SAMPLING OF STATE SPACE SYSTEMS WITH SEVERAL TIME DELAYS

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Abstract This article solves the problem of how to obtain a zero-order hold sampled version of a state space system containing several time delays at arbitrary positions. No assumption is made on commensurability of the time delays. It is shown that the condition for obtaining a finite dimensional sampled system for all sampling periods is that there are 'no signal loops' around any of the time delays. A short and constructive algorithm is presented for sampling such systems. All calculations can be performed using standard programs for sampling systems.

Keywords Delays; sampled data systems; state space; digital control; linear systems

1. INTRODUCTION

Many industrial processes contain several time delays. This is for instance common in chemical engineering processes, where time delays results from piping between units. The time behavior of such systems can often be adequately described by linear, continuous time, differential-difference equations (DDEs). Such equations have been the subject of much research, see e.g. [Choksy, 1960],[Marshall, 1979].

The control of linear time-delay systems is generally difficult both in theory and in practice. Often time delays put severe restrictions on achievable feedback performance. It is therefore important to have good methods for analysis and design of such systems. One possibility is to sample the system and use digital control. As we will see, it can then happen that the sampled system becomes finite dimensional. Further analysis and synthesis are then much simplified.

A continuous time linear system with a time delay is an infinite dimensional system. To model the delay one must store a function of time over a time interval equal the length of the time delay. It was therefore a surprise when it was found that the sampled version of the system in Fig 1 is finite-dimensional. This was noted as soon as computers were being used to implement control systems in the 1950s [Ragazzini and Franklin, 1958]. The method is by now classical and formulas appear in most text books, see e.g. [Åström and Wittenmark, 1990]. Algorithms are also included in most software packages for digital control. It is straightforward using these results to sample a multivariable linear system with time delays in control variables only. The solution consists of storing state variables and delayed input signals in a finite number of sampling points and showing that this information suffices to update the system equations.

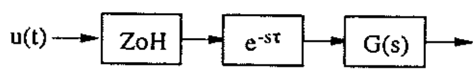

Fig. 1 Hold circuit, time delay and linear system

Note that state variables often have physical interpretations. To keep the engineering intuition from the continuous time model, it is advantageous to obtain a sampled system from which the state variables at the sampling points can be obtained.

Sampling of systems with internal time delays has received much less attention in the literature and few results have been obtained. The problem has only been solved for simple systems. The setup in Fig 1 was slightly generalized in [Araki *et al.*, 1984], [Wittenmark, 1985], [Fujinaka and Araki, 1987]. As a result the system in Fig 2 can also be sampled. Here the time delay is situated between two linear systems. The sampled system is finite dimensional in this case also. Note that the problem of sampling the system in Fig 2 can not be trivially solved by changing the order of $G_1(s)$ and the time delay and reducing the problem to the system in Fig1. The pulse transfer function between input and output will be the same, but the transformation changes the states of G_1 from $x_1(kh)$ to $x_1(kh-\tau)$ and one will hence not obtain a state space representation with the values of all state variables at the sampling points, see the discussion in [Wittenmark, 1985].

The question with several time delays at arbitrary positions in a multivariable linear system arises naturally. When is the sampled system finite dimensional ? This problem has not previously been

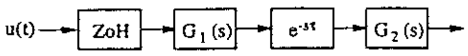

Fig. 2 The case with a single inner time delay

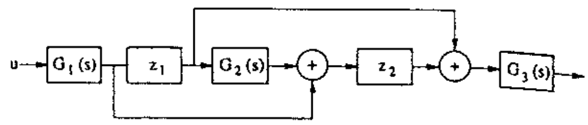

Fig. 3 A simple problem with several time delays. The system consists of three mixing tanks described by first order systems and has two transportation delays.

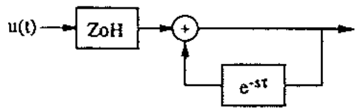

Fig. 4 This system becomes finite dimensional when sampled if and only if τ is a rational multiple of the sampling period h.

solved. One reason might be that the answer, as we will see, is that the sampled system is not always finite dimensional, see [Koepcke, 1965]. Sampling of general time delay systems can therefore be very hard and the success will depend on where the time delays are situated. In this paper we will describe what systems becomes finite dimensional when sampled and we will present a short and constructive state space algorithm for sampling such systems.

Since the problem with several inner time delays has not been solved before, it has been circumvented in different ways, e. g. by approximating delays with Taylor-series expansions or by neglecting the time delays. The success of all approximate methods will depend on the situation. The problem is generally harder the longer the time delays are. A comparison of some approximate methods used on an industrial example is made in [Hammarstrom and Gros, 1980].

From both a theoretical and practical viewpoint it is preferable with an exact representation of the sampled system. This is the aim of the current paper. In Section 2 we define notation and introduce an example to illustrate ideas. In Section 3 we show how a finite dimensional sampled system can be obtained for systems having no feedback loop around any time delay. We also give an example showing that this is not possible for all systems. In Section 4 we present necessary conditions for a system to be FDS for all values of time delays. Conclusions and open questions are presented in Section 5.

For a more detailed version of the paper see [Bernhardsson, 1992].

2. PROBLEM FORMULATION

We will assume that zero-order hold circuits are used. This means that control signals are held constant between sampling points:

$$u(t) = u(kh) \qquad t \in [kh, kh + h),$$

were h is the sampling period. We assume that the continuous time system is given in the following form:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=0}^{p} A_i x(t-\tau_i) + \sum_{i=p+1}^{r} B_i u(t-\tau_i) \\ &= \sum_{i=0}^{p} A_i z_i x(t) + \sum_{i=p+1}^{r} B_i z_i u(t) \qquad (1) \\ &= A(z)x(t) + B(z)u(t) \end{aligned}$$

where $z_i f(t) = f(t-\tau_i)$, $\tau_0 = 0$, $z = (z_1, \ldots, z_p)$ and $A(z) = A_0 + \sum_{i=0}^{p} A_i z_i$, $B(z) = \sum_{i=p+1}^{r} B_i z_i$

EXAMPLE 1
The system in Fig 3 describes a chemical engineering processes with two transport delays and three mixing tanks, described by first order systems. The system can be written as:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{bmatrix} z_1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix} z_2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} u$$

3. SAMPLING SYSTEMS WITH SEVERAL TIME DELAYS

Not all systems become finite dimensional when sampled. The phenomenon can also depend on the sample rate.

EXAMPLE 2
Consider the system in Fig 4. This system gives an infinite dimensional sampled system unless τ is a rational multiple of the sampling rate h. In fact the system has poles, at the zeros of $1 - e^{-s\tau}$, i.e. at $s_k = 2k\pi i/\tau$, $k = 0, \pm 1, \ldots$ After sampling, the poles are transformed to $e^{s_k h} = e^{2k\pi i h/\tau}$, $k = 0, \pm 1, \ldots$ and if h/τ is irrational there are infinitely many discrete time poles. The sampled system is therefore not finite dimensional. On the other hand if h/τ is rational it is easy to see that the sampled system is finite dimensional.

DEFINITION 1
We say that a differential-difference system of the form (1) is a finite dimensionably samplable (FDS) system, if it's zero order hold sampled version can be represented with a finite dimensional discrete time system for all values of the delays τ_i.

The following lemma is a useful description of the solution to (1).

LEMMA 1
The solution to (1) satisfies, $\forall s, t$,

$$x(s+t) = \phi(t,z)x(s) + \int_0^t \phi(t-r,z)B(z)u(s+r)dr$$

$$\phi(t,z) = \sum_{k=0}^{\infty} \frac{t^k}{k!}(A(z))^k$$

PROOF See [Hale, 1971],[Bernhardsson, 1992].

The following condition is the key to categorizing FDS-systems.

CONDITION FBF.
A system of the form (1) is said to be feedback free (FBF), if and only if, for all sets of indices $i_1, \dots i_p$ for which some of the strictly positive indices are equal ($i_r = i_s > 0$), we have

$$\prod_j A_{i_j} = 0 \qquad (2)$$

For instance we must have $A_1 A_0 A_1 = 0$ and $A_2^2 = 0$ but there is no restriction on e.g. $A_1 A_0^2 A_2 A_0$.

Condition FBF can easily be checked in a block diagram of the system. Write the system equations in a form where the matrices A_i all are of rank 1. This means that time delays of equal length, situated at different positions in the system, are treated as different. Condition FBF is then satisfied if there are no feedback loops in the system around any of the delays.

LEMMA 2
If (2) is satisfied there exist continuous functions $F_0(t), F_1(t), \dots F_{1\dots p}(t)$ such that

$$\begin{aligned}\phi(t,z) = {} & F_0(t) + F_1(t)z_1 + \dots + F_p(t)z_p + \\ & + F_{12}(t)z_1 z_2 + \dots + F_{p-1,p}(t)z_{p-1}z_p \\ & + \dots + F_{1\dots p}(t)z_1 \dots z_p \qquad \forall t, z \quad (3)\end{aligned}$$

SKETCH OF PROOF Treat $z_1, \dots, z_p$ as algebraic variables. The infinite sum

$$\phi(t,z) = \sum_{k=0}^{\infty} \frac{t^k}{k!}(A_0 + A_1 z_1 + \dots + A_p z_p)^k$$

then converges absolutely for all t and z. When we expand the terms in the sum, all terms where some z_i is multiplied by itself becomes zero. This is exactly (2). This leaves us with the terms stated in the theorem. The manipulations involved in collecting terms in this way are allowed.

One can give explicit expressions for the F:s.

$$\begin{aligned}F_0(t) &= e^{A_0 t} \\ F_1(t) &= e^{(A_0+A_1)t} - F_0(t) \\ &\vdots \\ F_p(t) &= e^{(A_0+A_p)t} - F_p(t) \\ F_{12}(t) &= e^{(A_0+A_1+A_2)t} - F_0(t) - F_1(t) - F_2(t) \\ &\vdots \\ F_{1\dots p}(t) &= e^{(A_0+A_1+\dots A_p)t} - F_0(t) - F_1(t) - \dots - F_{2\dots p}(t)\end{aligned}$$

We are now ready to show

THEOREM 1
If condition FBF, see (2), is satisfied, the system (1) is FDS. One finite dimensional sampled representation is given by

$$X_e(kh+h) = \Phi_e X_e(kh) + \Gamma_e U(kh)$$

Here X_e is an extended state space vector

$$X_e(t) = \begin{bmatrix} x(t) \\ x(t-\tau_1) \\ x(t-\tau_2) \\ \vdots \\ x(t-\tau_1-\dots-\tau_p) \end{bmatrix}$$

The matrix Φ_e equals

$$\begin{bmatrix} F_0(h) & . & F_{1\dots p}(h) \\ F_0(h-\tau_1) & . & F_{1\dots p}(h-\tau_1) \\ \vdots & & \vdots \\ F_0(h-\tau_1-\dots-\tau_p) & . & F_{1\dots p}(h-\tau_1-\dots-\tau_p) \end{bmatrix}$$

The vector U_e is given by

$$U_e = \begin{bmatrix} u(kh) \\ u(kh-h) \\ \vdots \\ u(kh-dh) \end{bmatrix} \qquad (d-1)h < \sum \tau_i \le dh$$

The matrix Γ_e is determined by using that $u(t)$ is constant between samples and that $\Gamma_e U_e$ equals

$$\begin{bmatrix} \int_0^h \phi(h-r,z)B(z)u(kh+r)dr \\ \int_0^{h-\tau_1} \phi(h-\tau_1-r,z)B(z)u(kh+r),dr \\ \vdots \\ \int_0^{h-\dots-\tau_p} \phi(h-\dots-\tau_p-r,z)B(z)u(kh+r)dr \end{bmatrix}$$

PROOF Using lemmas 1 and 2 with $s = kh$ and $t = h, h-\tau_1, \dots, t = h-\tau_1-\dots-\tau_p$, we can update the full state vector $X_e(kh+h)$ from the value of $X_e(kh)$ and the $d+1$ last values of $u(kh)$.

EXAMPLE 3
Using (3) on the example in Fig 3 gives

$$\phi(t,z) = F_0(t) + F_1(t)z_1 + F_2(t)z_2 + F_{12}(t)z_1 z_2 \quad (4)$$

where

$$F_0(t) = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} \; F_1(t) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{a_{21}(\alpha_1-\alpha_2)}{a_1-a_2} & 0 & 0 \\ \frac{-a_{31}(\alpha_1-\alpha_3)}{a_1-a_3} & 0 & 0 \end{bmatrix}$$

$$F_2(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{a_{31}(\alpha_1-\alpha_3)}{a_1-a_3} & \frac{a_{32}(\alpha_2-\alpha_3)}{a_2-a_3} & 0 \end{bmatrix}$$

where $\alpha_i = \exp(a_i t)$ and where F_{12} is given by a too long expression to write out here. We get

$$\begin{aligned} x(kh+t) = {} & F_0(t)x(kh) + F_1(t)x(kh-\tau_1) \\ & + F_2(t)x(kh-\tau_2) + F_{12}(t)x(kh-\tau_1-\tau_2) \\ & + \Gamma_0(t)u(kh) + \Gamma_1(t)u(kh-h) \qquad (5) \end{aligned}$$

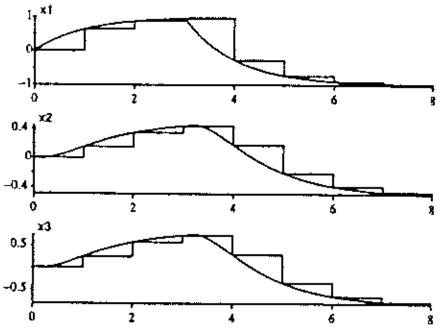

Fig. 5 Simulation verifying the equations for the sampled version of Ex 2.

If we assume for simplicity in notation that $\tau_1 + \tau_2 \leq h$ we get for $t \leq h$:

$$\Gamma_0(t) = \int_0^t F_0(t-s)B\,ds + \int_{\min(\tau_1,t)}^t F_1(t-s)B\,ds + \\ + \int_{\min(\tau_2,t)}^t F_2(t-s)B\,ds + \int_{\min(\tau_1+\tau_2,t)}^t F_{12}(t-s)B\,ds$$

and similarly for $\Gamma_1(t)$. Equation (5) can now be used for $t = h,\ h-\tau_1,\ h-\tau_2$ and $h-\tau_1-\tau_2$ to update the full X_e-vector. We will not present the full discrete time system since Φ_e is a 12 by 12-matrix. The representation is not of minimal order but can be reduced to such by any standard technique.

Fig 5 shows a simulation of the continuous time system and of the sampled data representation that results from Theorem 1. The system is started from zero initial condition at $t = 0$. The input is 1 until $t = 3$ and then -1. The plots perfectly confirms the calculations.

4. NECESSARY CONDITIONS

Theorem 1 shows that condition FBF is sufficient for a system of the form (1) to be FDS. We also have the following result of necessity which is proved in [Bernhardsson, 1992]:

THEOREM 2
Assume the system is given by (1). Then if the open loop system contains a feedback loop around any of the delays the system is not FDS for all sampling rates h and time delays τ_i.

Ex. 1 shows that a system with feedback loop around a time delay can be FDS for some special values of h and τ_i. Theorems 1,2 are therefore the best possible.

5. CONCLUSIONS/OPEN QUESTIONS

We have shown how to obtain a finite dimensional system when sampling a system containing several time delays. We have also shown that the condition for obtaining a finite dimensional system, for all h and τ_i, is that the system has no feedback loop around any time delay. The algorithm has been used on an example and the results have been verified by simulation.

It would be nice with an improved algorithm that guarantees a minimal order sampled representation.

It is possible to generalize the results of this paper to sampling of continuous time systems with white noise. Sampling of continuous time quadratic loss functions also gives rise to similar integrals. Both problems can be solved using Lemmas 1 and 2 above if the FBF-condition is satisfied.

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