Lecture 8: Linear Quadratic Gaussian Control

- Deterministic Linear Quadratic Control
- Riccati Equation
- Stochastic Linear Quadratic Control
- Algebraic Riccati equation
- Kalman filter

Deterministic Linear Quadratic Control

Minimize
$$\sum_{k=0}^{N-1} \left(x(k)^T Q_1 x(k) + 2x(k)^T Q_{12} u(k) + u(k)^T Q_2 u(k)\right) + x(N)^T Q_0 x(N)$$

subject to
$$x(k+1) = \Phi x(k) + \Gamma u(k), \qquad x(0) = x_0$$

Completion of squares

The scalar case Suppose $c > 0$.

$$ax^2 + 2bxu + cu^2 = x\left(a - \frac{b^2}{c}\right)x + \left(u + \frac{b}{c}x\right)c\left(u + \frac{b}{c}x\right)$$

is minimized by $u = -\frac{b}{c}x$. The minimum is $\left(a - b^2/c\right)x^2$.

The matrix case Suppose $Q_u \geq 0$.
Let L be such that $Q_u L = Q_{xu}^T$. Then

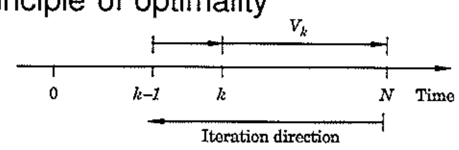
$$\begin{aligned} &x^T Q_x x + 2x^T Q_{xu} u + u^T Q_u u \\ &= x^T (Q_x - L^T Q_u L) x + (u + Lx)^T Q_u (u + Lx) \end{aligned}$$

is minimized by $u = -Lx$. The minimum is $x^T(Q_x - L^T Q_u L)x$.

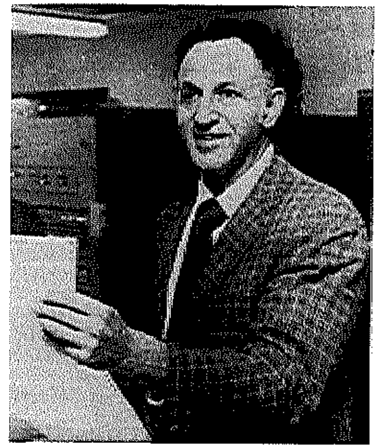
Note that $L = (Q_u)^{-1} Q_{xu}^T$ if Q_u is positive definite

Dynamic programming, Richard E. Bellman 1957

Principle of optimality



An optimal trajectory on the time interval $[0, N]$ must be optimal also on the subintervals $[0, k]$ and $[k, N]$.



Dynamic programming in linear quadratic control

Let $S(k)$ be defined by

$$x^T(k)S(k)x(k) := \min_{u(k),\dots,u(N-1)} \sum_{k}^{N-1} \{x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\} + x^T(N) Q_0 x(N)$$

Dynamic programming with $T_1 = k-1$, $T_2 = k$, $T_3 = N$ gives

$$x^T S(k-1) x = \min_u \{x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u + (\Phi x + \Gamma u)^T S(k)(\Phi x + \Gamma u)\}$$

$$= \min_u \Big\{ x^T \underbrace{(Q_1 + \Phi^T S(k)\Phi)}_{Q_x} x + 2x^T \underbrace{(Q_{12} + \Phi^T S(k)\Gamma)}_{Q_{xu}} u + u^T \underbrace{(Q_2 + \Gamma^T S(k)\Gamma)}_{Q_u} u \Big\}$$

Th completion of squares calculation then gives

$$S(k-1) = Q_x - L^T Q_u L \;=\; Q_1 + \Phi^T S(k)\Phi - L^T(Q_2 + \Gamma^T S(k)\Gamma)L$$
$$L = Q_u^{-1} Q_{xu}^T \;=\; (Q_2 + \Gamma^T S(k)\Gamma)^{-1}(Q_{12} + \Phi^T S(k)\Gamma)^T$$

Solution via the Riccati equation

Define $S(k)$ by the *Riccati equation*

$$S(k-1) = \Phi^T S(k)\Phi + Q_1 - L(k-1)^T(Q_2 + \Gamma^T S(k)\Gamma)L(k-1)$$

where

$$L(k-1) = (Q_2 + \Gamma^T S(k)\Gamma)^{-1}(\Gamma^T S(k)\Phi + Q_{12}^T)$$

Then the control law $u(k) = -L(k)x(k)$ minimizes the cost. The minimum is $x(0)^T S(0) x(0)$.

- Time-varying controller
- Often $N \to \infty$, or special Q_N

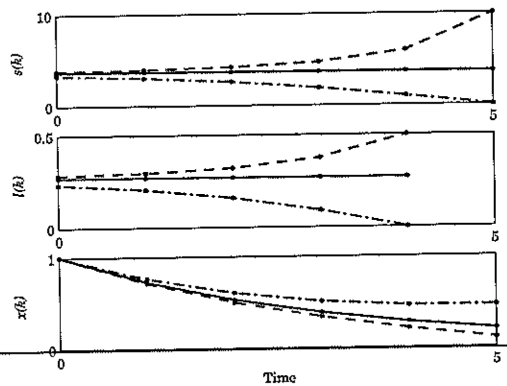
Jocopo Francesco Riccati, 1676–1754

LQ – First order system: $x(k+1) = x(k) + u(k)$

Loss function defined by q_1, q_2, q_0, and N

Riccati equation $\quad s(k) = s(k+1) + q_1 - \dfrac{s^2(k+1)}{q_2 + s(k+1)}, \quad s(N) = q_0$

Controller $\quad l(k) = \dfrac{s(k+1)}{s(k+1) + q_2}, \quad u(k) = -l(k)x(k)$



$x(0) = 1$
$q_0 = 10$ dashed
$q_0 = 3.70$ solid
$q_0 = 0$ dashed-dotted

Time-varying stochastic LQ control

Minimize $\mathbf{E}\sum_{k=0}^{N-1}\left(x^TQ_1x + 2x^TQ_{12}u + u^TQ_2u\right) + \mathbf{E}x(N)^TQ_0x(N)$

subject to $x(k+1) = \Phi x(k) + \Gamma u(k) + v(k)$

where $\mathbf{E}x(0) = m_0, \quad \mathbf{E}(x(0)-m_0)(x(0)-m_0)^T = R_0$
$\mathbf{E}(vv^T) = R_1$

Dynamic programming again

Define $S(k)$ by the Riccati equation and

$$V_k(x) = \min_{u(k),\dots,u(N-1)} \mathbf{E}\sum_{k}^{N-1}\left\{x^TQ_1x + 2x^TQ_{12}u + u^TQ_2u + x^T(N)Q_0x(N)\right\}$$

Then

$$\begin{aligned}
V_N(x) &= \mathbf{E}x^TQ_0x = \mathbf{E}x^TS(N)x \\
V_{N-1}(x) &= \min_u \mathbf{E}\left\{x^TQ_1x + 2x^TQ_{12}u + u^TQ_2u + V_N(\Phi x + \Gamma u + v)\right\} \\
&= \min_u \mathbf{E}\left\{x^TQ_1x + 2x^TQ_{12}u + u^TQ_2u \right.\\
&\qquad \left. + (\Phi x + \Gamma u)^TS(N)(\Phi x + \Gamma u)\right\} + \mathbf{E}v^TS(N)v \\
&= \mathbf{E}x^TS(N-1)x + \mathbf{E}v^TS(N)v \\
&\;\vdots \\
V_0(x) &= \mathbf{E}x^TS(0)x + \sum_{k=0}^{N-1}\mathbf{E}v(k)^TS(k+1)v(k)
\end{aligned}$$

Mean value of quadratic form

Assume $\mathbf{E}\, x = m$ and cov $x = R$

$$\begin{aligned}
\mathbf{E}(x-m)^TS(x-m) &= \mathbf{E}\,\mathrm{tr}\left[(x-m)^TS(x-m)\right] \\
&= \mathbf{E}\,\mathrm{tr}\left[S(x-m)(x-m)^T\right] \\
&= \mathrm{tr}\, S\mathbf{E}(x-m)(x-m)^T \\
&= \mathrm{tr}\, SR
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbf{E}\, x^TSx &= \mathbf{E}(x-m)^TS(x-m) + 2\mathbf{E}\, m^TSx - \mathbf{E}\, m^TSm \\
&= \mathbf{E}(x-m)^TS(x-m) + m^TSm \\
&= \mathrm{tr}\, SR + m^TSm
\end{aligned}$$

Solution to time-varying problem

Define $S(k)$ by the Riccati equation. Then the control law $u(k) = -L(k)x(k)$ minimizes the cost. The minimum for the time-varying problem is

$$\begin{aligned}
&\mathbf{E}x^TS(0)x + \sum_{k=0}^{N-1}\mathbf{E}v(k)^TS(k+1)v(k) \\
&= m_0^TS(0)m_0 + \mathrm{tr}\, S(0)R_0 + \sum_{k=0}^{N-1}\mathrm{tr}\, S(k+1)R_1
\end{aligned}$$

Note: The first term is identical to the deterministic case. The second term penalizes initial state uncertainty. The third term penalizes noise.

Solution to stationary problem

Minimize $\mathbf{E}\left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right)$

subject to $x(k+1) = \Phi x(k) + \Gamma u(k) + v(k)$

where v is white noise with $\mathbf{E}(vv^T) = R_1$

Define S by the *algebraic Riccati equation*

$$S = \Phi^T S \Phi + Q_1 - L^T (Q_2 + \Gamma^T S \Gamma) L$$

where $L = (Q_2 + \Gamma^T S \Gamma)^{-1} (\Gamma^T S \Phi + Q_{12}^T)$

Then the control law $u(k) = -Lx(k)$ minimizes

$$\mathbf{E}\left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right)$$

The minimum is $\mathrm{tr} S R_1$.

Theorem: Stability of closed-loop system

Assume that

$$Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix}$$

is positive definite and that there exists a positive-definite steady-state solution S to the algebraic Riccati equation. Then $u(k) = -Lx(k)$ gives an asymptotically stable closed-loop system

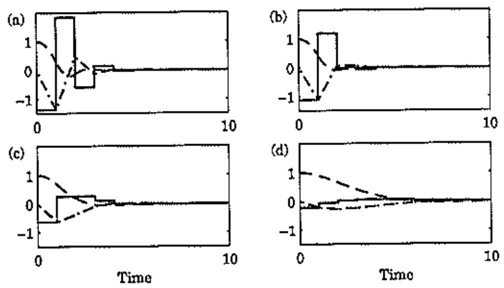
$$x(k+1) = (\Phi - \Gamma L) x(k)$$

BUT no guaranteed amplitude margin or phase margin

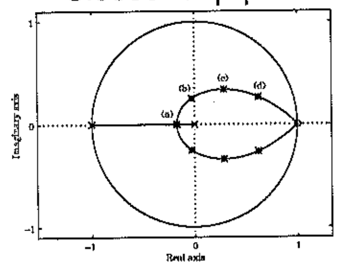
LQ – Double integrator

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad Q_2 = \rho \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad h = 1$$

States and inputs for
a) $\rho = 0.016$, b) $\rho = 0.05$, c) $\rho = 0.5$, d) $\rho = 10$



Closed loop poles



Summary

- Deterministic Linear Quadratic Control
- Riccati Equation
- Stochastic Linear Quadratic Control
- Algebraic Riccati equation

Linear Quadratic Gausian Control

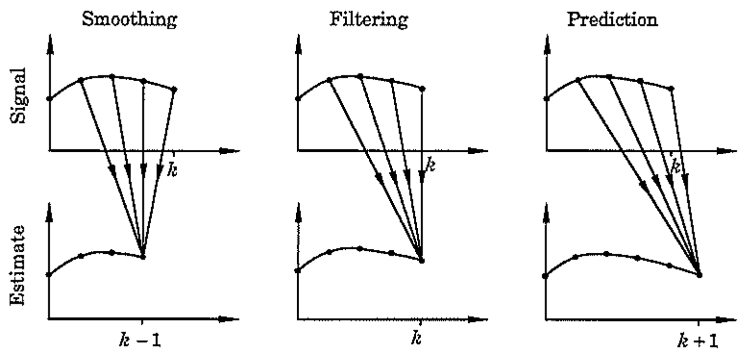
- A stochastic least squares problem
- The Kalman filter
- Separation
- LQG

Norbert Wiener, 1894–1964

Prediction and filtering

* Wiener (1949) Stationary I/O case
* Kalman and Bucy (1960) Time-varying state-space

Estimate $x(k+m)$ given $Y_k = \{y(i), u(i) \,|\, i \le k\}$



Examples

Smoothing To estimate the Wednesday temperature based on temperature measurements from Monday, Tuesday and Thursday

Filtering To estimate the Wednesday temperature based on temperature measurements from Monday, Tuesday and **Wednesday** (helps to reduce measurement error)

Prediction To predict the Wednesday temperature based on temperature measurements from Sunday, Monday and Tuesday

Su Mo Tu We Th

The Kalman filter problem

Consider the linear stochastic difference equation

$$
\begin{aligned}
x(k+1) &= \Phi x(k) + \Gamma u(k) + v(k) \\
y(k) &= Cx(k) + e(k)
\end{aligned}
$$

$$
\mathbf{E}x(0) = 0 \qquad \mathbf{E}x(0)x(0)^T = R_0 \qquad \mathbf{E}\begin{pmatrix} v(k) \\ e(k) \end{pmatrix}\begin{pmatrix} v(k) \\ e(k) \end{pmatrix}^T = \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix}
$$

Our objective is to estimate $x(k+1)$, $\hat{x}(k+1)$, by a linear combination of $y(0), y(1), \ldots, y(k)$.

A Filter Structure

$$
\begin{aligned}
x(k+1) &= \Phi x(k) + \Gamma u(k) + v(k) \\
y(k) &= Cx(k) + e(k)
\end{aligned}
$$

The following structure will be optimized with respect to $K(k)$:

$$
\hat{x}(k+1|k) = \Phi\hat{x}(k|k-1) + \Gamma u(k) + K(k)[y(k) - C\hat{x}(k|k-1)]
$$

The error dynamics become

$$
\begin{aligned}
\tilde{x}(k+1) &= x(k+1) - \hat{x}(k+1|k) \\
&= \Phi\tilde{x}(k) + v(k) - K(k)[y(k) - C\hat{x}(k|k-1)] \\
&= (\Phi - K(k)C)\tilde{x}(k) + v(k) - K(k)e(k) \\
&= \begin{pmatrix} I & -K(k) \end{pmatrix}\left(\begin{pmatrix} \Phi \\ C \end{pmatrix}\tilde{x}(k) + \begin{pmatrix} v(k) \\ e(k) \end{pmatrix}\right)
\end{aligned}
$$

Note that if $\tilde{x}(0) = 0$ then $\mathrm{E}\,\tilde{x}(k) = 0$ for all k.

How to choose $K(k)$

Minimize the variance of $\tilde{x}(k)$.

$$
\begin{aligned}
P(k+1) &= \min_{K(0),\ldots,K(k)} \mathbf{E}\tilde{x}(k+1)\tilde{x}(k+1)^T \\
&= \min_{K(k)} \begin{pmatrix} I & -K(k) \end{pmatrix}\left(\begin{pmatrix} \Phi \\ C \end{pmatrix} P(k) \begin{pmatrix} \Phi \\ C \end{pmatrix}^T + \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix}\right)\begin{pmatrix} I \\ -K^T(k) \end{pmatrix} \\
&= \min_{K(k)} \begin{pmatrix} I & -K(k) \end{pmatrix}\begin{pmatrix} \Phi P(k)\Phi^T + R_1 & \Phi P(k)C^T + R_{12} \\ CP(k)\Phi^T + R_{12}^T & CP(k)C^T + R_2 \end{pmatrix}\begin{pmatrix} I \\ -K^T(k) \end{pmatrix}
\end{aligned}
$$

A least squares problem!
Solution by differentiation or completion of squares:

$$
K(k) = (\Phi P(k)C^T + R_{12})(CP(k)C^T + R_2)^{-1}
$$

Kalman filter – The solution

$$
\begin{aligned}
\hat{x}(k+1|k) &= \Phi\hat{x}(k|k-1) + \Gamma u(k) + K(k)(y(k) - C\hat{x}(k|k-1)) \\
K(k) &= (\Phi P(k)C^T + R_{12})(CP(k)C^T + R_2)^{-1} \\
P(k+1) &= \Phi P(k)\Phi^T + R_1 - K(k)(CP(k)C^T + R_2)K^T(k) \qquad P(0) = R_0
\end{aligned}
$$

- $P(k) = P(k|k-1)$ covariance of prediction error
- Minimizing the prediction error for each coefficient of the state independently!
- Allows time-varying systems

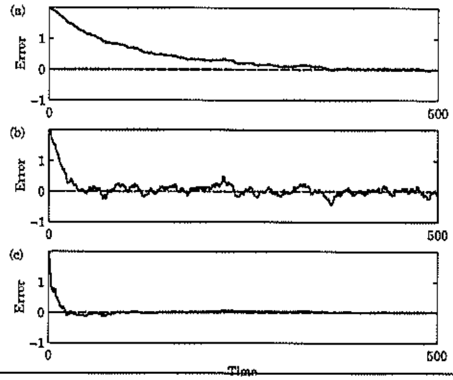
Example 1 – Kalman filter

$$x(k+1) = x(k) \qquad x(0) \in \mathrm{N}(-2, 0.5)$$
$$y(k) = x(k) + e(k) \qquad \mathrm{cov}\ e = \sigma^2 = 1$$
$$K(k) = \frac{P(k)}{\sigma^2 + P(k)} \qquad \left(K(k) = \frac{1}{k+3}\right) \qquad P(k+1) = \frac{\sigma^2 P(k)}{\sigma^2 + P(k)}$$

a) $K = 0.01$

b) $K = 0.05$

c) Optimal $K(k)$



Example 2 – Kalman filter

$$y(k) = \frac{q+c}{q+a} e(k) = \frac{c-a}{q+a} e(k) + e(k) \qquad |c| < 1$$

State-space representation

$$x(k+1) = -ax(k) + e(k) \qquad R_1 = R_2 = R_{12} = \sigma^2$$
$$y(k) = (c-a)x(k) + e(k)$$

Kalman filter in steady-state

$$K = \frac{\sigma^2 - aP(c-a)}{(c-a)^2 P + \sigma^2} \qquad P = a^2 P + \sigma^2 - \frac{(\sigma^2 - aP(c-a))^2}{(c-a)^2 P + \sigma^2}$$

One solution is $P = 0$ and $K = 1$

$$\hat{x}(k+1|k) = -c\hat{x}(k|k-1) + y(k)$$

How about if $|c| > 1$?

Linear Quadratic Gaussian (LQG) control

Recall: Deterministic LQ Control

Minimize

$$x^T S(k) x = \min_{u(k),\dots,u(N-1)} \sum_{k}^{N-1} \{x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\} + x^T(N) Q_0 x(N)$$

subject to $x(k+1) = \Phi x(k) + \Gamma u(k), \qquad x(0) = x_0$

The Riccati equation

$$x^T S(k-1) x = \min_u \{x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u + (\Phi x + \Gamma u)^T S(k)(\Phi x + \Gamma u)\}$$

$$\Rightarrow \begin{cases} S(k-1) = Q_1 + \Phi^T S(k)\Phi - L^T(Q_2 + \Gamma^T S(k)\Gamma)L, & S(N) = Q_0 \\ L = (Q_2 + \Gamma^T S(k)\Gamma)^{-1}(Q_{12} + \Phi^T S(k)\Gamma)^T \end{cases}$$

Recall: Stochastic LQ Control

Minimize $\quad \mathbf{E}\sum_{k=0}^{N-1} (x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u) + \mathbf{E}x(N)^T Q_0 x(N)$

subject to $\quad x(k+1) = \Phi x(k) + \Gamma u(k) + v(k)$

where $\quad \mathbf{E}x(0) = 0, \quad \mathbf{E}x(0)x(0)^T = R_0, \quad \mathbf{E}(vv^T) = R_1$

The control law $u(k) = -L(k)x(k)$ gives the minimum

$$\min_{u(k),\dots,u(N-1)} \mathbf{E}\sum_{k}^{N-1} \{x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u + x^T(N) Q_0 x(N)\}$$
$$= \mathrm{tr}[S(0)R_0] + \sum_{k=0}^{N-1} \mathrm{tr}[S(k+1)R_1]$$

The LQG control problem

Given the linear stochastic difference equation

$$x(k+1) = \Phi x(k) + \Gamma u(k) + v(k)$$
$$y(k) = Cx(k) + e(k)$$

$$\mathbf{E}x(0) = 0 \qquad \mathbf{E}x(0)x(0)^T = R_0 \qquad \mathbf{E}\begin{pmatrix} v(k) \\ e(k) \end{pmatrix}\begin{pmatrix} v(k) \\ e(k) \end{pmatrix}^T = \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix}$$

find a linear control law

$$y(0), y(1), \ldots, y(k-1) \quad \mapsto \quad u(k)$$

that minimizes

$$\mathbf{E}\sum_{k=0}^{N-1}(x^TQ_1x + 2x^TQ_{12}u + u^TQ_2u) + \mathbf{E}x(N)^TQ_0x(N)$$

The idea of separation

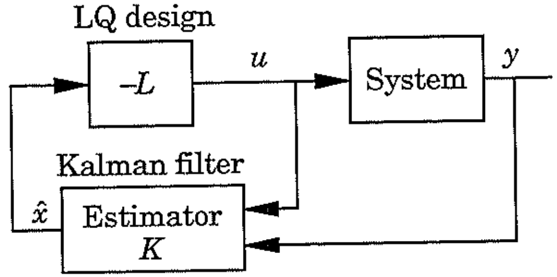
- The state feedback control law is independent of R_0, R_1
- The Kalman filter minimizes $\tilde{x}^TQ\tilde{x}$ independently of $Q \geq 0$

This makes it possible to use the control law $u(k) = -L(k)\hat{x}(k)$ with the dynamics

$$x(k+1) = \Phi x(k) - \Gamma L(k)x(k) + \Gamma L(k)\tilde{x}(k) + v(k)$$

and to view the term $\Gamma L(k)\tilde{x}(k)$ as part of the noise.

The idea of separation



Solve LQ and filtering separately
Choosing $u(k) = -L(k)\hat{x}(k|k-1)$ achieves the minimal loss

$$\mathrm{tr}S(0)R_0 + \mathbf{E}\Big(\sum_{k=0}^{N-1}\mathrm{tr}S(k+1)[v(k)v(k)^T + \tilde{x}(k)\tilde{x}(k)^T] + [L\tilde{x}(k)]^TP(k)[L\tilde{x}(k)]\Big)$$
$$= \mathrm{tr}S(0)R_0 + \sum_{k=0}^{N-1}\mathrm{tr}S(k+1)R_1 + \sum_{k=0}^{N-1}\mathrm{tr}P(k)L^T(k)(\Gamma^TS(k+1)\Gamma + Q_2)L(k)$$

Duality between control and estimation

Optimal control	State estimation
k	$N-k$
Φ	Φ^T
Γ	C^T
Q_0	R_0
Q_1	R_1
Q_{12}	R_{12}
S	P
L	K^T

Summary

- A stochastic least squares problem
- The Kalman filter
- Separation
- LQG