Chapter 12

Models for Sampled Data Systems

The Shift Operator

Forward shift operator

$$\boxed{q(f[k]) \triangleq f[k+1]}$$

In terms of this operator, the model given earlier becomes:

$$q^n y[k] + \overline{a}_{n-1} q^{n-1} y[k] + \cdots + \overline{a}_0 y[k] = \overline{b}_m q^m u[k] + \cdots + \overline{b}_0 u[k]$$

For a discrete time system it is also possible to have discrete state space models. In the shift domain these models take the form:

$$qx[k] = \mathbf{A}_q x[k] + \mathbf{B}_q u[k]$$
$$y[k] = \mathbf{C}_q x[k] + \mathbf{D}_q u[k]$$

Z-Transform

Analogously to the use of Laplace Transforms for continuous time signals, we introduce the Z-transform for discrete time signals.

Consider a sequence {*y*[*k*]; *k* = 0, 1, 2, …]. Then the Z-transform pair associated with {*y*[*k*]} is given by

$$\mathcal{Z}\left[y[k]\right] = Y(z) = \sum_{k=0}^{\infty} z^{-k} y[k]$$

$$\mathcal{Z}^{-1}\left[Y(z)\right] = y[k] = \frac{1}{2\pi j}\oint z^{k-1} Y(z) dz$$

How do we use Z-transforms ?

We saw earlier that Laplace Transforms have a remarkable property that they convert differential equations into algebraic equations.

Z-transforms have a similar property for discrete time models, namely they convert difference equations (expressed in terms of the shift operator q) into algebraic equations.

We illustrate this below for a discrete high-order difference equation model:

Discrete Delta Domain Models

The shift operator (*as described above*) is used in the vast majority of digital control and digital signal processing work. However, in some applications the shift operator can lead to difficulties. The reason for these difficulties are explained below.

Consider the first order continuous time equation

$$\rho y(t) + y(t) = \frac{dy(t)}{dt} + y(t) = u(t)$$

and the corresponding discretized shift operator equation is of the form:

$$a_2 q y(t_k) + a_1 y(t_k) = b_1 u(t_k)$$

Expanding the differential explicitly as a limiting operation, we obtain the following form of the continuous time equation:

$$\lim_{\Delta \to 0} \left(\frac{y(t+\Delta) - y(t)}{\Delta} \right) + y(t) = u(t)$$

If we now compare the discrete model to the approximate expanded form, namely

$$a_2 y(t+\Delta) + a_1 y(t) = b_1 u(t); \quad \text{where } \Delta = t_{k+1} - t_k$$

we then see that the fundamental difference between continuous and discrete time is that the discrete model describes absolute displacements (i.e. $y(t+\Delta)$ in terms of $y(t)$, etc.) whereas the differential equation describes the increment

$$\left(\text{i.e.} \frac{y(t+\Delta) - y(t)}{\Delta} \right)$$

This fundamental difficulty is avoided by use of an alternative operator; namely the *Delta operator*:

$$\delta\left(f(k\Delta)\right) \triangleq \frac{f((k+1)\Delta) - f(k\Delta)}{\Delta}$$

For sampled signals, an important feature of this operation is the observation that

$$\lim_{\Delta\to 0}\left[\delta\{f(k\Delta)\}\right] = \rho(f(t))$$

i.e., the Delta operator acts as a derivative in the limit as the sampling period →0. Note, however, that *no approximations* will be involved in employing the Delta operator for finite sampling periods since we will derive *exact* model descriptions relevant to this operator at the given sampling rate.

We next develop an alternative discrete transform (*which we call the Delta transform*) which is the appropriate transform to use with the Delta operator, i.e.

Time Domain	*Transfer Domain*
q	Z-transform
δ	delta transform

Discrete Delta Transform

We define the Discrete Delta Transform pair as:

$$\mathcal{D}\left[y(k\Delta)\right] \stackrel{\triangle}{=} Y_\delta(\gamma) = \sum_{k=0}^{\infty}(1+\gamma\Delta)^{-k} y(k\Delta)\Delta$$

$$\mathcal{D}^{-1}\left[Y_\delta(\gamma)\right] = y(k\Delta) = \frac{1}{2\pi j}\oint (1+\gamma\Delta)^{k-1} Y_\delta(\gamma) d\gamma$$

The Discrete Delta Transform can be related to Z-transform by noting that

$$Y_\delta(\gamma) = \Delta Y_q(z)\Big|_{z=\Delta\gamma+1}$$

where $Y_q(z) = Z[(k\Delta)]$. Conversely

$$Y_q(z) = \frac{1}{\Delta} Y_\delta(\gamma)\Big|_{\gamma=\frac{z-1}{\Delta}}$$

- The next slide shows a table of Delta transform pairs;
- The slide after next lists some Delta transform properties.

$f[k] \quad (k \geq 0)$	$\mathcal{D}[f[k]]$	Region of Convergence
1	$\frac{1+\Delta\gamma}{\gamma}$	$\left\lvert\gamma + \frac{1}{\Delta}\right\rvert > \frac{1}{\Delta}$
$\frac{1}{\Delta}\delta_K[k]$	1	$\lvert\gamma\rvert < \infty$
$\mu[k] - \mu[k-1]$	$\frac{1}{\Delta}$	$\lvert\gamma\rvert < \infty$
k	$\frac{1+\Delta\gamma}{\Delta\gamma^2}$	$\left\lvert\gamma + \frac{1}{\Delta}\right\rvert > \frac{1}{\Delta}$
k^2	$\frac{(1+\Delta\gamma)(2+\Delta\gamma)}{\Delta^2\gamma^3}$	$\left\lvert\gamma + \frac{1}{\Delta}\right\rvert > \frac{1}{\Delta}$
$e^{\alpha\Delta k} \quad \alpha \in \mathbb{C}$	$\frac{1+\Delta\gamma}{\gamma - \frac{e^{\alpha\Delta}-1}{\Delta}}$	$\left\lvert\gamma + \frac{1}{\Delta}\right\rvert > \frac{e^{\alpha\Delta}}{\Delta}$
$ke^{\alpha\Delta k} \quad \alpha \in \mathbb{C}$	$\frac{(1+\Delta\gamma)e^{\alpha\Delta}}{\Delta\left(\gamma - \frac{e^{\alpha\Delta}-1}{\Delta}\right)^2}$	$\left\lvert\gamma + \frac{1}{\Delta}\right\rvert > \frac{e^{\alpha\Delta}}{\Delta}$
$\sin(\omega_o\Delta k)$	$\frac{(1+\Delta\gamma)\omega_o \mathrm{sinc}(\omega_o\Delta)}{\gamma^2 + \Delta\phi(\omega_o,\Delta)\gamma + \phi(\omega_o,\Delta)}$ where $\mathrm{sinc}(\omega_o\Delta) = \frac{\sin(\omega_o\Delta)}{\omega_o\Delta}$ and $\phi(\omega_o,\Delta) = \frac{2(1-\cos(\omega_o\Delta))}{\Delta^2}$	$\left\lvert\gamma + \frac{1}{\Delta}\right\rvert > \frac{1}{\Delta}$
$\cos(\omega_o\Delta k)$	$\frac{(1+\Delta\gamma)(\gamma + 0.5\Delta\phi(\omega_o,\Delta))}{\gamma^2 + \Delta\phi(\omega_o,\Delta)\gamma + \phi(\omega_o,\Delta)}$	$\left\lvert\gamma + \frac{1}{\Delta}\right\rvert > \frac{1}{\Delta}$

Table 12.3: *Delta Transform Table*

$f[k]$	$\mathcal{D}[f[k]]$	Names
$\sum_{i=1}^{l} a_i f_i[k]$	$\sum_{i=1}^{l} a_i F_i(\gamma)$	Partial fractions
$f_1[k+1]$	$(\Delta\gamma+1)(F_1(\gamma)-f_1[0])$	Forward shift
$\frac{f_1[k+1]-f_1[k]}{\Delta}$	$\gamma F_1(\gamma)-(1+\gamma\Delta)f_1[0]$	Scaled difference
$\sum_{l=0}^{k-1} f[l]\Delta$	$\frac{1}{\gamma}F(\gamma)$	Reimann sum
$f[k-1]$	$(1+\gamma\Delta)^{-1}F(\gamma)+f[-1]$	Backward shift
$f[k-l]\mu[k-l]$	$(1+\gamma\Delta)^{-l}F(\gamma)$	
$kf[k]$	$-\frac{1+\gamma\Delta}{\Delta}\frac{dF(\gamma)}{d\gamma}$	
$\frac{1}{k}f[k]$	$\int_{\gamma}^{\infty}\frac{F(\zeta)}{1+\zeta\Delta}d\zeta$	
$\lim_{k\to\infty} f[k]$	$\lim_{\gamma\to 0}\gamma F(\gamma)$	Final value theorem
$\lim_{k\to 0} f[k]$	$\lim_{\gamma\to\infty}\frac{\gamma F(\gamma)}{1+\gamma\Delta}$	Initial value theorem
$\sum_{l=0}^{k-1} f_1[l]f_2[k-l]\Delta$	$F_1(\gamma)F_2(\gamma)$	Convolution
$f_1[k]f_2[k]$	$\frac{1}{2\pi j}\oint F_1(\zeta)F_2\left(\frac{\gamma-\zeta}{1+\zeta\Delta}\right)\frac{d\zeta}{1+\zeta\Delta}$	Complex convolution
$(1+a\Delta)^k f_1[k]$	$F_1\left(\frac{\gamma-a}{1+a\Delta}\right)$	

Table 12.4: *Delta Transform properties. Note that $F_i(\gamma) = D[fi[k]]$, $\mu[k]$ denotes, as usual, a unit step, $f[\infty]$ must be well defined and the convolution property holds provided that $f_1[k] = f_2[k] = 0$ for all $k < 0$.*

Why is the Delta Transform sometimes better than the Z-Transform?

As can be seen from by comparing the Z-transform given in Table 12.1 with those for the Laplace Transform given in Table 4.1, expressions in Laplace and Z-transform do not exhibit an obvious structural equivalence. Intuitively, we would expect such an equivalence to exist when the discrete sequence is obtained by sampling a continuous time signal.

We will show that this indeed happens if we use the alternative delta operator.

In particular, by comparing the entries in Table 12.3 (*The Delta Transform*) with those in Table 4.1 (*The Laplace Transform*) we see that a key property of Delta Transforms is that they converge to the associated Laplace Transform as $\Delta \rightarrow 0$, i.e.

$$\lim_{\Delta \rightarrow 0} Y_\delta(\gamma) = Y(s)\Big|_{s=\gamma}$$

We illustrate this property by a simple example:

Example 12.9

Say that $\{y[k]\}$ arises from sampling, at period Δ, a continuous time exponential $e^{\beta t}$. Then

$$y[k] = e^{\beta k \Delta}$$

and, from Table 12.3

$$Y_\delta(\gamma) = \frac{1 + \gamma\Delta}{\gamma - \left[\frac{e^{\beta\Delta} - 1}{\Delta}\right]}$$

In particular, note that as $\Delta \rightarrow 0$, $Y_\delta(\gamma) \rightarrow \frac{1}{\gamma - \beta}$ which is the Laplace transform of $e^{\beta t}$.

Hence we confirm the close connections between the Delta and Laplace Transforms.

How do we use Delta Transforms?

We saw earlier in this chapter that Z-transforms could be used to convert discrete time models expressed in terms of the shift operator into algebraic equations. Similarly, the Delta Transform can be used to convert difference equations (*expressed in terms of the Delta operator*) into algebraic equations. The Delta Transform also provides a smooth transition from discrete to continuous time as the sampling rate increases.

We next examine several properties of discrete time models, beginning with the issue of stability.

Discrete System Stability

Relationship to Poles

We have seen that the response of a discrete system (in the shift operator) to an input $U(z)$ has the form

$$Y(z) = G_q(z)U(z) + \frac{f_q(z, x_o)}{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)}$$

where $\alpha_1 \ldots \alpha_n$ are the poles of the system.

We then know, via a partial fraction expansion, that $Y(z)$ can be written as

$$Y(z) = \sum_{j=1}^{n} \frac{\beta_j z}{z-\alpha_j} + \text{terms depending on } U(z)$$

where, for simplicity, we have assumed non repeated poles.

The corresponding time response is

$$y[k] = \beta_j \left[\alpha_j\right]^k + \text{ terms depending on the input}$$

Stability requires that $[\alpha_j]^k \rightarrow 0$, which is the case if $[\alpha_j] < 1$.

Hence stability requires the poles to have magnitude less than 1, i.e. to lie inside a unit circle centered at the origin.

Delta Domain Stability

We have seen that the delta domain is simply a shifted and scaled version of the Z-Domain, i.e. $\gamma = \frac{Z-1}{\Delta}$ and $Z = \gamma\Delta + 1$. It follows that the Delta Domain stability boundary is simply a shifted and scaled version of the Z-domain stability boundary. In particular, the delta domain stability boundary is a circle of radius $^1/_\Delta$ centered on - $^1/_\Delta$ in the γ domain. Note again the close connection between the continuous s-domain and discrete δ-domain, since the δ-stability region approaches the s-stability region (OLHP) as $\Delta \to 0$.

Using Continuous State Space Models

Next we show how a discrete model can be developed when the plant is described by a continuous time state space model

$$\frac{dx(t)}{dt} = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}x(t)$$

Then, using the solution formula (*see Chapter 3*) the sampled state response over an interval Δ is given by

$$x((k+1)\Delta) = e^{\mathbf{A}\Delta}x(k\Delta) + \int_0^{\Delta} e^{\mathbf{A}(\Delta-\tau)}\mathbf{B}u(\tau)d\tau$$

Now using the fact that $u(\tau+k\Delta)$ is equal to $u(k\Delta)$ for $0 \leq \tau < \Delta$ we have

$$x((k+1)\Delta) = \mathbf{A}_q x(k\Delta) + \mathbf{B}_q u(k\Delta)$$

where

$$\mathbf{A}_q = e^{\mathbf{A}\Delta}$$

$$\mathbf{B}_q = \int_0^{\Delta} e^{\mathbf{A}(\Delta-\tau)}\mathbf{B}d\tau$$

Also the output is

$$y(k\Delta) = \mathbf{C}_q x(k\Delta) \qquad \text{where} \qquad \mathbf{C}_q = \mathbf{C}$$

Shift form

The discrete time state space model derived above can be expressed compactly using the forward shift operator, *q*, as

$$qx[k] = \mathbf{A}_q x[k] + \mathbf{B}_q u[k]$$
$$y[k] = \mathbf{C}_q x[k]$$

where

$$\mathbf{A}_q \triangleq e^{\mathbf{A}\Delta} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}\Delta)^k}{k!}$$

$$\mathbf{B}_q \triangleq \int_0^{\Delta} e^{\mathbf{A}(\Delta-\tau)}\mathbf{B}d\tau = \mathbf{A}^{-1}\left[e^{\mathbf{A}\Delta} - I\right] \quad \text{if} \quad \mathbf{A} \quad \text{is nonsingular}$$

$$\mathbf{C}_q \triangleq \mathbf{C}$$

$$\mathbf{D}_q \triangleq \mathbf{D}$$

Delta Form

Alternatively, the discrete state space model can be expressed in Delta form as

$$\delta x(t_k) = \mathbf{A}_\delta x(t_k) + \mathbf{B}_\delta u(t_k)$$
$$y(t_k) = \mathbf{C}_\delta x(t_k) + \mathbf{D}_\delta u(t_k)$$

where $\mathbf{C}_\delta = \mathbf{C}_q = \mathbf{C}$, $\mathbf{D}_\delta = \mathbf{D}_q = \mathbf{D}$ and

$$\mathbf{A}_\delta \triangleq \frac{e^{\mathbf{A}\Delta} - I}{\Delta}$$
$$\mathbf{B}_\delta \triangleq \mathbf{\Omega}\mathbf{B}$$
$$\mathbf{\Omega} = \frac{1}{\Delta}\int_0^\Delta e^{\mathbf{A}\tau} d\tau = I + \frac{\mathbf{A}\Delta}{2!} + \frac{\mathbf{A}^2\Delta^2}{3!} + \ldots$$

Some Comparisons of Shift and Delta Forms

For the delta form we have

$$\lim_{\Delta \to 0} \mathbf{A}_\delta = \mathbf{A} \qquad \lim_{\Delta \to 0} \mathbf{B}_\delta = \mathbf{B}$$

For the shift form

$$\lim_{\Delta \to 0} \mathbf{A}_q = I \qquad \lim_{\Delta \to 0} \mathbf{B}_q = \mathbf{0}$$

Indeed, this reconfirms one of the principal advantages of the delta form, namely that it converges to the underlying continuous time model as the sampling period approaches zero. Note that this is not true of the alternative shift operator form.

- The chapter has introduced two discrete operators:
  - the shift operator, q, defined by $qx[k] \triangleq x[k+1]$
  - the δ-operator, δ, defined by $\delta x[k] \triangleq \frac{x[k+1]-x[k]}{\Delta}$
- Thus, $\delta = \frac{q-1}{\Delta}$, or $q = \delta\Delta + 1$.
- Due to this conversion possibility, the choice is largely based on preference and experience. Comparisons are outlined below.

- The shift operator, q,
  - is the traditional operator;
  - is the operator many engineers feel more familiar with;
  - is used in the majority of the literature.
- The δ-operator, δ, has the advantages of:
  - emphasizing the link between continuous and discrete systems (resembles a differential);
  - δ-expressions converge to familiar continuous expressions as $\Delta \to 0$, which is intuitive;
  - is numerically vastly superior at fast sampling rates when properly implemented.

- Analysis of digital systems relies on discrete time versions of the continuous operators:
  - the discrete version of the differential operator is difference operator;
  - the discrete version of the Laplace Transform is either the Z-transform (associated with the shift operator) or the γ-transform (associated with the δ-operator).
- With the help of these operators,
  - continuous time differential equation models can be converted to discrete time difference equation models;
  - continuous time transfer or state space models can be converted to discrete time transfer or state space models in either the shift or δ operators.