Lecture 3: State-feedback, observers, reference values, and integrators

- Problem formulation
- Pole placement
- Observers
- Output feedback
- Reference values
- A larger example

Problem formulation

- Model: Continuous-time, sample with $h \quad \Rightarrow$

$$x(kh+h) = \Phi x(kh) + \Gamma u(kh)$$

- Disturbances: Sporadic pulse disturbances $x(0) = x_0$
- Criterion: $x(t) \to 0$ reasonably fast with reasonable inputs u. Choose closed loop poles
- Admissible controls: Linear controllers, all states available

More complicated problems later

Problem formulation cont'd

Design parameters

- Closed loop poles
- Sampling interval

Evaluation

- Compare $x(t)$ and $u(kh)$ with specifications
- Trade-off between control magnitude and speed of response
- Subjective judgements

Example – Double integrator

$$x(kh+h) = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} x(kh) + \begin{pmatrix} h^2/2 \\ h \end{pmatrix} u(kh)$$

Linear state-feedback controller

$$u(kh) = -Lx(kh) = -l_1 x_1(kh) - l_2 x_2(kh)$$

Closed-loop system becomes

$$\begin{aligned} x(kh+h) &= (\Phi - \Gamma L)x(kh) \\ &= \begin{pmatrix} 1 - l_1 h^2/2 & h - l_2 h^2/2 \\ -l_1 h & 1 - l_2 h \end{pmatrix} x(kh) \end{aligned}$$

Characteristic equation

$$z^2 + \left(\frac{l_1 h^2}{2} + l_2 h - 2\right) z + \left(\frac{l_1 h^2}{2} - l_2 h + 1\right) = 0$$

Example cont'd

Characteristic equation

$$z^2 + \left(\frac{l_1 h^2}{2} + l_2 h - 2\right) z + \left(\frac{l_1 h^2}{2} - l_2 h + 1\right) = 0$$

Desired characteristic equation

$$z^2 + p_1 z + p_2 = 0$$

Linear equations for l_1 and l_2

$$\frac{l_1 h^2}{2} + l_2 h - 2 = p_1 \qquad \frac{l_1 h^2}{2} - l_2 h + 1 = p_2$$

Solution

$$l_1 = \frac{1}{h^2}(1 + p_1 + p_2) \qquad l_2 = \frac{1}{2h}(3 + p_1 - p_2)$$

General case

Basic problem:
Find L such that $\Phi - \Gamma L$ has prescribed eigenvalues

Solvable
$\Leftrightarrow$ $(\Phi,\ \Gamma)$ reachable
$\Leftrightarrow$ $W_c = [\Gamma,\ \Phi\Gamma,\ \ldots,\ \Phi^{n-1}\Gamma]$ has full rank

- Matlab `L = place(Phi,Gamma,neweigs)`
- Unique solution, linear in l_i
- L depends on h
- How to choose the specifications?
  - Use the continuous time counterpart
  - Damping ζ and natural frequency ω (ω_0) of dominating poles

Solution via controllable form

$$z = Tx, \qquad \tilde{\Phi} = T\Phi T^{-1}, \quad \tilde{\Gamma} = T\Gamma$$

$$u = -\tilde{L}z = -\tilde{L}Tx = -Lx$$

$$\det(zI - \Phi) = A(z) \qquad \det(zI - (\Phi - \Gamma L)) = P(z)$$

$$\tilde{\Phi} = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \qquad \tilde{\Gamma} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Desired characteristic equation obtained for

$$u = -\tilde{L}z = -\begin{pmatrix} \tilde{l}_1 & \tilde{l}_2 & \ldots & \tilde{l}_n \end{pmatrix} z$$
$$= -\begin{pmatrix} p_1 - a_1 & p_2 - a_2 & \cdots & p_n - a_n \end{pmatrix} z$$

- How to get T?

How to find T?

$$L = \tilde{L}T = \begin{pmatrix} p_1 - a_1 & p_2 - a_2 & \cdots & p_n - a_n \end{pmatrix} T$$
$$\tilde{W}_c = [T\Gamma,\ T\Phi T^{-1}T\Gamma, \ldots, T\Phi^{n-1}\Gamma] = TW_c$$

Solvable for T if the system is reachable

$$\tilde{W}_c^{-1} = \begin{pmatrix} 1 & a_1 & \cdots & a_{n-1} \\ 0 & 1 & \cdots & a_{n-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$T^{-1} = W_c \tilde{W}_c^{-1} = \begin{pmatrix} \Gamma & a_1\Gamma + \Phi\Gamma & \ldots & a_{n-1}\Gamma + \cdots + \Phi^{n-1}\Gamma \end{pmatrix}$$
$$L = \begin{pmatrix} p_1 - a_1 & p_2 - a_2 & \cdots & p_n - a_n \end{pmatrix} \cdot$$
$$\begin{pmatrix} \Gamma & a_1\Gamma + \Phi\Gamma & \ldots & a_{n-1}\Gamma + \cdots + \Phi^{n-1}\Gamma \end{pmatrix}^{-1}$$

Ackermann's formula

$$A(\tilde{\Phi}) = \tilde{\Phi}^n + a_1\tilde{\Phi}^{n-1} + \cdots + a_n I = 0$$
$$P(\tilde{\Phi}) = \tilde{\Phi}^n + p_1\tilde{\Phi}^{n-1} + \cdots + p_n I = (p_1 - a_1)\tilde{\Phi}^{n-1} + \cdots + (p_n - a_n)I$$

For controllable form and $0 \le k < n$

$$\begin{pmatrix} 0 & \dots & 1 \end{pmatrix} \tilde{\Phi}^k = \begin{pmatrix} 0 & \dots & \underset{n-k}{1} & \dots & 0 \end{pmatrix}$$

so that

$$\begin{pmatrix} 0 & \dots & 1 \end{pmatrix} P(\tilde{\Phi}) = \begin{pmatrix} p_1 - a_1 & p_2 - a_2 & \cdots & p_n - a_n \end{pmatrix} = \tilde{L}$$

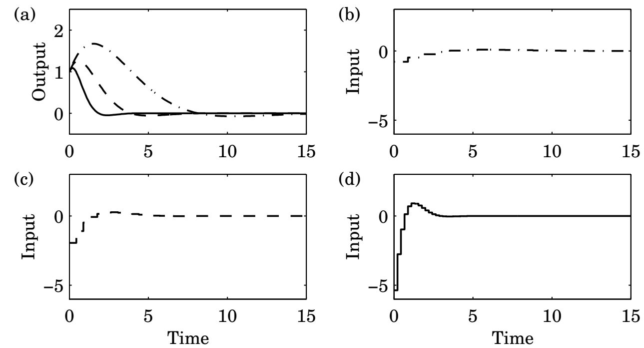
thus

$$L = \tilde{L}T = \begin{pmatrix} 0 & \dots & 1 \end{pmatrix} \overbrace{P(T\Phi T^{-1})}^{TP(\Phi)T^{-1}} T = \begin{pmatrix} 0 & \dots & 1 \end{pmatrix} TP(\Phi)$$
$$= \begin{pmatrix} 0 & \dots & 1 \end{pmatrix} \tilde{W}_c W_c^{-1} P(\Phi) = \begin{pmatrix} 0 & \dots & 1 \end{pmatrix} W_c^{-1} P(\Phi)$$

Double integrator

Use the continuous time counterpart

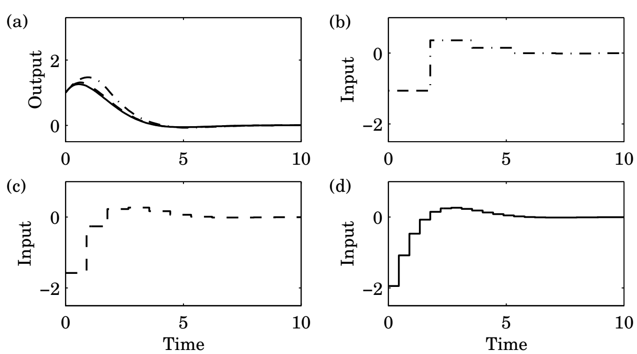
$$s^2 + 2\zeta\omega s + \omega^2$$



Change of ω ($\omega h = 0.44$) $x^T(0) = [1\ 1]$, b) $\omega = 0.5$, c) $\omega = 1$, d) $\omega = 2$

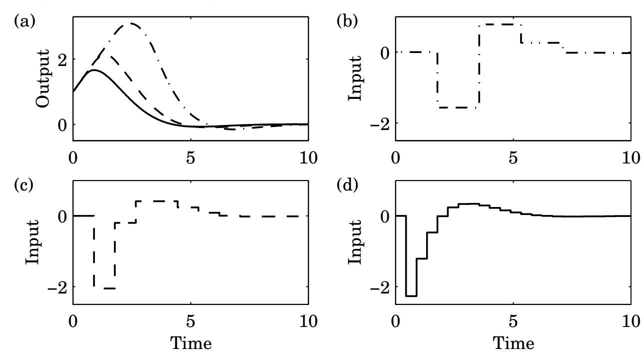
Double integrator cont'd

Change of h (N)
b) $N = 5$, c) $N = 10$, d) $N = 20$



Double integrator cont'd

Other initial value $x^T(0^+) = [1\ 1]$
b) $N = 5$, c) $N = 10$, d) $N = 20$

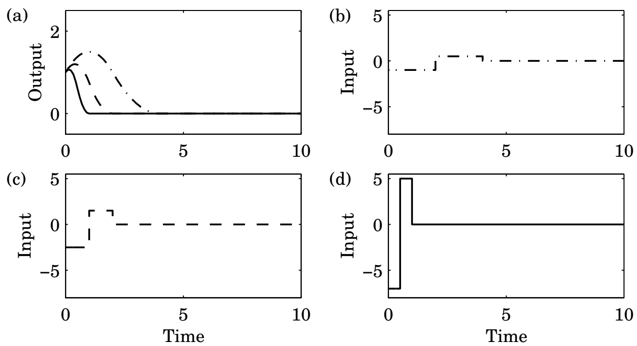


Rule of thumb: ωh =0.2 – 0.6

Deadbeat control

Choose $P(z) = z^n$
Remaining design parameter h

Double integrator $x^T(0) = [1\ 1]$



b) $h = 2$, c) $h = 1$, d) $h = 0.5$

Observers

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = Cx(k)$$

Assume only the output available

- Direct calculation
- Kalman form
- Luenberger form

Main question:
How to choose the observer poles?

Reconstruction using dynamical model

Consider the model

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = Cx(k)$$

Introduce "feedback" from measured $y(k)$

$$\hat{x}(k+1 \mid k) = \Phi\hat{x}(k \mid k-1) + \Gamma u(k) + \underline{K[y(k) - C\hat{x}(k \mid k-1)]}$$

Form the estimation error $\tilde{x} = x - \hat{x}$

$$\tilde{x}(k+1 \mid k) = \Phi\tilde{x}(k \mid k-1) - KC\tilde{x}(k \mid k-1) = [\Phi - KC]\tilde{x}(k \mid k-1)$$

- Choose K to get good convergence
- Any eigenvalues possible, provided W_o full rank
- Trade-off against noise amplification

Luenberger observer
Reduced order observer

Introduce direct term in the observer

$$\begin{aligned}\hat{x}(k \mid k) &= \Phi\hat{x}(k-1 \mid k-1) + \Gamma u(k-1) \\ &\quad + K\Big(y(k) - C(\Phi\hat{x}(k-1 \mid k-1) + \Gamma u(k-1))\Big) \\ &= (I - KC)\left(\Phi\hat{x}(k-1 \mid k-1) + \Gamma u(k-1)\right) + Ky(k)\end{aligned}$$

Reconstruction error can be written as

$$\tilde{x}(k \mid k) = x(k) - \hat{x}(k \mid k) = (\Phi - KC\Phi)\tilde{x}(k-1 \mid k-1)$$

Look at

$$\begin{aligned}\tilde{y}(k) &= y(k) - C\hat{x}(k \mid k) = C\tilde{x}(k \mid k) = C(I - KC)\Phi\tilde{x}(k-1 \mid k-1) \\ &= (C - CKC)\Phi\tilde{x}(k-1 \mid k-1) = \underbrace{(I - CK)}_{=0?}C\Phi\tilde{x}(k-1 \mid k-1)\end{aligned}$$

Why reduced order?

New coordinates

$$z = \begin{pmatrix} y \\ \bar{z} \end{pmatrix} = \begin{pmatrix} C \\ \bar{C} \end{pmatrix} x = Tx$$

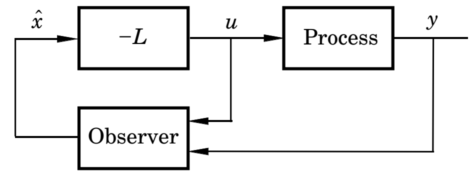
The error can be written as

$$\tilde{z}(k \mid k) = \begin{pmatrix} \tilde{y}(k \mid k) \\ \tilde{\bar{z}}(k \mid k) \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{\bar{z}}(k \mid k) \end{pmatrix}$$

where $\dim \tilde{\bar{z}} = \dim x - \dim y = n - p$

Only $\hat{\bar{z}}(k \mid k) = \dots$ needed

Output feedback



$$\hat{x}(k+1 \mid k) = \Phi\hat{x}(k \mid k-1) + \Gamma u(k) + K(y(k) - C\hat{x}(k \mid k-1))$$
$$u(k) = -L\hat{x}(k \mid k-1)$$

$$\hat{x}(k+1 \mid k) = (\Phi - KC - \Gamma L)\hat{x}(k \mid k-1) + Ky(k)$$
$$u(k) = -L\hat{x}(k \mid k-1)$$

The transfer function of the controller

$$G_r(z) = -L(zI - \Phi + \Gamma L + KC)^{-1}K$$

Closed loop system

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$\tilde{x}(k+1 \mid k) = (\Phi - KC)\tilde{x}(k \mid k-1)$$
$$u(k) = -L(x(k) - \tilde{x}(k \mid k-1))$$

Eliminate $u(k)$

$$\begin{pmatrix} x(k+1) \\ \tilde{x}(k+1 \mid k) \end{pmatrix} = \begin{pmatrix} \Phi - \Gamma L & \Gamma L \\ 0 & \Phi - KC \end{pmatrix} \begin{pmatrix} x(k) \\ \tilde{x}(k \mid k-1) \end{pmatrix}$$

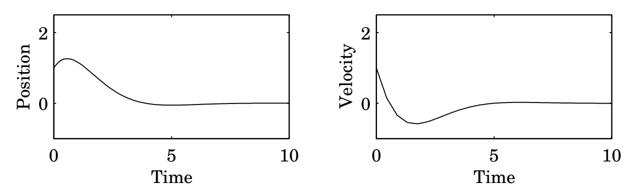
Separation

$$\text{Process poles:} \quad A_r(z) = \det(zI - \Phi + \Gamma L)$$
$$\text{Observer poles:} \quad A_o(z) = \det(zI - \Phi + KC)$$

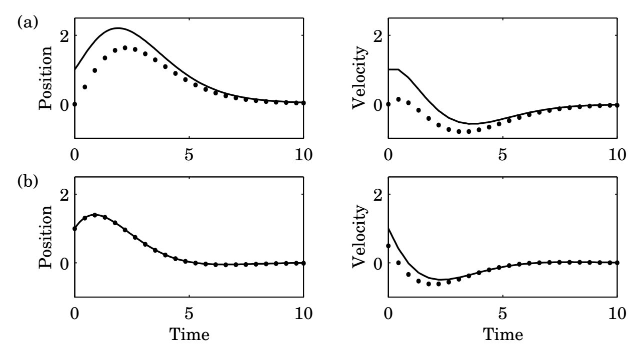
May also use observer with direct term

Example – Double integrator

Measured states



Estimated states
a) Kalman
b) Luenberger



General disturbances

Dynamical systems with initial values gives the disturbance v

$$\frac{dx}{dt} = Ax + Bu + v$$
$$\frac{dw}{dt} = A_w w \qquad v = C_w w$$
$$\frac{d}{dt}\begin{pmatrix} x \\ w \end{pmatrix} = \begin{pmatrix} A & C_w \\ 0 & A_w \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u$$

Sampling gives

$$\begin{pmatrix} x(k+1) \\ w(k+1) \end{pmatrix} = \begin{pmatrix} \Phi & \Phi_{xw} \\ 0 & \Phi_w \end{pmatrix}\begin{pmatrix} x(k) \\ w(k) \end{pmatrix} + \begin{pmatrix} \Gamma \\ 0 \end{pmatrix} u(k)$$

General disturbances, cont'd

Combined feedback and feedforward

$$u(k) = -Lx(k) - L_w w(k)$$

Closed-loop system

$$x(k+1) = (\Phi - \Gamma L)x(k) + \underbrace{(\Phi_{xw} - \Gamma L_w)}_{\approx 0?} w(k)$$
$$w(k+1) = \Phi_w w(k)$$

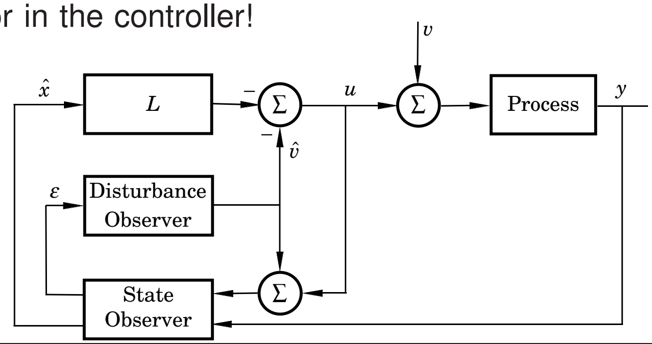
w uncontrollable from u! w observable?

Unmeasurable disturbances – Step input load

$\Phi_w = 1$ i.e. $v(k+1) = v(k)$ and $\Phi_{xw} = \Gamma$

$$u(k) = -L\hat{x}(k) - \overbrace{L_w}^{=1} \hat{w}(k) = -L\hat{x}(k) - \hat{w}(k)$$
$$\hat{x}(k+1) = \Phi\hat{x}(k) + \Gamma(\hat{w}(k) + u(k)) + K(y(k) - C\hat{x}(k))$$
$$\hat{w}(k+1) = \hat{w}(k) + K_w(y(k) - C\hat{x}(k))$$

Integrator in the controller!



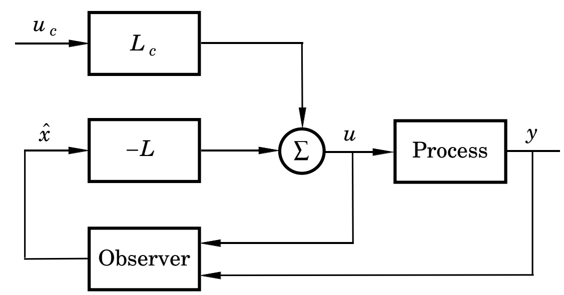
Servo case

Try first

$$u(k) = -L\hat{x}(k) + L_c u_c(k)$$

The closed loop system

$$x(k+1) = (\Phi - \Gamma L)x(k) + \Gamma L\tilde{x}(k) + \Gamma L_c u_c(k)$$
$$\tilde{x}(k+1) = (\Phi - KC)\tilde{x}(k)$$
$$y(k) = Cx(k)$$



Servo case, cont'd

Observer error not reachable from u_c, i.e. cancellation of

$$A_o = \det(zI - \Phi + KC)$$

Closed loop system from reference to output

$$H_{cl}(z) = C(zI - \Phi + \Gamma L)^{-1}\Gamma L_c = L_c \frac{B(z)}{A_c(z)}$$

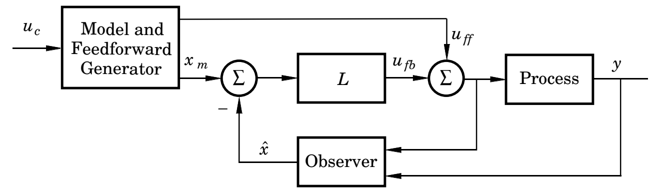
Open-loop pulse transfer function

$$H(z) = C(zI - \Phi)^{-1}\Gamma = \frac{B(z)}{A(z)}$$

Notice: Same zeros! Why?

Model and feedforward

Use a two-degree-of-freedom controller



Reference trajectories for states

$$x_m(k+1) = \Phi_m x_m(k) + \Gamma_m u_c(k)$$
$$y_m(k) = C_m x_m(k)$$

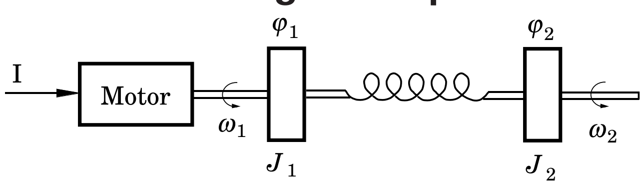
Controller

$$u(k) = L(x_m(k) - \hat{x}(k)) + u_{ff}(k)$$

Feedforward signal

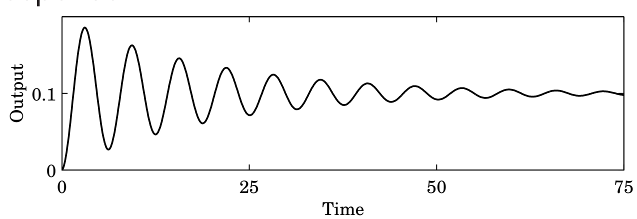
$$u_{ff}(k) = \frac{H_m(q)}{H(q)} u_c(k)$$

A design example



$$x_1 = \varphi_1 - \varphi_2 \qquad x_2 = \omega_1/\omega_0 \qquad x_3 = \omega_2/\omega_0$$

Impulse response



No friction $\Rightarrow$ Integrator $\Rightarrow y \not\to 0$ after an impulse

Design

Open-loop system: $\omega_p = 1$ and $\zeta_p = 0.05$

Specifications: $\omega_m = 0.5$ and $\zeta_m = 0.7$

Sampling interval: $h = 0.5 \quad \Rightarrow \omega_N = 6$

$$u(kh) = -L\hat{x}(kh \mid kh - h) + L_c u_c(kh)$$

Desired closed loop poles (in continuous time)

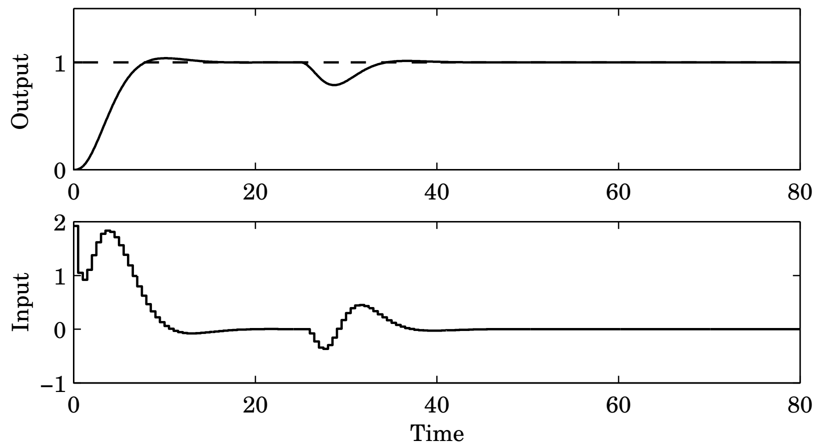
$$(s^2 + 2\zeta_m \omega_m s + \omega_m^2)(s + \alpha_1 \omega_m) = 0 \quad \alpha_1 = 2$$

Observer design

$$(s^2 + 2\zeta_m \alpha_0 \omega_m s + (\alpha_0 \omega_m)^2)(s + \alpha_0 \alpha_1 \omega_m) = 0 \quad \alpha_0 = 2$$

Design

Feedback from observed states. Observer twice as fast as closed loop dynamics.



Summary

- State feedback
  - Reachability
  - Ackermann's formula
- Observers
  - Full model
  - Reduced order
  - Delays in the estimator
- General disturbances
- The servo case
- Two degree-of-freedom controller