Lecture 2: z-transform and I/O models

- Shift operator
- I/O models
- Direct sampling
- z-transform
- Poles and zeros
- Selection of sampling interval
- Frequency response of sampled-data systems
- Lyapunov theory for discrete-time systems

Shift-operator

Forward shift operator

$$qf(k) = f(k+1)$$

Backward shift (delay) operator

$$q^{-1}f(k) = f(k-1)$$

The range of the shift operator is double infinite sequences

Compare with the differential operator $p = \frac{d}{dt}$

Shift-operator calculus

$$\begin{aligned} y(k+na) + a_1 y(k+na-1) + \cdots + a_{na} y(k) \\ = b_0 u(k+nb) + \cdots + b_{nb} u(k) \end{aligned}$$

where $na \geq nb$. Using the shift operator gives

$$(q^{na} + a_1 q^{na-1} + \cdots + a_{na}) y(k) = (b_0 q^{nb} + \cdots + b_{nb}) u(k)$$

Introduce the polynomials

$$\begin{aligned} A(z) &= z^{na} + a_1 z^{na-1} + \cdots + a_{na} \\ B(z) &= b_0 z^{nb} + b_1 z^{nb-1} + \cdots + b_{nb} \end{aligned}$$

the difference equation can be written as

$$\begin{aligned} A(q)y(k) &= B(q)u(k) \\ y(k) &= \frac{B(q)}{A(q)} u(k) \end{aligned}$$

Reciprocal polynomials

$$\begin{aligned} y(k+na) + a_1 y(k+na-1) + \cdots + a_{na} y(k) \\ = b_0 u(k+nb) + \cdots + b_{nb} u(k) \end{aligned}$$

can be written as

$$\begin{aligned} y(k) + a_1 y(k-1) + \cdots + a_{na} y(k-na) \\ = b_0 u(k-d) + \cdots + b_{nb} u(k-d-nb) \end{aligned}$$

Pole excess $d = na - nb$
Reciprocal polynomial

$$A^*(z) = 1 + a_1 z + \cdots + a_{na} z^{na} = z^{na} A(z^{-1})$$

The system description in the backward shift operator

$$\begin{aligned} A^*(q^{-1}) y(k) &= B^*(q^{-1}) u(k-d) \\ y(k) &= \frac{B^*(q^{-1})}{A^*(q^{-1})} u(k-d) \end{aligned}$$

Pulse-transfer function operator

State-space system

$$x(k+1) = qx(k) = \Phi x(k) + \Gamma u(k)$$

Use the shift operator

$$(qI - \Phi)x(k) = \Gamma u(k)$$

Eliminate $x(k)$

$$y(k) = Cx(k) + Du(k) = \big(C(qI - \Phi)^{-1}\Gamma + D\big)u(k)$$

Pulse-transfer operator

$$H(q) = C(qI - \Phi)^{-1}\Gamma + D$$

In the backward-shift operator

$$H^*(q^{-1}) = C(I - q^{-1}\Phi)^{-1}q^{-1}\Gamma + D = H(q)$$

SISO systems

$$H(q) = C(qI - \Phi)^{-1}\Gamma + D = \frac{B(q)}{A(q)}$$

If no common factors

$$\begin{aligned} \deg A &= n \\ A(q) &= \det[qI - \Phi] \end{aligned}$$

and

$$\begin{aligned} y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) \\ = b_0 u(k) + \cdots + b_n u(k-n) \end{aligned}$$

where a_i are the coefficients of the characteristic polynomial of Φ.

Poles, zeros, and system order

$$H(q) = C(qI - \Phi)^{-1}\Gamma + D = \frac{B(q)}{A(q)}$$

Poles: $A(q) = 0$
Zeros: $B(q) = 0$
System order: $\deg A(q)$

Important to use the forward shift operator for poles/zeros, system order, and stability.

The backward shift operator is suited for causality considerations.

Example – Double integrator with delay

$h = 1$ and $\tau = 0.5$ gives

$$\Phi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \Gamma_1 = \begin{pmatrix} 0.375 \\ 0.5 \end{pmatrix} \quad \Gamma_0 = \begin{pmatrix} 0.125 \\ 0.5 \end{pmatrix}$$

Then

$$\begin{aligned} H(q) &= C(qI - \Phi)^{-1}(\Gamma_0 + \Gamma_1 q^{-1}) \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\begin{pmatrix} q-1 & 1 \\ 0 & q-1 \end{pmatrix}}{(q-1)^2} \begin{pmatrix} 0.125 + 0.375q^{-1} \\ 0.5 + 0.5q^{-1} \end{pmatrix} \\ &= \frac{0.125(q^2 + 6q + 1)}{q(q^2 - 2q + 1)} = \frac{0.125(q^{-1} + 6q^{-2} + q^{-3})}{1 - 2q^{-1} + q^{-2}} \end{aligned}$$

Order: 3
Poles: 0, 1, and 1
Zeros: $-3 \pm \sqrt{8}$

How to get $H(q)$ from $G(s)$?

Use Table 2.1

Zero-order hold sampling of a continuous-time system, $G(s)$.

$$H(q) = \frac{b_1 q^{n-1} + b_2 q^{n-2} + \cdots + b_n}{q^n + a_1 q^{n-1} + \cdots + a_n}$$

$G(s)$	$H(q)$
$\frac{1}{s}$	$\frac{h}{q-1}$
$\frac{1}{s^2}$	$\frac{h^2(q+1)}{2(q-1)^2}$
$\frac{a}{s+a}$	$\frac{1-\exp(-ah)}{q-\exp(-ah)}$

z-transform

Definition of z-transform
Consider the discrete-time signal $\{f(kh) : k = 0, 1, \ldots\}$.

$$\mathcal{Z}(f(kh)) = F(z) = \sum_{k=0}^{\infty} f(kh) z^{-k}$$

The inverse transform is given by

$$f(kh) = \frac{1}{2\pi i} \oint F(z) z^{k-1}\, dz$$

where the contour of integration encloses all singularities of $F(z)$. Maps a *semi-infinite time sequence* into a function of a complex variable

Example

Let $y(kh) = kh$ for $k \geq 0$. Then

$$\begin{aligned} Y(z) &= 0 + hz^{-1} + 2hz^{-2} + \cdots \\ &= h(z^{-1} + 2z^{-2} + \cdots \\ &= \frac{hz}{(z-1)^2} \end{aligned}$$

- Similarities with Laplace transform
- Common in applied mathematics
- How the theory of sampled-data systems started

Properties of z-transform

1. Definition.
   $F(z) = \sum_{k=0}^{\infty} f(kh) z^{-k}$
2. Time shift.
   $\mathcal{Z} q^{-n} f = z^{-n} F$
   $\mathcal{Z}\{q^n f\} = z^n (F - F_1)$
   where $F_1(z) = \sum_{j=0}^{n-1} f(jh) z^{-j}$
3. Initial value theorem.
4. Final-value theorem.
5. Convolution.
   $\mathcal{Z}(f * g) = \mathcal{Z} \sum_{n=0}^{k} f(n) g(k-n) = (\mathcal{Z} f)(\mathcal{Z} g)$

Pulse-transfer function

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = Cx(k) + Du(k)$$

Take the z-transform of both sides

$$z\left(\sum_{k=0}^{\infty} z^{-k}x(k) - x(0)\right) = \sum_{k=0}^{\infty} \Phi z^{-k}x(k) + \sum_{k=0}^{\infty} \Gamma z^{-k}u(k)$$

Hence

$$z(X(z) - x(0)) = \Phi X(z) + \Gamma U(z)$$
$$X(z) = (zI - \Phi)^{-1}(zx(0) + \Gamma U(z))$$

$$Y(z) = C(zI - \Phi)^{-1}zx(0) + (C(zI - \Phi)^{-1}\Gamma + D)U(z)$$

Pulse-transfer function

$$H(z) = C(zI - \Phi)^{-1}\Gamma + D$$

Why both q and z?

- Could be sufficient with only the shift operator q
- Many books contain the z-transform
- Must be aware of the difficulties with z-transform
- Remember q operator and z complex variable

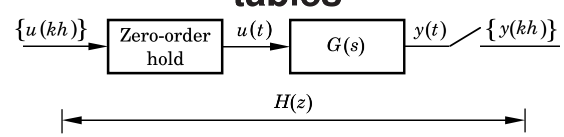
A warning

!!!Use the z-transform tables correctly!!!!

$f(kh)$	$\mathcal{L}f(t)$	$\mathcal{Z}f(kh)$
$\delta(k)$ (pulse)	–	1
$1 \quad k \geq 0$ (step)	$\frac{1}{s}$	$\frac{z}{z-1}$
kh	$\frac{1}{s^2}$	$\frac{hz}{(z-1)^2}$
$\frac{1}{2}(kh)^2$	$\frac{1}{s^3}$	$\frac{h^2z(z+1)}{2(z-1)^3}$
$e^{-kh/T}$	$\frac{T}{1+sT}$	$\frac{z}{z-e^{-h/T}}$
$1 - e^{-kh/T}$	$\frac{1}{s(1+sT)}$	$\frac{z(1-e^{-h/T})}{(z-1)(z-e^{-h/T})}$

Warning. Notice that $\mathcal{Z}f$ in the table does not give the zero-order-hold sampling of a system with the transfer function $\mathcal{L}f$.

Calculation of $H(z)$ given $G(s)$ using z-transform tables



1. Determine the step response of the system with the transfer function $G(s)$.
2. Determine the corresponding z-transform of the step response using the table.
3. Divide by the z-transform of the step function.

$$Y(s) = \frac{G(s)}{s} \to \tilde{Y} = \mathcal{Z}(\mathcal{L}^{-1}Y)$$
$$\to H(z) = (1 - z^{-1})\tilde{Y}(z)$$

Double integrator – Sampling using table

Transfer function $G(s) = 1/s^2$

Introduce the step

$$Y(s) = \frac{1}{s^3}$$

Use the table

$$\tilde{Y} = \mathcal{Z}(\mathcal{L}^{-1}Y) = \frac{h^2 z(z+1)}{2(z-1)^3}$$

Get the pulse transfer function

$$H(z) = (1 - z^{-1})\tilde{Y}(z) = \frac{h^2(z+1)}{2(z-1)^2}$$

Formula for $H(z)$

The following formula can be derived:

$$H(z) = \frac{z-1}{z} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{sh}}{z - e^{sh}} \frac{G(s)}{s} \, ds$$

If $G(s)$ goes to zero at least as fast as $|s|^{-1}$ for a large s and has distinct poles (none at the origin)

$$H(z) = \sum_{s=s_i} \frac{1}{z - e^{sh}} \operatorname{Res} \left\{ \frac{e^{sh} - 1}{s} \right\} G(s)$$

where s_i are the poles of $G(s)$
Multiple poles influence the calculations of the residues.

Modified z-transform

Can be used to determine intersample behavior

Definition: Modified z-transform

$$\tilde{F}(z, m) = \sum_{k=0}^{\infty} z^{-k} f(kh - h + mh), \qquad 0 \le m \le 1$$

The inverse transform is given by

$$f(nh - h + mh) = \frac{1}{2\pi i} \int_{\Gamma} \tilde{F}(z, m) z^{n-1} dz$$

Γ encloses all singularities of the integrand

Interpretation of poles and zeros

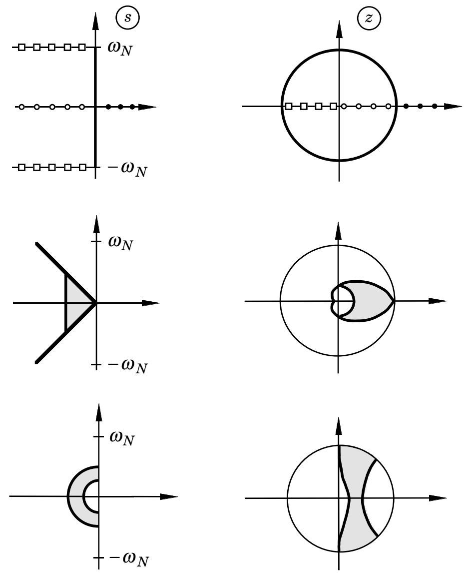
Poles:

- A pole $z = a$ is associated with the time function $z(k) = a^k$
- A pole $z = a$ is an eigenvalue of Φ

Zeros:

- A zero $z = a$ implies that the transmission of the input $u(k) = a^k$ is blocked by the system
- A zero is related to how inputs and outputs are coupled to the states

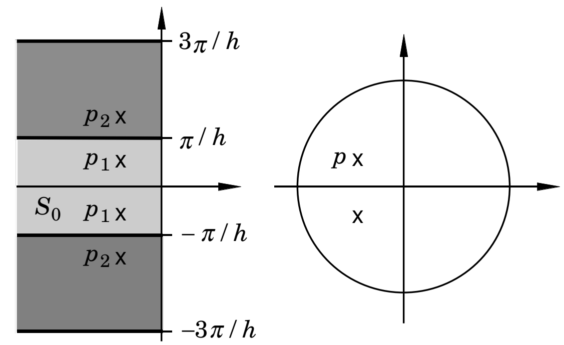
Transformation of poles $\lambda_i(\Phi) = e^{\lambda_i(A)h}$



New evidence of alias problem

$$z = e^{sh}$$

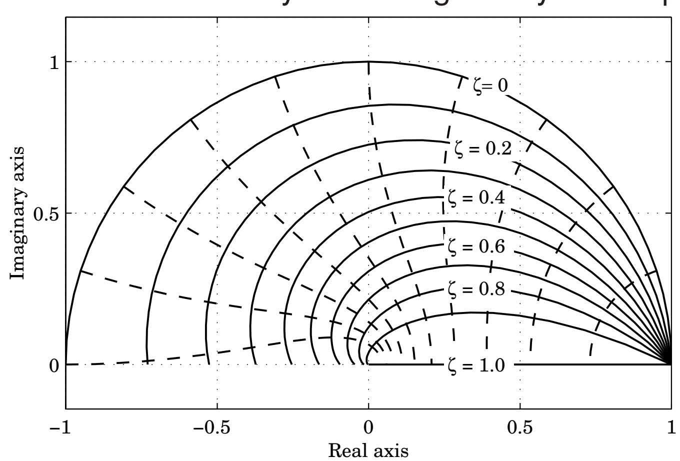
Several points in the s-plane is mapped into the same point in the z-plane.
The map is not bijective



Sampling of a second order system

$$\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Poles of the discrete-time system are given by the mapping



Transformation of zeros

More difficult than poles
In general, more sampled zeros than continuous
For short sampling periods $z_i \approx e^{s_i h}$
For large s then $G(s) \approx s^{-d}$
where $d = \deg A(s) - \deg B(s)$
The $r = d - 1$ sampling zeros go to the zeros of the polynomials Z_d

d	Z_d
1	1
2	$z + 1$
3	$z^2 + 4z + 1$
4	$z^3 + 11z^2 + 11z + 1$
5	$z^4 + 26z^3 + 66z^2 + 26z + 1$

Systems with unstable inverse

Continuous-time system is nonminimum phase if it has right half-plane zeros or time delays.

A discrete-time system is in many books defined to be non-minimum phase if it has zeros outside the unit disc

We will use the following notation:

Definition – Unstable inverse
A discrete-time system has an unstable inverse if it has zeros outside the unit disc

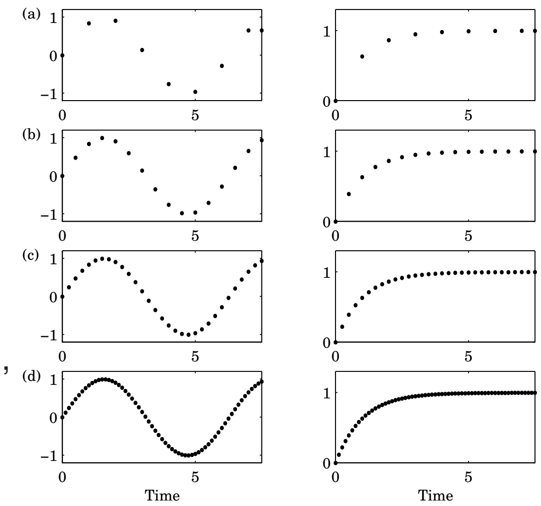
Nonminimum phase $\overset{?}{\leftrightarrow}$ Unstable inverse

Selection of sampling period

Number of samples per rise time

$$N_r = \frac{T_r}{h} \approx 4 - 10$$

The rise times of the signals are $T_r = 1$.
a) $N_r = 1$, b) $N_r = 2$, c) $N_r = 4$, d) $N_r = 8$



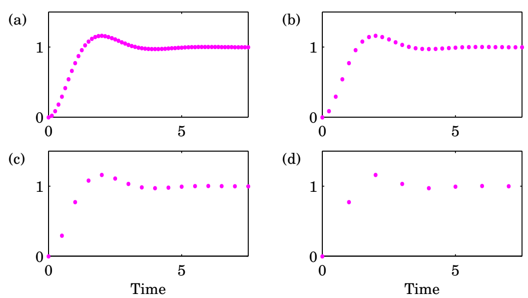
Second order system

$$N_r = \frac{T_r}{h} \approx 4 - 10$$

Corresponds to (for dominating modes)

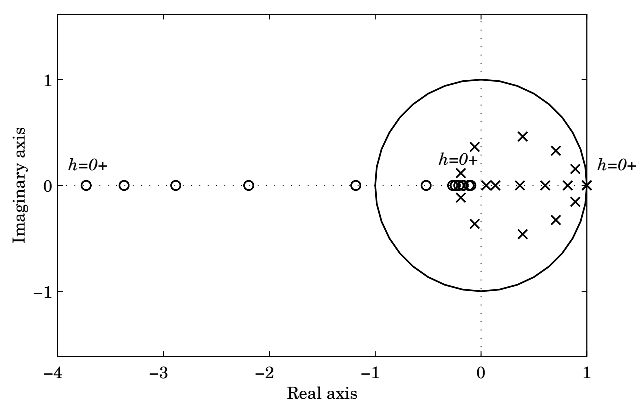
$$\omega_0 h \approx 0.2 - 0.6$$

$\zeta = 0.5$, $\omega_0 = 1.83$ gives $T_r = 1$;
a) $h = 0.125$ ($\omega_0 h = 0.23$)
b) $h = 0.25$ ($\omega_0 h = 0.46$)
c) $h = 0.5$ ($\omega_0 h = 0.92$)
d) $h = 1.0$ ($\omega_0 h = 1.83$)



Pole-zero variation with h

$$G(s) = \frac{1}{(s+1)(s^2+s+1)}$$



$h = 0.0001, 0.2, 0.5, 1, 2,$ and 3

Nyquist and Bode diagrams

Nyquist curve: $H(e^{i\omega h})$
for $\omega h \in [0, \pi]$, i.e. up to ω_N

- Periodic
- Interpretation
- Higher order harmonics
- Discuss more in connection with Chapter 7

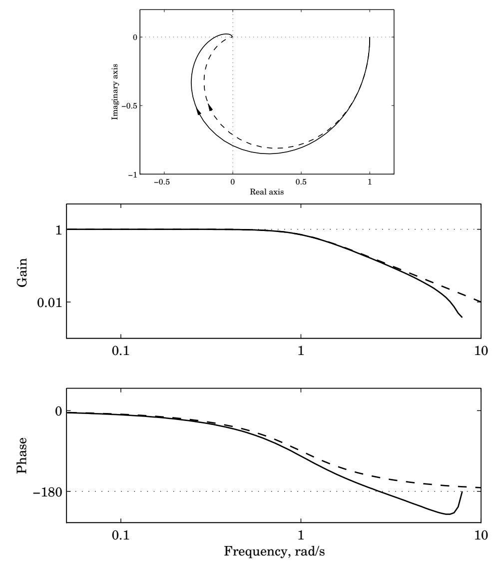
Example

$$G(s) = \frac{1}{s^2 + 1.4s + 1}$$

Zero-order hold sampling
$h = 0.4$

$$H(z) = \frac{0.066z + 0.055}{z^2 - 1.450z + 0.571}$$

Continuous-time (dashed), discrete-time (full)



A. M. Lyapunov
1857–1918

Lyapunov theory

Consider the system

$$x(k+1) = f(x(k)), \qquad f(0) = 0$$

Monotonic convergence $\|x(k+1)\| < \|x(k)\|$ a too strong condition for stability

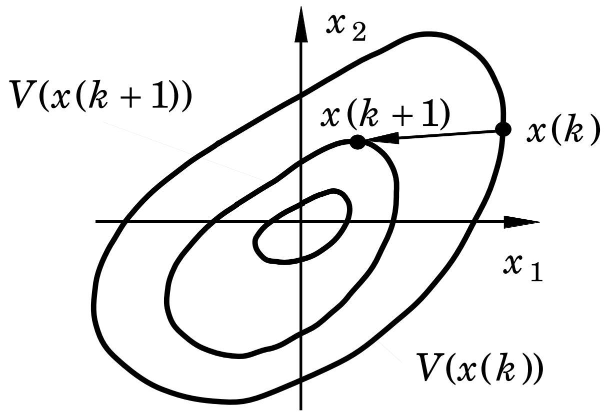
Find other "norm", a Lyapunov function $V(x)$

- $V(x)$ is continuous in x and $V(0) = 0$
- $V(x)$ is positive definite
- $\Delta V(x) = V(f(x)) - V(x)$ is negative definite
- $V(x) \to \infty, \quad |x| \to \infty$

Existence of Lyapunov function implies asymptotic stability for the *solution* $x = 0$

Geometric interpretation

$$x(k+1) = f(x(k)), \qquad f(0) = 0$$



Linear system

$$x(k+1) = \Phi x(k)$$

$$V(x) = x^T P x \qquad P > 0$$

$$\Delta V(x) = V(\Phi x) - V(x) = x^T \Phi^T P \Phi x - x^T P x$$

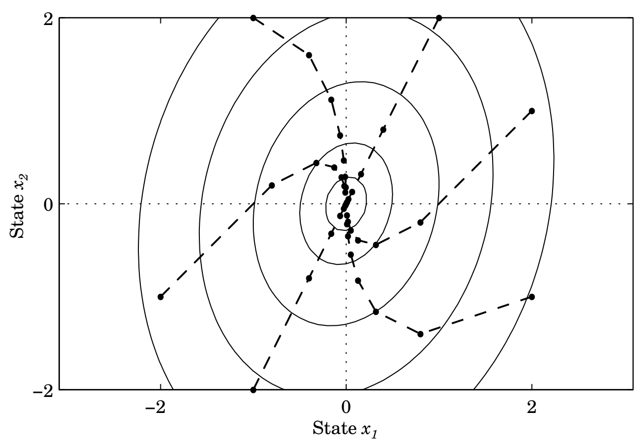
$$= x^T \left(\Phi^T P \Phi - P\right) x = -x^T Q x$$

V is a Lyapunov function iff there exists a $P > 0$ that satisfies the *Lyapunov equation*

$$\Phi^T P \Phi - P = -Q \qquad Q > 0$$

Example

$$\Phi = \begin{pmatrix} 0.4 & 0 \\ -0.4 & 0.6 \end{pmatrix} \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



Summary

- Piecewise constant input and periodic sampling gives time-invariant discrete-time system
- Solution of the system equation, $\lambda(\Phi)$
- Shift operator q and pulse transfer *operator*
- z-transform and pulse transfer *function*
- Be careful with z-transform tables
- Poles, zeros, and system order
- Selection of sampling period

$$N_r = \frac{T_r}{h} \approx 4 - 10$$

$$\omega_0 h \approx 0.2 - 0.6$$

- Frequency function