we stabilize the system? Are the optimal controllers linear? Can they be computed efficiently?

These questions will be adressed during the first two lectures.

Control Synthesis from a Decentralized Perspective



Can local controllers be designed without knowledge of the entire system?
What level of performance can be achieved this way?

This will be the main topic in of lecture 3-4.

A Course of Six Lectures

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1. Introduction
   Fixed modes, Team theory, Witsenhausen's counterexample

2. Partial nestedness and quadratic invariance
   Control with information delays
   Example: Tele-operation

3. Dual decomposition
   The saddle algorithm
   Example: The Internet protocol

4. Distributed MPC
   Example: Water Supply Network

5. Spatially invariant systems.

6. Distributed control of positive systems. Consensus algorithms
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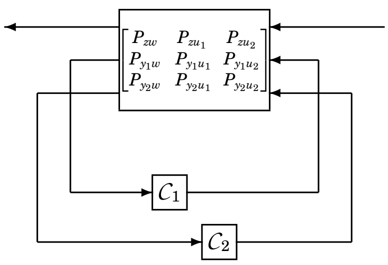
Outline of Lecture 2

- **Partial Nestedness [Ho/Chu 1972]**
- Quadratic Invariance [Rotkowitz/Lall 2006]
- Example: Tele-operation [Kristalny/Cho 2012]

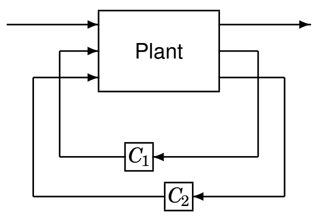
An incentive for signalling



If one controller has information useful for the other, then there is an incentive to encode this information in the control inputs.

This “signalling” creates complicated nonlinear control laws.

The signalling incentive sometimes disappears!



[Yu-Chi Ho and K'ai-Ching Chu (1972)]:

If a decision-makers action affects our information, then knowing what he knows will yield linear optimal solutions

The condition is called “partial nestedness”.

Standard linear quadratic optimal control

Find $u = Lx$ to minimize $\mathbf{E}(|x|^2 + |u|^2)$ when

$$x(k+1) = Ax(k) + Bu(k) + w(k) \qquad \mathbf{E}w(k)w(k)^T = I$$

Notation: $\begin{bmatrix} X_{xx} & X_{xu} \\ X_{ux} & X_{uu} \end{bmatrix} = \mathbf{E}\begin{bmatrix} x \\ u \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}^T$

Solution by convex optimization:

Minimize $\quad$ trace (X_{xx}) + trace (X_{uu})

subject to $\quad X_{xx} = \begin{bmatrix} A & B & I \end{bmatrix} \underbrace{\begin{bmatrix} X_{xx} & X_{xu} & 0 \\ X_{ux} & X_{uu} & 0 \\ 0 & 0 & I \end{bmatrix}}_{\succ 0} \begin{bmatrix} A^T \\ B^T \\ I \end{bmatrix}$

Then put $u(k) = Lx(k)$ where $LX_{xx} = X_{ux}$.

Control with disturbance measurements

Find $u = Lx + Mw$ to minimize $\mathbf{E}(|x|^2 + |u|^2)$ when

$$x(k+1) = Ax(k) + Bu(k) + w(k) \qquad \mathbf{E}w(k)w(k)^T = I$$

Solution by convex optimization:

Minimize $\quad$ trace (X_{xx}) + trace (X_{uu})

subject to $\quad X_{xx} = \begin{bmatrix} A & B & I \end{bmatrix} \underbrace{\begin{bmatrix} X_{xx} & X_{xu} & 0 \\ X_{ux} & X_{uu} & X_{uw} \\ 0 & X_{wu} & I \end{bmatrix}}_{\succ 0} \begin{bmatrix} A^T \\ B^T \\ I \end{bmatrix}$

Then put $u(k) = X_{ux}X_{xx}^{-1}x(k) + X_{uw}w(k)$

A one-step delay information pattern

Find $u = \begin{bmatrix} L_1x + M_1w_1 \\ L_2x + M_2w_2 \end{bmatrix}$ to minimize $\mathbf{E}(|x|^2 + |u|^2)$ when $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and

$$x(k+1) = Ax(k) + Bu(k) + w(k) \qquad \mathbf{E}w(k)w(k)^T = I$$

Is the problem partially nested?

A one-step delay information pattern

Find $u = Lx + \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} w$ to minimize $\mathbf{E}(|x|^2 + |u|^2)$ when

$$x(k+1) = Ax(k) + Bu(k) + w(k) \qquad \mathbf{E}w(k)w(k)^T = I$$

Solution by convex optimization:

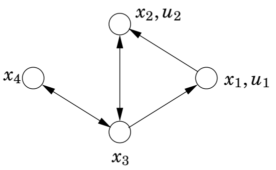
Minimize $\quad$ trace (X_{xx}) + trace (X_{uu})

subject to $\quad X_{xx} = \begin{bmatrix} A & B & I \end{bmatrix} \underbrace{\begin{bmatrix} X_{xx} & X_{xu} & 0 \\ X_{ux} & X_{uu} & X_{uw} \\ 0 & X_{wu} & I \end{bmatrix}}_{\succ 0} \begin{bmatrix} A^T \\ B^T \\ I \end{bmatrix}$

$$X_{uw} = X_{wu}^T = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

Then put $u(k) = X_{ux}X_{xx}^{-1}x(k) + X_{uw}w(k)$

A Team Problem with Delay Constraints



Find μ_1, μ_2 to minimize the stationary variance

$$\mathbf{E}\sum_{i,j}(|x_i|^2 + |u_j|^2)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} \Phi_{11} & 0 & \Phi_{13} & 0 \\ \Phi_{21} & \Phi_{22} & \Phi_{23} & 0 \\ 0 & \Phi_{32} & \Phi_{33} & \Phi_{34} \\ 0 & 0 & \Phi_{43} & \Phi_{44} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} \Gamma_1 u_1(k) + w_1(k) \\ \Gamma_2 u_2(k) + w_2(k) \\ w_3(k) \\ w_4(k) \end{bmatrix}$$

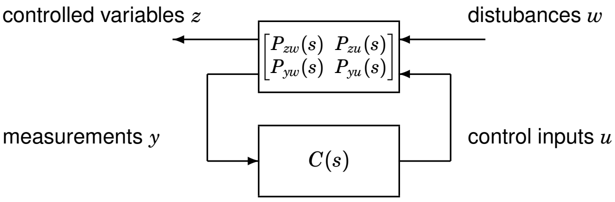
$$u_1(k) = \mu_1\Big(y_1(k), y_2(k-2), y_3(k-1), y_4(k-2)\Big)$$
$$u_2(k) = \mu_2\Big(y_1(k-1), y_2(k), y_3(k-1), y_4(k-2)\Big) \qquad y_i(k) = \begin{bmatrix} C_ix_i(k) \\ C_ix_i(k-1) \\ C_ix_i(k-2) \\ \vdots \end{bmatrix}$$

Is it partially nested?

Outline of Lecture 2

- Partial Nestedness [Ho/Chu 1972]
- **Quadratic Invariance [Rotkowitz/Lall 2006]**
- Example: Tele-operation [Kristalny/Cho 2012]

The Youla parametrization for stable plants



Original problem:

Minimize $\|P_{zw} - P_{zu}C(I - P_{yu}C)^{-1}P_{yw}\|$ over stabilizing C

Equivalent problem:

Minimize $\|P_{zw} + P_{zu}QP_{yw}\|$ over stable Q

Synthesis by convex optimization

A general control synthesis problem can be stated as a convex optimization problem in the variables $Q_0, \dots, Q_m$. The problem has a quadratic objective, with linear and quadratic constraints:

Minimize$_{Q_k}$ $\quad \int_{-\infty}^{\infty} |P_{zw}(i\omega) + P_{zu}(i\omega)\overbrace{\sum_k Q_k\phi_k(i\omega)}^{Q(i\omega)} P_{yw}(i\omega)|^2 d\omega$ } quadratic objective

subject to $\quad$ step response $w_i \to z_j$ is smaller than f_{ijk} at time t_k; step response $w_i \to z_j$ is bigger than g_{ijk} at time t_k } linear constraints

Bode magnitude $w_i \to z_j$ is smaller than h_{ijk} at ω_k } quadratic constraints

Once the variables $Q_0, \dots, Q_m$ have been optimized, the controller is obtained as $C(s) = \left[I - Q(s)P_{yu}(s)\right]^{-1}Q(s)$

Youla parameterization with constraints

[Rotkowitz, Lall (2002)]: Let S be a linear space.

Original problem:

$$\text{Minimize}_{C\in S}\ \|P_{zw} - P_{zu}C(I - P_{yu}C)^{-1}P_{yw}\|$$

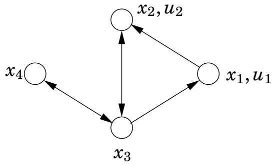
Modified problem:

$$\text{Minimize}_{Q\in S}\ \|P_{zw} + P_{zu}QP_{yw}\|$$

Condition for equivalence between the two:
The two are equivalent if S is *quadratically invariant* under P_{yu}, i.e.

$$CP_{yu}C \in S \text{ for all } C \in S$$

A Team Problem with Delay Constraints



Find μ_1, μ_2 to minimize the stationary variance

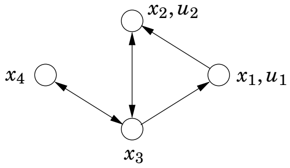
$$\mathbf{E}\sum_{i,j}(|x_i|^2 + |u_j|^2)$$

$$\begin{bmatrix} x_1(k+1)\\ x_2(k+1)\\ x_3(k+1)\\ x_4(k+1)\end{bmatrix} = \begin{bmatrix}\Phi_{11} & 0 & \Phi_{13} & 0\\ \Phi_{21} & \Phi_{22} & \Phi_{23} & 0\\ 0 & \Phi_{32} & \Phi_{33} & \Phi_{34}\\ 0 & 0 & \Phi_{43} & \Phi_{44}\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\\ x_3(k)\\ x_4(k)\end{bmatrix} + \begin{bmatrix}\Gamma_1 u_1(k) + w_1(k)\\ \Gamma_2 u_2(k) + w_2(k)\\ w_3(k)\\ w_4(k)\end{bmatrix}$$

$$u_1(k) = \mu_1\Big(y_1(k), y_2(k-2), y_3(k-1), y_4(k-2)\Big)$$
$$u_2(k) = \mu_2\Big(y_1(k-1), y_2(k), y_3(k-1), y_4(k-2)\Big)$$
$$y_i(k) = \begin{bmatrix} C_i x_i(k)\\ C_i x_i(k-1)\\ C_i x_i(k-2)\\ \vdots\end{bmatrix}$$

Is it quadratically invariant?

Convexity in distributed control



Minimize $\|P_{zw} + P_{zu}QP_{yw}\|^2$ where

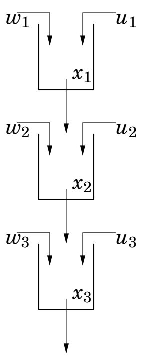
$$Q(z) = C(I - P_{yu}C)^{-1}$$
$$P_{yu}(z) = \begin{bmatrix} p_{11}(z) & z^{-2}p_{12}(z)\\ z^{-1}p_{21}(z) & p_{22}(z)\\ z^{-2}p_{31}(z) & z^{-1}p_{32}(z)\\ z^{-3}p_{41}(z) & z^{-2}p_{42}(z)\end{bmatrix}$$
$$C(z) = \begin{bmatrix} c_{11}(z) & z^{-2}c_{12}(z) & z^{-1}c_{13}(z) & z^{-2}c_{14}(z)\\ z^{-1}c_{21}(z) & c_{22}(z) & z^{-1}c_{23}(z) & z^{-2}c_{24}(z)\end{bmatrix}$$

Convexity in distributed control

[Bamieh, Voulgaris (2002)] and [Rotkowitz, Lall (2002)]:

The distributed control synthesis problem becomes convex when *communication links propagate information at least as fast as the process does.*

Mini-problem: Irrigation system



$$\begin{bmatrix}x_1(t+1)\\ x_2(t+1)\\ x_3(t+1)\\ x_4(t+1)\end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0\\ * & * & 0 & 0\\ 0 & * & * & 0\\ 0 & 0 & * & *\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix} + \begin{bmatrix}B_1u_1 + w_1\\ B_2u_2 + w_2\\ B_3u_3 + w_3\\ B_4u_4 + w_4\end{bmatrix}$$

What control structures are quadratically invariant?

$$\begin{bmatrix}u_1(t)\\ u_2(t)\\ u_3(t)\\ u_4(t)\end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0\\ * & * & 0 & 0\\ 0 & * & * & 0\\ 0 & 0 & * & *\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix}$$

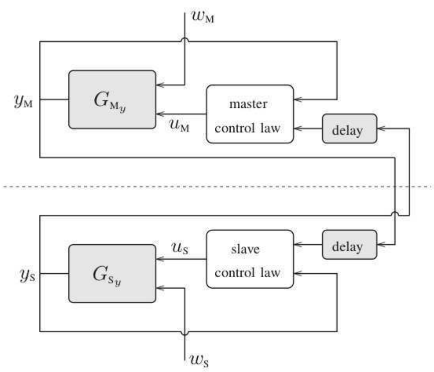
$$\begin{bmatrix}u_1(t)\\ u_2(t)\\ u_3(t)\\ u_4(t)\end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0\\ * & * & 0 & 0\\ * & * & * & 0\\ * & * & * & *\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix}$$

$$\begin{bmatrix}u_1(t)\\ u_2(t)\\ u_3(t)\\ u_4(t)\end{bmatrix} = \begin{bmatrix} * & * & * & *\\ 0 & * & * & *\\ 0 & 0 & * & *\\ 0 & 0 & 0 & *\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix}$$

Outline of Lecture 2

- Partial Nestedness [Ho/Chu 1972]
- Quadratic Invariance [Rotkowitz/Lall 2006]
- **Example: Tele-operation [Kristalny/Cho 2012]**

Tele-operation



Next Lecture

Lecture 2	Partial nestedness and quadratic invariance Control with information delays
Lecture 3	**Dual decomposition** **The saddle algorithm** **Example: The Internet protocol**