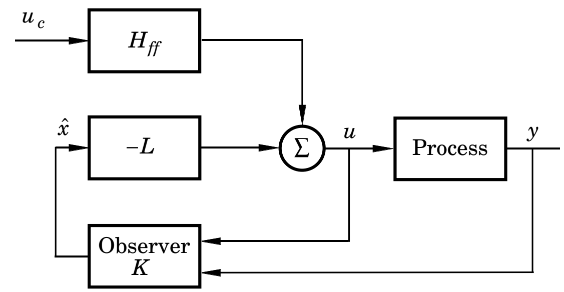
Lecture 4: Input/output pole-placement

- Simple design procedure
- "Remove" factors in A and B
- Shape command signal response
- Example
- "Add" factors in R and S
- Practical aspects
- More examples
- Sensitivity

State-space design – output feedback



$$A_c = \det(qI - \Phi + \Gamma L) \qquad A_o = \det(qI - \Phi + KC)$$

$$y = H_{ff}\frac{BA_o}{A_cA_o}u_c = H_m u_c = \frac{B_m}{A_m}u_c$$

- Choose $H_{ff} = \frac{B_m/A_m}{B/A_c}$
- Special case $A_m = A_c$
- Watch out for B!

Problem formulation

* Process

$$H(z) = \frac{B(z)}{A(z)}$$

* Observer polynomial $\quad A_o(z) \quad$ stable

* Desired characteristic equation

$$A_{cl}(z) = A_c(z)A_o(z) \qquad A_c \text{ controller polynomial, stable}$$

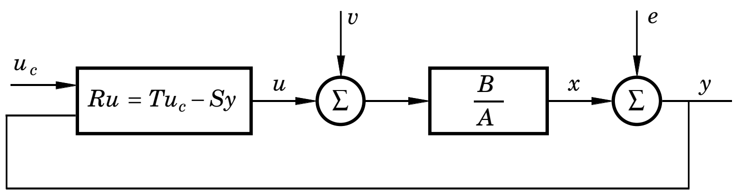
* Controller

$$R(q)u(k) = T(q)u_c(k) - S(q)y(k)$$

Causality implies

$$\deg R \geq \deg T \qquad \deg R \geq \deg S$$

A formal solution



$$A(q)y(k) = B(q)u(k)$$
$$R(q)u(k) = T(q)u_c(k) - S(q)y(k)$$

Closed loop system

$$y = \frac{BT}{AR + BS}u_c$$

Desired input-output relation

$$\frac{BT}{AR + BS} = \frac{BT}{A_cA_o} = \frac{t_0B}{A_c}$$

Problem: How to determine R, S, and T?

Simple pole-placement design

Data: Model: $B(z)/A(z)$, $A(z)$ and $B(z)$ do not have any common factors. Specifications: Desired closed-loop characteristic polynomial $A_{cl}(z)$.

Step 1. Find $R(z)$ and $S(z)$ with $\deg S(z) \le \deg R(z)$ such that

$$A(z)R(z) + B(z)S(z) = A_{cl}(z)$$

Step 2. Factor the closed-loop characteristic polynomial as $A_{cl}(z) = A_c(z)A_o(z)$, where $\deg A_o(z) \le \deg R(z)$, and choose

$$T(z) = t_0 A_o(z)$$

where $t_0 = A_c(1)/B(1)$. The control law is

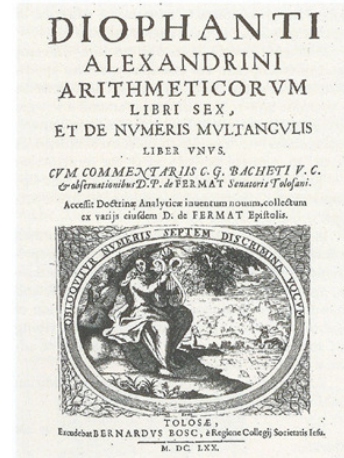
$$R(q)u(k) = T(q)u_c(k) - S(q)y(k) \quad \Rightarrow \quad A_c(q)y(k) = t_0 B(q)u_c(k)$$

Diophantine equation

$$A(z)X(z) + B(z)Y(z) = C(z)$$

Diophantine (Diophantus $\approx$ A.D. 300), Aryabhatta, Bezout

- Two unknowns, one equation?!
- When has the Diophantine equation a (unique) solution?
- An algebraic digression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An algebraic digression

Assume x and y integers

$$3x + 2y = 5$$

Some solutions

$x:$	-5	-3	-1	1	3	5	7
$y:$	10	7	4	1	-2	-5	-8

General solution

$$x = x_0 + 2n \qquad n \text{ integer}$$
$$y = y_0 - 3n$$

Unique solution if

$$0 \le x < 2 \quad \text{or} \quad 0 \le y < 3$$

No solution to

$$4x + 6y = 1$$

Integers and polynomials are rings

Main result

Diophantine equation

$$A(z)X(z) + B(z)Y(z) = C(z)$$

Theorem

- Solution exists if and only if greatest common factor of A and B also a factor in C
- Many solutions. If X_0 and Y_0 is a solutions then for arbitrary Q
  $$X = X_0 + QB$$
  $$Y = Y_0 - QA$$
  is also a solution
- Uniqueness if
  $$\deg X < \deg B \qquad \text{or} \qquad \deg Y < \deg A$$

Compatibility conditions

$$AR + BS = A_o A_c$$
$$Ru = -Sy + Tu_c$$

* Causality

$$\deg R \geq \deg S$$
$$\deg R \geq \deg T$$

  Equality implies no delay in the controller
* Uniqueness (Minimum degree solution)

$$\deg S < \deg A$$

$$\Downarrow$$

$$\deg S = \deg R = \deg T = \deg A_o = n - 1$$
$$\deg A_c = n$$

Solution of Diophantine equation

$$A(z)X(z) + B(z)Y(z) = C(z)$$
$$(a_0 z^n + \cdots + a_n)(x_0 z^{n-1} + \cdots + x_{n-1})$$
$$+(b_0 z^n + \cdots + b_n)(y_0 z^{n-1} + \cdots + y_{n-1})$$
$$= c_0 z^{2n-1} + \cdots + c_{2n-1})$$

$$\underbrace{\begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 & b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 & b_{n-1} & b_{n-2} & \cdots & b_0 \\ a_n & a_{n-1} & \cdots & a_1 & b_n & b_{n-1} & \cdots & b_1 \\ 0 & a_n & \cdots & a_2 & 0 & b_n & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n & 0 & 0 & \cdots & b_n \end{pmatrix}}_{\mathcal{S}} \begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \\ y_0 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} c_0 \\ \vdots \\ c_{n-1} \\ c_n \\ \vdots \\ c_{2n-1} \end{pmatrix}$$

Sylvester matrix, Common factors

Cancellation of poles and zeros

Why? Why not? Factorize

$$A = A^+ A^- \qquad B = B^+ B^-$$

A^+ and B^+ are “nice” polynomials

Cancellation of A^+ and B^+ implies

$$R = B^+ \bar{R} \qquad S = A^+ \bar{S} \qquad T = A^+ \bar{T}$$
$$A_{cl} = AR + BS = A^+ B^+ (A^- \bar{R} + B^- \bar{S}) = A^+ B^+ \bar{A}_{cl}$$

Introduce (quite arbitrarily)

$$A_c = B^+ \bar{A}_c \qquad A_o = A^+ \bar{A}_o$$

Design equation

$$A^- \bar{R} + B^- \bar{S} = \bar{A}_{cl} = \bar{A}_c \bar{A}_o$$

Design procedure

Desired closed-loop characteristic equation

$$A^- \bar{R} + B^- \bar{S} = \bar{A}_{cl} = \bar{A}_c \bar{A}_o$$

Minimum degree solution if $\deg \bar{S} < \deg A^-$ $\quad \Rightarrow$

$$B^+ \bar{R} u = A^+ \bar{T} u_c - A^+ \bar{S} y$$

Interpretation

$$u = \frac{A^+}{B^+} \left(\frac{\bar{T}}{\bar{R}} u_c - \frac{\bar{S}}{\bar{R}} y \right)$$

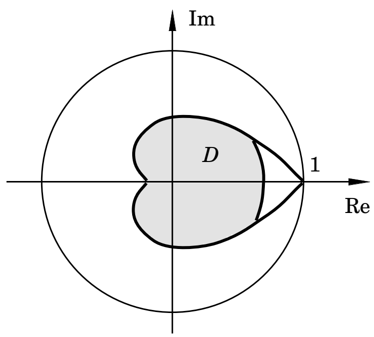
Cancel some poles and zeros, and then make the design.

The simple case $T = t_0 A_o$

$$\frac{BT}{A_{cl}} = \frac{t_0 B^+ B^- A_o}{A_c A_o} = \frac{t_0 B^-}{\bar{A}_c}$$

Practical limitations on B^+

- Don't cancel all zeros within the unit circle!
- Avoid zeros on the negative real axis
- Avoid poorly damped zeros
- Cancel only in shaded area D



Separation of disturbance and command resp.

Compare state-feedback design and feedforward from reference signal. Desired command signal response

$$y_m = H_m u_c = \frac{B_m}{A_m} u_c$$

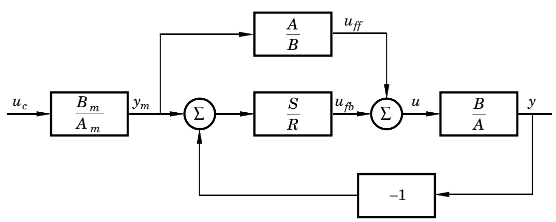
Limitation: $B_m = \bar{B}_m B^-$ Try the controller

$$R = A_m B^+ \bar{R} \qquad S = A_m A^+ \bar{S} \qquad T = \bar{B}_m \bar{A}_o \bar{A}_c A^+$$

If common factors between A_m and $\bar{A}_c$, cancel before implementation of the controller

$$\frac{BT}{AR + BS} = \frac{B^+ B^- \bar{B}_m \bar{A}_o \bar{A}_c A^+}{A^+ A^- A_m B^+ \bar{R} + B^+ B^- A_m A^+ \bar{S}} = \frac{B^- \bar{B}_m \bar{A}_o \bar{A}_c}{A_m \underbrace{(A^- \bar{R} + B^- \bar{S})}_{\bar{A}_o \bar{A}_c}}$$

Alternativ formulation



$$u = \frac{T}{R} u_c - \frac{S}{R} y = \frac{A^+ \bar{B}_m}{A_m B^+} \cdot \frac{\bar{A}_o \bar{A}_c}{\bar{R}} u_c - \frac{S}{R} y$$
$$= \frac{A^+ \bar{B}_m}{A_m B^+} \cdot \frac{A^- \bar{R} + B^- \bar{S}}{\bar{R}} u_c - \frac{S}{R} y = \frac{\bar{B}_m A}{A_m B^+} u_c + \frac{S}{R} (y_m - y)$$

Feedforward and feedback Gives the closed loop system

$$y = \frac{B_m}{A_m} u_c$$

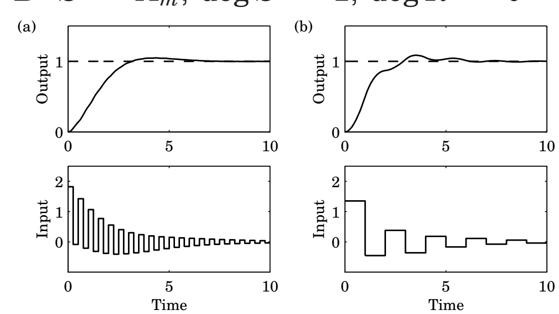
The combined generation of y_m and u_{ff} requires great care.

Example – Motor

$$H(z) = \frac{K(z-b)}{(z-1)(z-a)} \qquad b < 0! \qquad H_m(z) = \frac{z(1 + p_1 + p_2)}{z^2 + p_1 z + p_2}$$

Cancels the zero
$B^+ = z - b$, $B^- = K$, $\bar{B}_m = B_m/K$, $A^+ = 1$, $A_o = 1$, and $\bar{A}_c = A_m$.
Control law using $A\bar{R} + B^- \bar{S} = A_m$, $\deg S = 1$, $\deg \bar{R} = 0$

a) $h = 0.25$, b) $h = 1$



Motor example – No cancellation

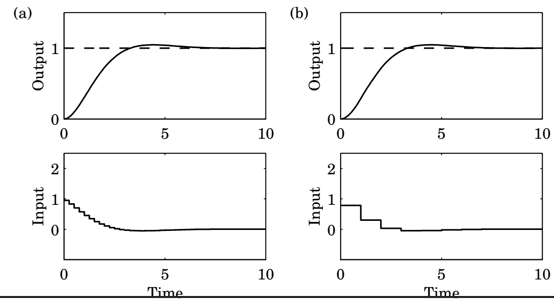
$$H_m(z) = \frac{1+p_1+p_2}{1-b}\,\frac{z-b}{z^2+p_1z+p_2}$$

$B^+ = 1$, $B^- = K(z-b)$, $A^+ = 1$, $A_c = A_m$, $A_o = z$, and $\bar{B}_m = \frac{1+p_1+p_2}{K(1-b)}$

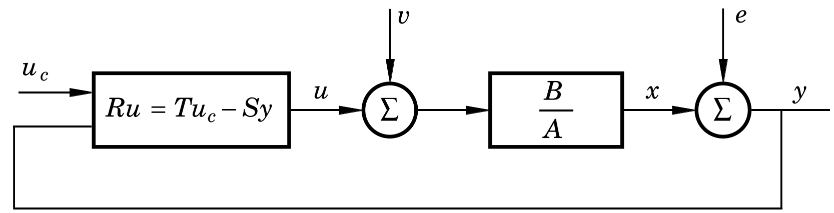
Control law using $AR + B^-S = A_mA_o$, $\deg S = 1$, $\deg R = 1$

$$u(k) = t_0u_c(k) - s_0y(k) - s_1y(k-1) - r_1u(k-1)$$

Note: The same structure as for cancellation a) $h = 0.25$, b) $h = 1$



"Add" factors in R ans S



$$x = \frac{BT}{AR+BS}u_c + \frac{BR}{AR+BS}v - \frac{BS}{AR+BS}e$$
$$u = \frac{AT}{AR+BS}u_c - \frac{AS}{AR+BS}v - \frac{AS}{AR+BS}e$$

- Integrators $R = B^+R_d\bar{R}$, $R_d = (z-1)^l$
- Notch filter $S = A^+S_d\bar{S}$ where e.g.

$$S_d(z) = z^2 - 2z\cos\omega h + 1$$

Full design algorithm

Data: B/A, B_m/A_m, R_d, S_d, and A_{cl}

Step 1: Factor $A = A^-A^+$, $B = B^-B^+$, $B_m = B^-\bar{B}_m$, $A_{cl} = A^+B^+A_m\bar{A}_{cl}$

Step 2: Solve

$$A^-R_d\underline{\bar{R}} + B^-S_d\underline{\bar{S}} = \bar{A}_{cl} \qquad \deg\bar{S} = \deg A^-R_d - 1$$

Step 3: Control law

$$Ru = Tu_c - Sy R = A_mB^+R_d\bar{R}, \quad S = A_mA^+S_d\bar{S} \quad T = \bar{B}_mA^+\bar{A}_{cl}$$

Degree conditions:

$$\deg A_m - \deg B_m \geq \deg A - \deg B = d$$
$$\deg A_{cl} = (\deg AR = \deg A + \deg S)$$
$$= 2\deg A + \deg A_m + \deg R_d + \deg S_d - 1$$

Practical aspects

* Solution of the Diophantine equation
* Zero cancellations
* How to choose A_c and A_o?
* Magnitude of u

$$\begin{matrix} y = Hu \\ y = H_mu_c \end{matrix} \quad\Rightarrow\quad u = \frac{H_m}{H}u_c$$

* Selection of h $\quad (A_oA_cB^+)$

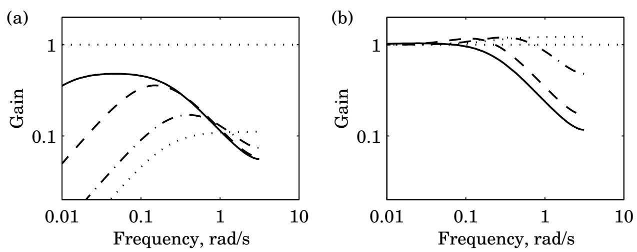
$$\omega h = 0.2\text{–}0.6$$

* Response to load and noise
* Influence of observer polynomial

Observer polynomial

Consider

$$H(z) = \frac{0.1}{z-1} \qquad H_m(z) = \frac{0.2}{z-0.8}$$



$A_o(z) = z - \alpha$ Transmission to x from a) Load v and b) Measurement error e
$\alpha = 1$ (full), $\alpha = 0.9$ (dashed), $\alpha = 0.5$ (dash-dotted), $\alpha = 0$ (dotted)

Harmonic oscillator

Process model

$$G(s) = \frac{\omega_0^2}{s^2 + \omega_0^2} \qquad \omega_0 = 1$$

Sampled pulse-transfer operator

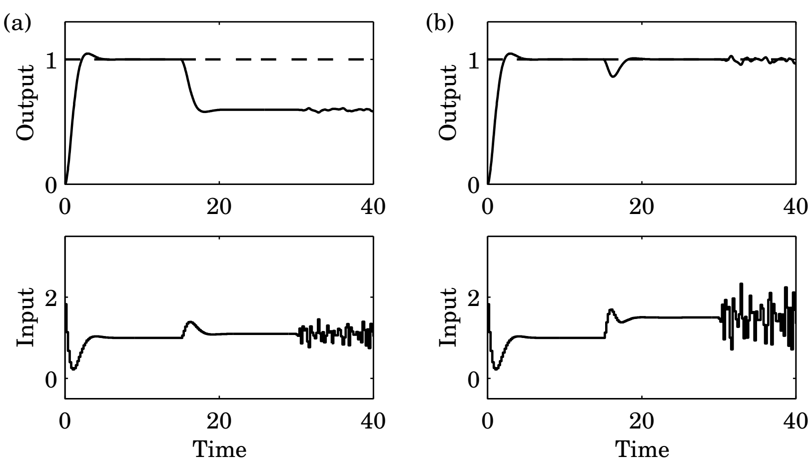
$$H(q) = \frac{(1-\beta)(q+1)}{q^2 - 2\beta q + 1} = \frac{B(q)}{A(q)} \qquad \beta = \cos(\omega_0 h)$$

Specifications (nominal design)

- No zero cancellation
- A_c : $\quad s^2 + 2\zeta\omega s + \omega^2 = 0 \qquad \omega = 1.5 \quad \zeta = 0.7$
- A_o : $\quad s^2 + 2\zeta_{obs}\omega_{obs} s + \omega_{obs}^2 \qquad \omega_{obs} = 3 \quad \zeta_{obs} = 0.7$
- Sampling interval $h = 0.2$

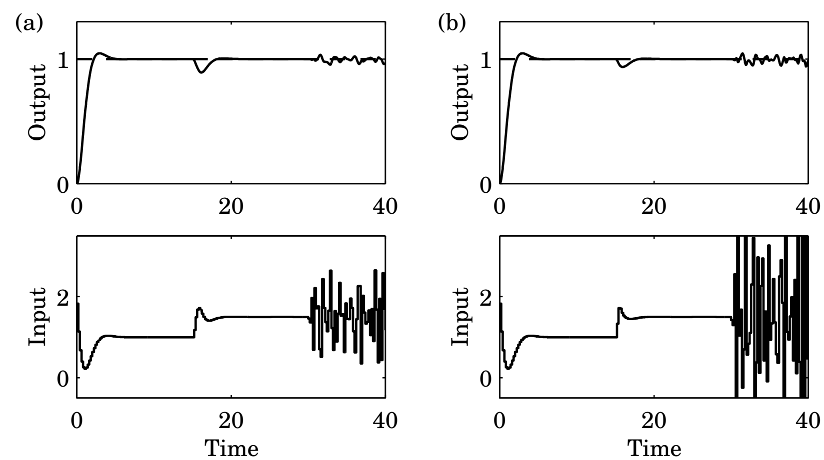
Harmonic oscillator cont'd

Nominal design a) Without, b) With integrator



Harmonic oscillator cont'd

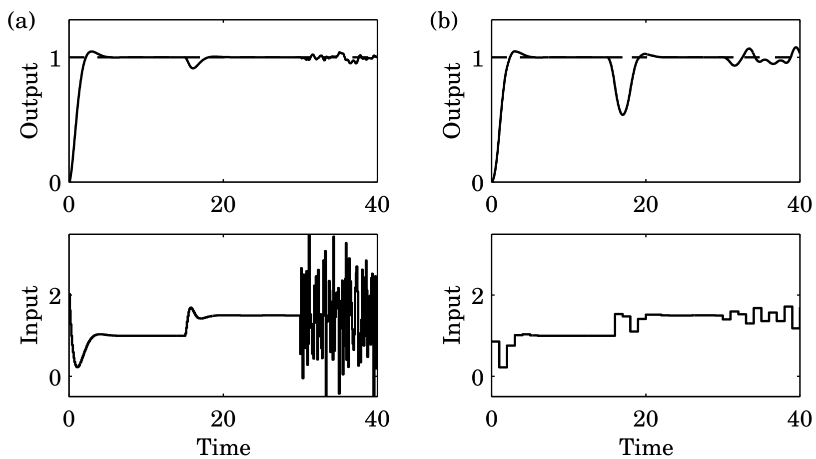
Changing observer dynamics
a) $\omega_{obs} = 4$ b) $\omega_{obs} = 8$



Harmonic oscillator cont'd

Changing sampling interval

Nominal $\omega_{obs}h = 0.6$ a) $h = 0.1$ b) $h = 1$

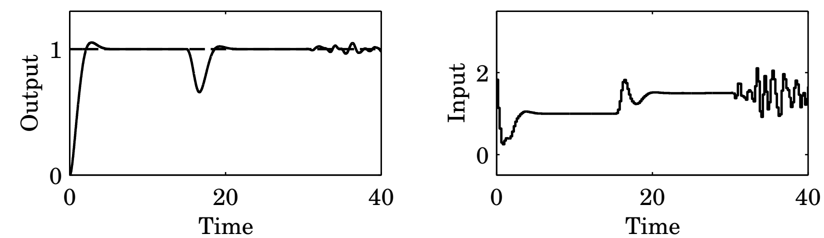


Harmonic oscillator cont'd

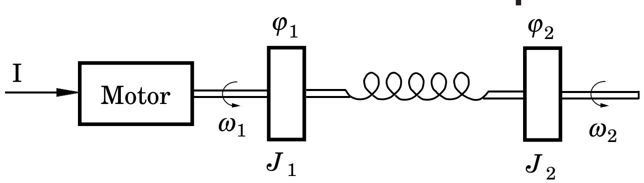
Antialiasing filter 6th order Bessel

Nominal design gives unstable system

Approximate the filter with a delay and redo the design



Robot mechanism example



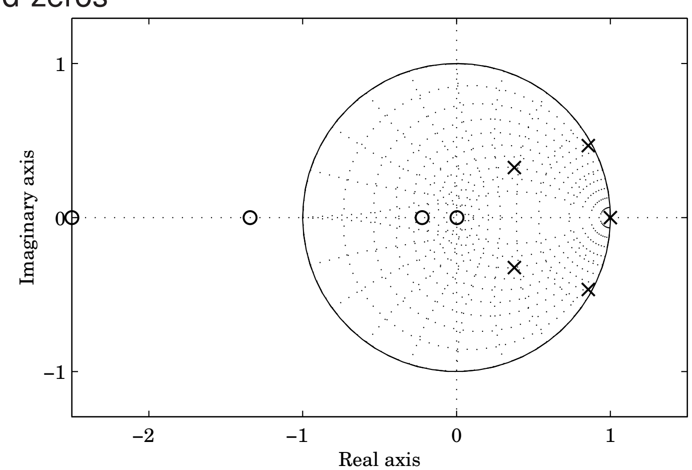
Antialiasing filter, second order with $\omega_f = 2$

$$A(q) = \underbrace{(q^2 - 0.7505q + 0.2466)}_{\text{filter}}\underbrace{(q^2 - 1.7124q + 0.9512)(q-1)}_{\text{process}}$$

$$B(q) = 0.1420 \cdot 10^{-3}(q+12.1314)(q+1.3422)(q+0.2234)(q-0.0023)$$

Robot mechanism example

Poles and zeros



Notch filter design

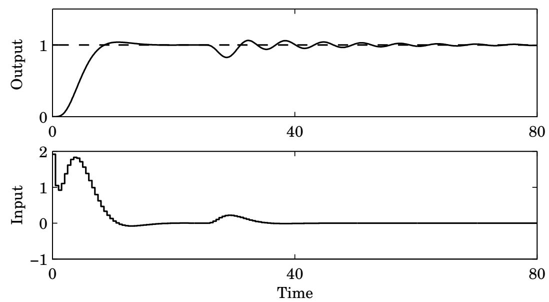
- Sample, $h = 0.5$, $A_c(s) = (s^2 + 2\zeta_m \omega_m s + \omega_m^2)(s + \alpha_1 \omega_m)$ and keep the antialiasing dynamics
- $\deg A_o = 2$ Same poles as A_f
- Include the oscillatory part

$$A^+(z) = z^2 - 1.712z + 0.9512$$

$$AR + BS = A^+ A_c A_o$$

4th order controller $\Rightarrow$ 9th order closed loop system

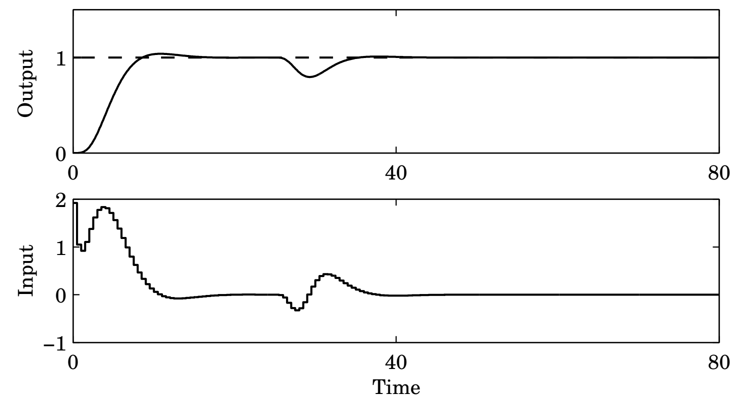
A_n factor in A but not in B $\Rightarrow$ factor in S



Active damping

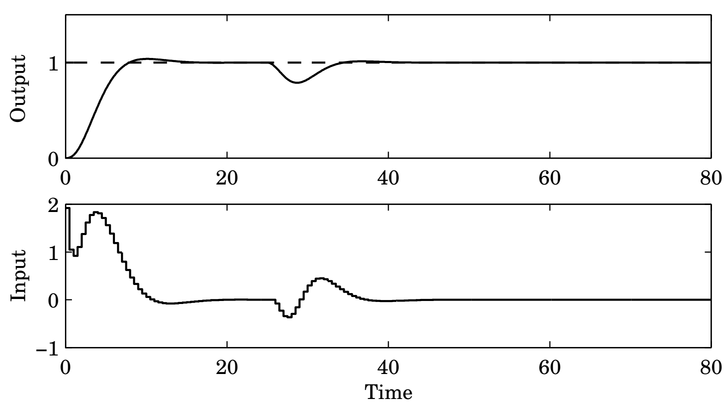
Damp the oscillatory modes,

$\zeta_p = 0.05 \rightarrow 0.7$



Comparison

State feedback design and full observer
(No antialiasing filter)



Sensitivity

The design is done for $H = B/A$ but the true system is $H^0 = B^0/A^0$

Problem: How sensitive is the closed loop system?

Theorem
The closed loop system is stable if

$$\frac{|H(z) - H^0(z)|}{|H(z)|} \le \frac{1}{|H_m(z)|} \left| \frac{H_{ff}(z)}{H_{fb}(z)} \right| = \frac{1}{|H_m(z)|} \left| \frac{T}{S} \right|$$

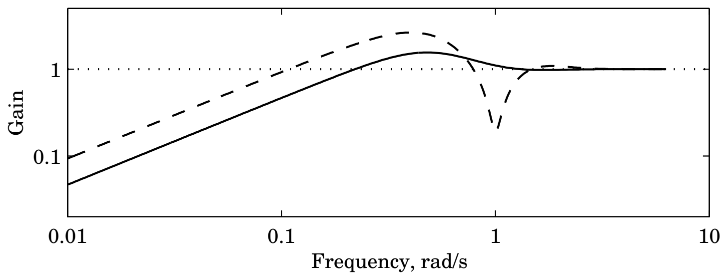
for $|z| = 1$

Right hand side depends on known quantities!

Robot mechanism

Sensitivity function for notch design and active damping design

$$\mathcal{S} = \frac{AR}{A_{cl}}$$



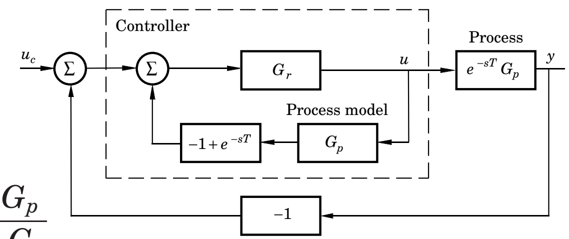
Notch (full), active (dashed)

Smith-predictor

One way of reducing the effect of delays

- Design a regulator G_r as if there is no delay.

$$\frac{G_r G_p}{1 + G_r G_p}$$



- Find G_r' such that

$$\frac{e^{sT}G_r' G_p}{1 + G_r' G_p} = \frac{e^{sT}G_r G_p}{1 + G_r G_p}$$

- Good for discrete-time systems

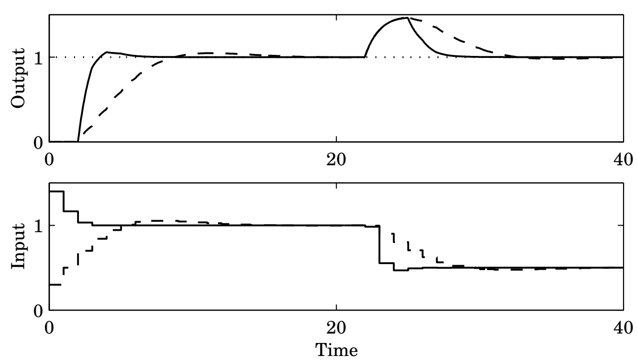
Smith-predictor – Example

First order system with time-delay

$$y(k+1) = 0.37y(k) + 0.63u(k-2)$$

No delay: PI with $K = 0.4$ and $T_i = 0.4$

Smith pred. (full) and PI-contr, $K = 0.1$, $T_i = 0.5$ (dashed)



Summary

- Convenient method for design
- Design parameters A_c, A_o, and h
- Relate A_c and A_o to physical process
- Be careful with zero cancellations
- Many other methods can be interpreted as pole-placement