Lecture 6:
Sampling of signals

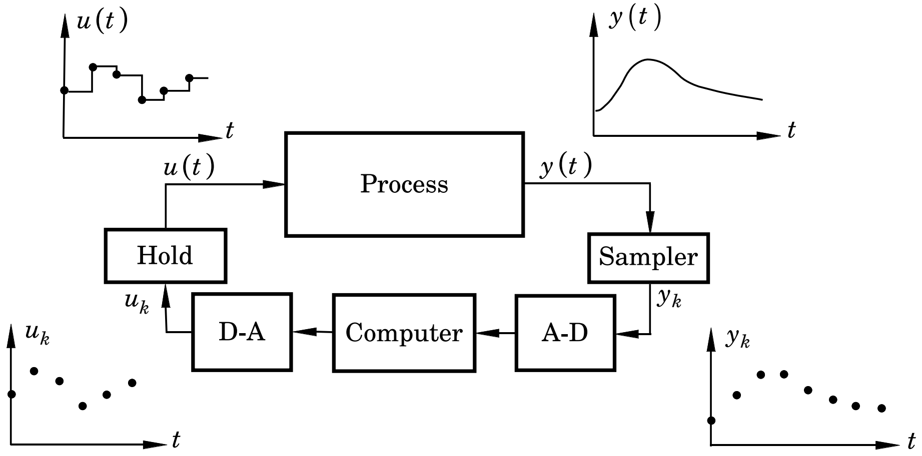
Operators and transform

- Sampling of signals
- Process-oriented models Operators and transforms
  - Match 1: z vs q
  - Match 2: q vs δ

Are there any important differences?
Is it necessary to learn about them all?

Computer-based process control

Relation between continous-time and discrete-time signals



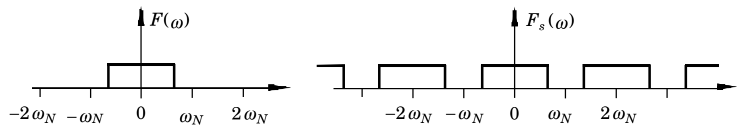
Sampling theorem

When is information lost by sampling ?

Theorem (Shannon 1949)
A continuous-time signal with Fourier transform $F(\omega) = 0$ for $|\omega| > \omega_0$, is uniquely defined by its sampled values, provided the sampling frequency satisfies $\omega_N = \omega_s/2 > \omega_0$
The signal can be reconstructed by the interpolation formula

$$f(t) = \sum_{k=-\infty}^{\infty} f(kh) \frac{\sin \omega_s (t - kh)/2}{\omega_s (t - kh)/2}$$

Idea behind the proof



Expand the periodic function $F_s(\omega)$ as a Fourier series

$$F_s(\omega) = \frac{1}{h} \sum_{k=-\infty}^{\infty} F(\omega + k\omega_s) = \sum_{k=-\infty}^{\infty} f(kh) e^{-ikh\omega}$$

But

$$F(\omega) = \begin{cases} hF_s(\omega) & |\omega| \le \frac{\omega_s}{2} \\ 0 & |\omega| > \frac{\omega_s}{2} \end{cases}$$

Thus

$$\{f(kh)\} \Rightarrow F_s(\omega) \Rightarrow F(s) \Rightarrow f(t)$$

Reconstruction

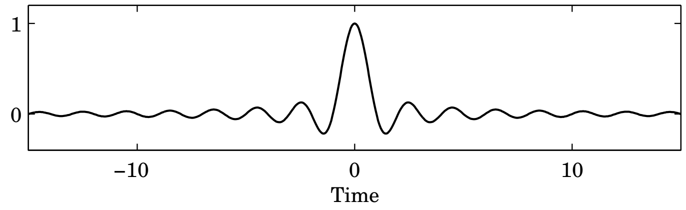
- Shannon

$$f(t) = \sum_{k=-\infty}^{\infty} f(kh) \frac{\sin(\omega_s(t-kh)/2)}{\omega_s(t-kh)/2}$$

- Zero order hold
- First order hold
- Predictive first order hold

Shannon reconstruction

$$f(t) = \sum_{k=-\infty}^{\infty} f(kh) \frac{\sin(\omega_s(t-kh)/2)}{\omega_s(t-kh)/2}$$



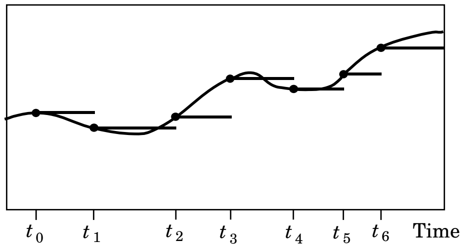
Properties?

$$\hat{f}(nh+\tau) = \sum_{k=-\infty}^{n+d} f(kh) h(nh+\tau-kh) \qquad h(\tau) = \frac{\sin \omega_s \tau/2}{\omega_s \tau/2}$$

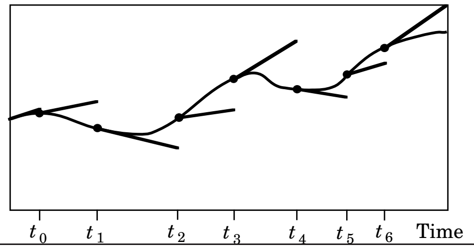
Delay acceptable?

Zero and first order hold

Zero order hold



First order hold



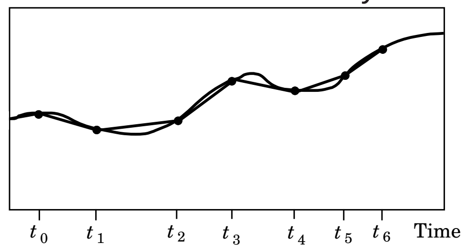
Predictive first order hold

Use

- Forward difference instead of backward difference

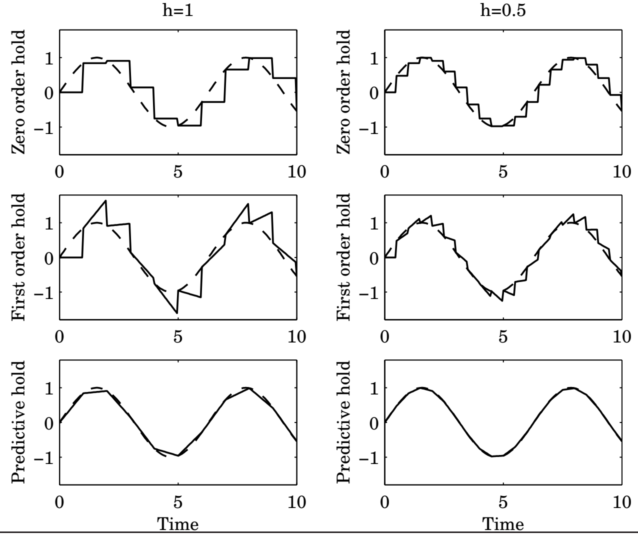
$$u(kh+\tau) = u(kh) + \frac{\tau}{h}(u(kh+h) - u(kh))$$

- Model of the controller. New causality conditions.



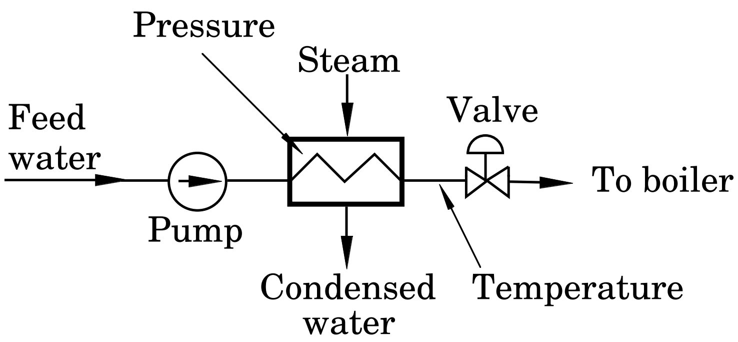
Comparision

Sinusoidal signal with $h = 1$ and $h = 0.5$

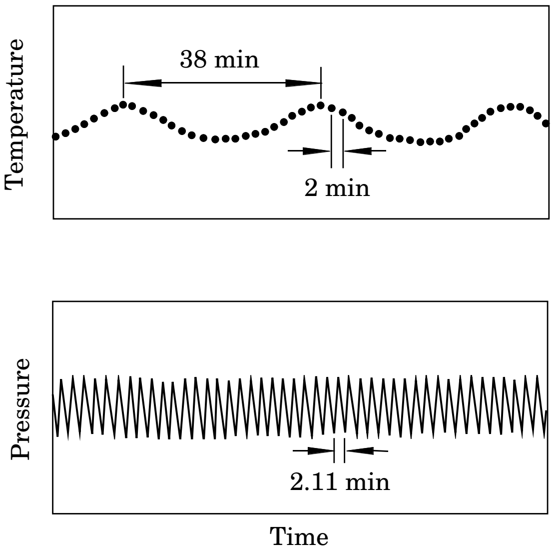


Aliasing – Example

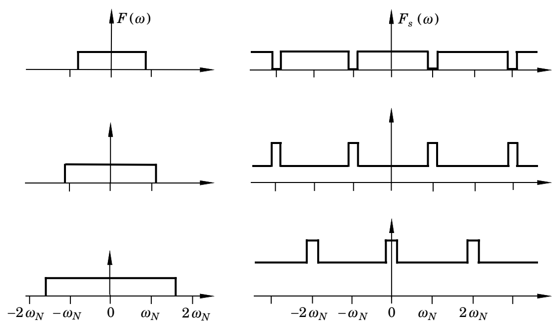
Feedwater heating in a ship boiler



Aliasing – Example cont'd



Frequency folding



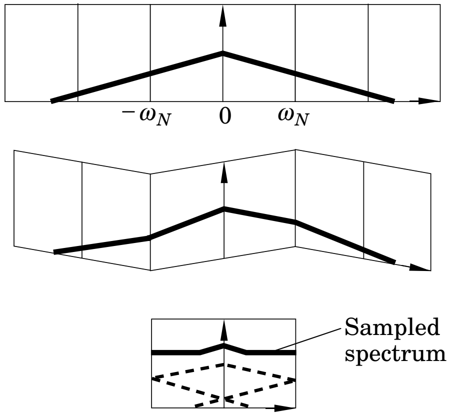
$\omega_N = \omega_s/2$ Nyquist frequency, periodic and NOT time-invariant system

$$\omega_{sampled} = n\omega_s \pm \omega$$

Fundamental alias given by

$$\omega = |(\omega_1 + \omega_N) \bmod (\omega_s) - \omega_N|$$

Frequency folding



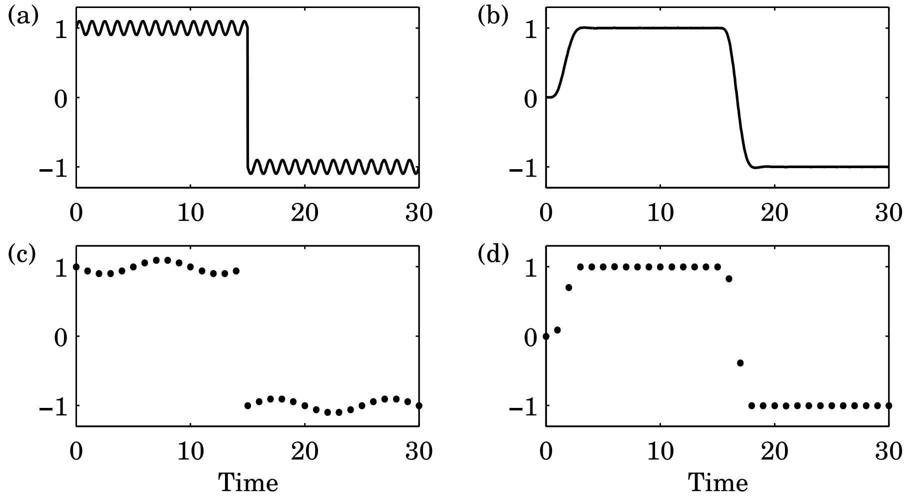
Pre- and postsampling filters

Typical control problem:

"Decrease the influence of a low-frequency process disturbance despite high-frequency measurement noise."

- Frequency separation
- Prefilter ω_N
  - Bessel, Butterworth, ITAE filters
- Postsampling filters
  - Avoid exciting mechanical resonances
  - Higher order hold

Example – Prefiltering

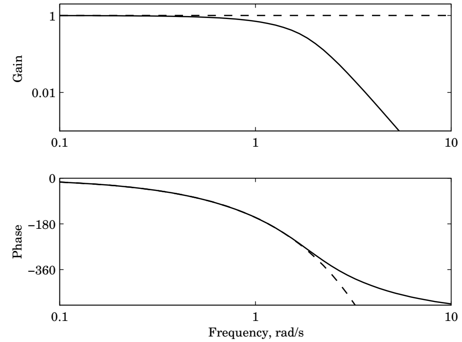


$\omega_d = 0.9$, $\omega_N = 0.5$, $\omega_{alias} = 0.1$ 6th order Bessel with $\omega_B = 0.25$

Consequence of using prefilter

Included the filter in the process model (Exception: Fast sampling)

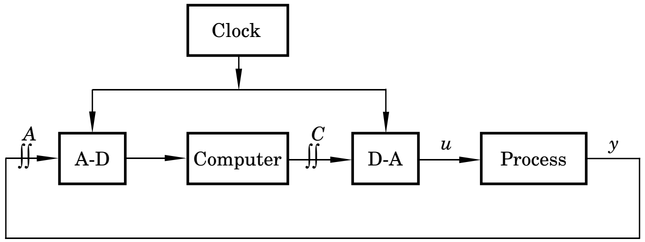
Advantage of Besselfilter: Can be approximated with a delay



6th order Bessel $\omega_B = 1$ (full), time delay with $T_d = 2.7$ (dashed)

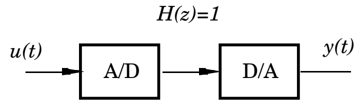
Process oriented models

Modulation model



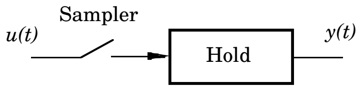
- Look at the analog signals $y(t)$
- Compare stroboscopic model $y(kh)$
- New phenomena
  - Periodic system
  - New frequencies

Model of sample and hold



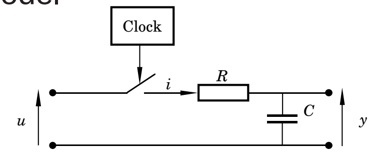
$$m(t) = \begin{cases} 1 & \text{if switch is closed} \\ 0 & \text{if switch is open} \end{cases}$$

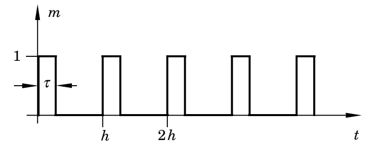
$$i = \frac{u-y}{R}\, m$$



$$C\,\frac{dy(t)}{dt} = i(t) = \frac{u(t)-y(t)}{R}\, m(t)$$

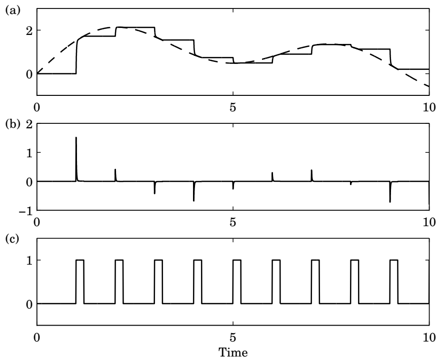
Model



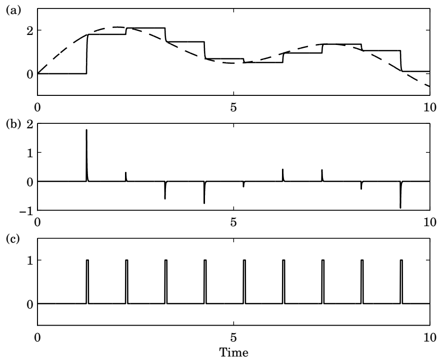


Compromise between τ and RC

$\tau = 0.2$ and $RC = 0.01$



$\tau = 0.05$ and $RC = 0.01$

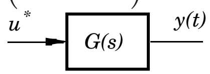


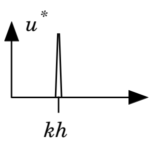
Mathematical model

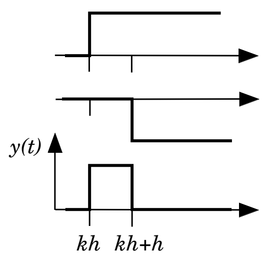
Look at continuous-time signals

$$u^* = u \cdot m \qquad m(t) = \sum_{k=-\infty}^{\infty} \delta(t-kh)$$

Hold circuit $G_{zoh}(s) = \frac{1}{s}\left(1 - e^{-sh}\right)$

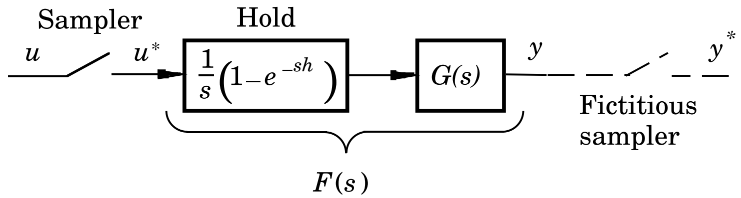






Input-output relations

Use transform theory



$$U^*(s) = \int_0^\infty e^{-st} u^*(t)\, dt = \sum_{k=0}^{\infty} e^{-skh}\, u(kh)$$

$$Y(s) = F(s) \sum_{k=0}^{\infty} e^{-skh}\, u(kh) \neq [\ldots] U(s)$$

Input-output relations, cont'd

But (Theorem 7.3)

$$Y^*(s) = [F(s)U^*(s)]^* = F^*(s)U^*(s)$$
$$Y(z) = \tilde{F}(z)U(z)$$
$$\tilde{F}(z) = F^*(s)|_{s=(\ln z)/h}$$

This is the historic way to approach sample-data systems

Pulse transfer function formalism

* z-transform for a continuous-time function $f(t)$

$$\mathcal{Z}(f(t)) = \tilde{F}(z) = \sum_{k=0}^{\infty} z^{-k} f(kh)$$

* Theorem 7.2

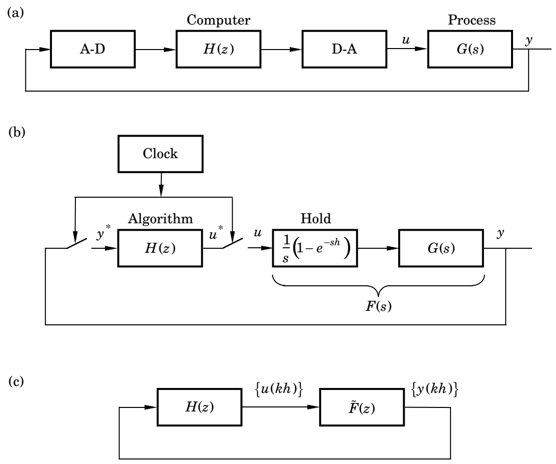
$$F^*(s) = \mathcal{L}(f^*(t)) = \tilde{F}(e^{sh})$$
$$\tilde{F}(z) = \mathcal{Z}(f(t))$$

* Fictitious sampler (Theorem 7.3)

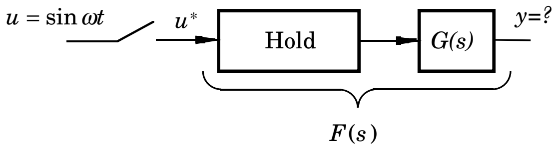
$$m(t)\,\{f(t) * [m(t)g(t)]\} = [m(t)f(t)] * [m(t)g(t)]$$
$$[F(s)G^*(s)]^* = F^*(s)G^*(s)$$

A formalism

- Each A-D is represented by an ideal sample
- Each D-A is represented by a hold circuit



Frequency response



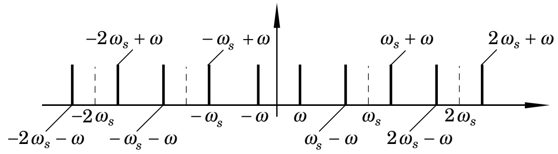
Formal Fourier series expansion

$$m(t) = \sum_{k=-\infty}^{\infty} \delta(t - kh) = \frac{1}{h}\left(1 + 2\sum_{k=1}^{\infty} \cos k\omega_s t\right) \qquad u(t) = \sin(\omega t + \varphi)$$

$$u^*(t) = \frac{1}{h}\left(\sin(\omega t + \varphi) + 2\sum_{k=1}^{\infty} \cos(k\omega_s t)\sin(\omega t + \varphi)\right)$$

$$= \frac{1}{h}\left(\sin(\omega t + \varphi) + \sum_{k=1}^{\infty}\big(\sin(k\omega_s t + \omega t + \varphi) - \sin(k\omega_s t - \omega t - \varphi)\big)\right)$$

Frequency response cont'd



$\Rightarrow y$ contains many frequencies

The *fundamental* frequency

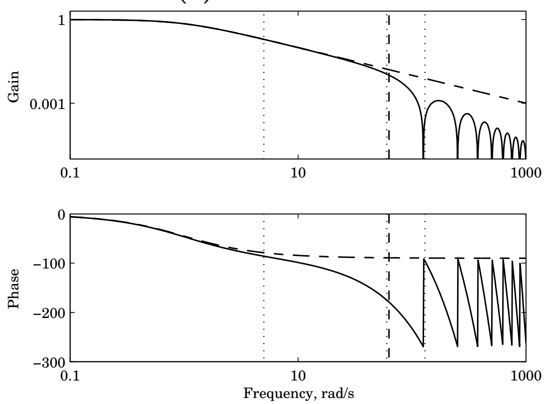
$$\hat{F}(i\omega) = \begin{cases} \dfrac{1}{h}\, F(i\omega) & \omega \neq k\omega_N \\ \dfrac{2}{h}\, F(i\omega)e^{i(\pi/2-\varphi)}\sin\varphi & \omega = k\omega_N \end{cases}$$

- The other frequencies? Interference
- Synchronization
- Generalization $F(i\omega) \to H(e^{i\omega h})F(i\omega)$

Example – Frequency response

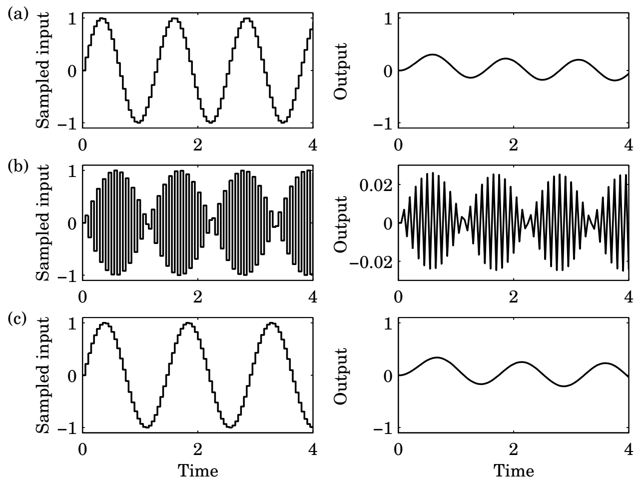
$$G(s) = \frac{1}{s+1} \qquad h = 0.05 \qquad \omega_N = 62.8$$

Bode diagram of ZoH+$G(s)$



Example – Frequency response

Output



Summary

- Periodic time-varying system
- Filter before sampling
- Intersample behavior is complex
- Pulse transfer function formalism
- Multirate systems

Operators and transforms

Match 1: z vs q

- Subtle differences, but convenient to differ between

  z complex number

  q shift operator
- Compare s vs $p = \frac{d}{dt}$

Match 2: q vs δ

Tschauner (1963):

"This autor prefers the ζ-transformation because of its following beutiful properties: that all relations of sampled-dat systems have a form always showing the asymptotic connection to the corresponding relations of (analogous) continuous systems"

Jury:

"However, I wish to indicate, that the ζ-transform as applied to sampled data is more adequate than the z-transform, are rather optimistic."

Peterka, Gawthrop,
Goodwin – Middleton (1990): Digital Control and Estimation:
A Unified Approach

The δ-operator

$$\delta = \frac{q-1}{h}$$

$\delta \equiv q$ Theoretically

$\delta \neq q$ Numerically

Motivation – Short sampling times

Properties of the δ-operator

$$H(q) = \frac{B(q)}{A(q)} = \frac{B(\delta h + 1)}{A(\delta h + 1)} = \frac{\bar{B}(\delta)}{\bar{A}(\delta)} \bar{H}(\delta)$$

$$\lim_{h \to 0} \bar{H}(\delta) = G(\delta)$$

Example: $G(s) = 1/s^2$

$$H(q) = \frac{h^2(q+1)}{2(q-1)^2} = \frac{1 + \delta h/2}{\delta^2} = \bar{H}(\delta)$$

Also "sampling zero" $\delta = -2/h \to -\infty$

$\delta =$ shift of origin + scaling

Stability area $= C(-1/h, 1/h) \quad \to \quad$ LHP, but depends on h!

Summary

- Make a difference between z and q
- δ-operator have good numerical properties
- Organizing of the code
- Coefficient sensitivity