A Course on Distributed Control

Anders Rantzer and Enrico Lovisari

Schedule

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Mo April  8 at 1315-1430 lecture
Mo April 15 at 1315-1600 lecture and exercises
Fr April 26 at 0915-1200 lecture and exercises
Tu   May  7 at 1315-1600 lecture and exercises
Mo   May 13 at 1315-1600 lecture and exercises
Mo   May 20 at 1315-1600 lecture and exercises
Mo   May 27 at 1315-1500 exercises
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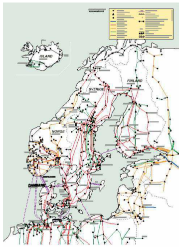
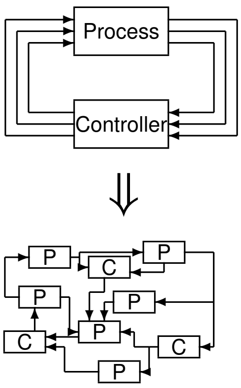
Building theoretical foundations for distributed control

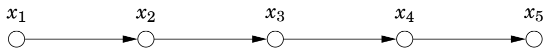




We need methodology for

- Decentralized specifications
- Decentralized design
- Verification of global behavior

Example 1: A vehicle formation



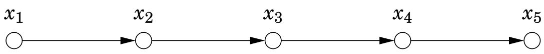
Each vehicle obeys the independent dynamics

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} B_1u_1(t)+w_1(t) \\ B_2u_2(t)+w_2(t) \\ B_3u_3(t)+w_3(t) \\ B_4u_4(t)+w_4(t) \end{bmatrix}$$

The objective is to make $\mathbf{E}|Cx_{i+1} - Cx_i|^2$ small for $i = 1, \dots, 4$.

How do we optimize?
What information needs to be communicated?

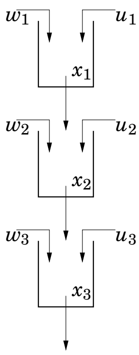
Example 2: A supply chain for fresh products



Fresh products degrade with time:

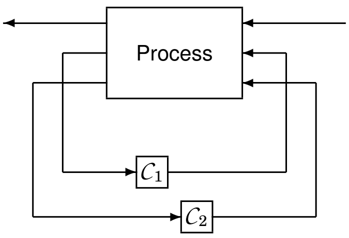
$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} -u_1(t)+w_1(t) \\ u_1(t)-u_2(t) \\ u_2(t)-u_3(t) \\ u_3(t)+w_4(t) \end{bmatrix}$$

Example 3: Water distribution systems



$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} B_1u_1+w_1 \\ B_2u_2+w_2 \\ B_3u_3+w_3 \\ B_4u_4+w_4 \end{bmatrix}$$

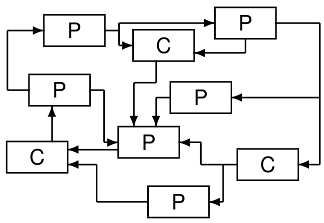
Control with Information Constraints



Can we stabilize the system? Are the optimal controllers linear? Can they be computed efficiently?

These questions will be adressed during the first two lectures.

Control Synthesis from a Decentralized Perspective



Can local controllers be designed without knowledge of the entire system?
What level of performance can be achieved this way?

This will be the main topic in of lecture 3-4.

A Course of Six Lectures

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1. Introduction
   Fixed modes, Team theory, Witsenhausen's counterexample

2. Partial nestedness and quadratic invariance
   Control with information delays
   Example: Tele-operation

3. Dual decomposition
   The saddle algorithm
   Example: The Internet protocol

4. Distributed MPC
   Example: Water Supply Network

5. Spatially invariant systems.

6. Distributed control of positive systems. Consensus algorithms
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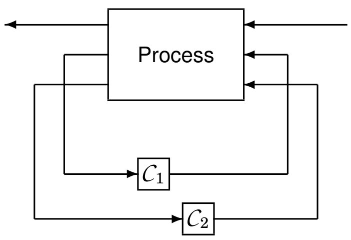
Outline of Lecture 1

- Introduction
- **Fixed modes. [Wang/Davison 1973]**
- Team theory. [Radner 1962]
- Witsenhausen's counterexample. [Witsenhausen 1968]

Control with Information Constraints



Can we stabilize the system?

Theorem (Wang/Davison 1973) on fixed modes

$$x(t+1) = Ax(t) + \sum_{i=1}^{m} B_i u_i(t)$$

$$\begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} = \begin{bmatrix} C_1 x(t) \\ \dots \\ C_m x(t) \end{bmatrix}$$

has a stabilizing controller of the form

$$U_1(z) = C_1(z)Y_1(z) \qquad \dots \qquad U_m(z) = C_m(z)Y_m(z)$$

if and only if there are no unstable "fixed modes", i.e. if

$$A + \begin{bmatrix} B_1 & \dots & B_m \end{bmatrix} \begin{bmatrix} K_1 & & \\ & \ddots & \\ & & K_m \end{bmatrix} \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix}$$

has no unstable eigenvalues that cannot be affected by $K_1, \dots, K_m$.

Proof sketch

A fixed mode implies a solution x of the eigenvalue equation

$$\lambda x = \left(A + \begin{bmatrix} B_1 & \dots & B_m \end{bmatrix} \begin{bmatrix} K_1 & & \\ & \ddots & \\ & & K_m \end{bmatrix} \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix} \right) x$$

that remains valid for all K_i. It follows that

$$\lambda x = \left(A + \begin{bmatrix} B_1 & \dots & B_m \end{bmatrix} \begin{bmatrix} C_1(\lambda) & & \\ & \ddots & \\ & & C_m(\lambda) \end{bmatrix} \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix} \right) x$$

so the same pole must be unaffected by $C_1, \dots, C_m$.

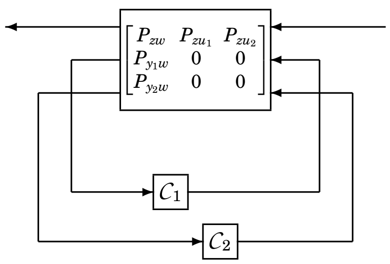
If an eigenvalue can be moved by some K_i, then it is observable and an controllable by the the corresponding pair (B_i, C_i), so it can be stabilized using C_i. This can be used repeatedly to stabilize all eigenvalues.

Outline of Lecture 1

- Introduction
- Fixed modes. [Wang/Davison 1973]
- **Team theory. [Radner 1962]**
- Witsenhausen's counterexample. [Witsenhausen 1968]

Team Decision Problems

Each decision maker has his own set of measurements.
A common performance objective should be optimized.

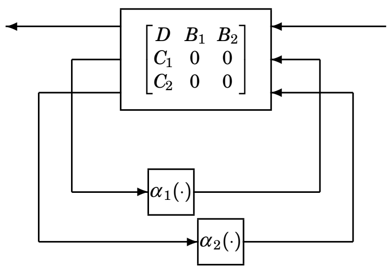


Are the optimal controllers C_1 and C_2 linear time-invariant?
Can they be computed efficiently?

Team Decision Problems

Minimize $\quad \mathbf{E}\,|Dw + B_1\alpha_1(C_1w) + B_2\alpha_2(C_2w)|^2$

when w is a normal distributed stochastic variable

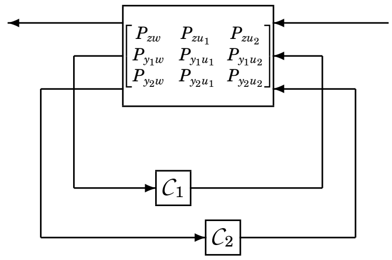


By [Radner 1962], the optimal α_1 and α_2 are linear.
Proof: Convexity gives optimality when gradient is zero.

Outline of Lecture 1

- Introduction
- Fixed modes. [Wang/Davison 1973]
- Team theory. [Radner 1962]
- **Witsenhausen's counterexample. [Witsenhausen 1968]**

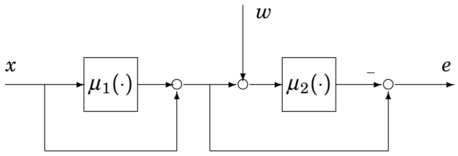
An incentive for signalling



If one controller has information useful for the other, then there is an incentive to encode this information in the control inputs.

This "signalling" creates complicated nonlinear control laws.

The Witsenhausen counterexample



Minimize $\mathbf{E}\left(\left|x + \mu_1(x) - \mu_2(x + \mu_1(x) + w)\right|^2 + \left|\mu_1(x)\right|^2\right)$

when x and w are given Gaussian variables.

The best controllers are not linear, because for a fixed output variance of μ_1, a non-Gaussian signal can transfer more information than a Gaussian one.

[Witsenhausen (1968) A counterexample in stochastic control]

Next Lecture

Lecture 1	Introduction Fixed modes Team theory Witsenhausen's counterexample
Lecture 2	**Partial nestedness and quadratic invariance** **Control with information delays**