Positive Systems: An Introduction

Christian Grussler

April 24, 2013

Outline

Positive Systems

Let us consider a continuous linear time-invariant system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$.

Definition: Positive system

A linear system (A, B, C, D) is called (internally) *positive* if and only if its state and output are non-negative for every non-negative input and every non-negative initial state.

Theorem: Positivity [Luenberger, 1979]

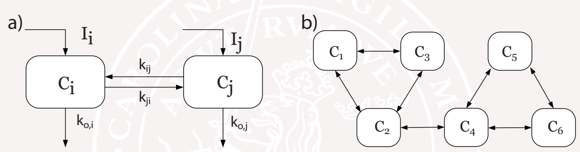
A (cont.) linear system (A, B, C, D) is positive if and only if A is a Metzler-matrix and $B, C, D \geqq 0$.

"[...]the positivity property just defined, is always nothing but the immediate consequence of the nature of the phenomenon we are dealing with. A huge number of examples are just before our eyes." (Farina, 2002)

- Network flows: traffic, transport, communication, etc.
- Social science: population models
- Biology/Medicine: Nitrade models, proteins, etc.
- Economy: stochastic models, money flow, etc.
- Discretization of PDEs: heat equation
- Process Control

Involves at least three of our **LCCCgroups**:
$\longrightarrow$ PowerWindBuild, DistTraffic, EmbedCloud.

Example: Compartmental System



$$\dot{x}_i(t) = -k_{o,i}x_i(t) + \sum_{j \neq i}^{n} [k_{ij}x_j(t) - k_{ji}x_i(t)] + \underbrace{\sum_{j=1}^{m} b_{ij}u_j(t)}_{I_i :=}$$

Non-negative matrices

Theorem: Perron-Frobenius (1907,1912)

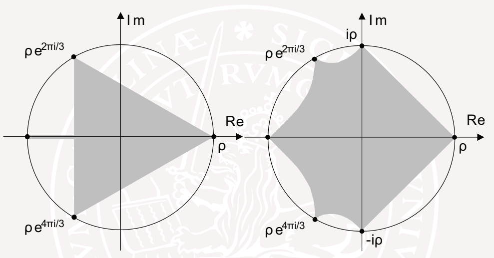
Let $A \gg 0$, then there exists a $\lambda_0 > 0$ and a $x_0 \gg 0$ such that

1. $Ax_0 = \lambda_0 x_0$
2. $\lambda_0 > |\lambda|, \quad \forall \lambda \in \sigma(A) \setminus \{\lambda_0\}$.

In case of $A \geqq 0$, the same statements can be made by replacing the strict relations with $\geq$ and $\geqq$, respectively.

Remark: If $A \gg 0$, then λ_0 and the number of non-negative eigenvectors is simple.
Moreover, the dominant eigenvalues of a non-negative matrix A are all the roots of $\lambda^k - \rho(A)^k = 0$ for some $k = 1, \dots, n$.

Karpelevich: On the characteristic roots of matrices with non-negative elements (1951).



The intersections points are: $\rho e^{2\pi i \frac{a}{b}}$, where a and b are coprime integers with $0 \leq a \leq b \leq n$, $n > 1$.

The arcs are given by: $\lambda^{b_2}(\lambda^{b_1} - s)^{\left[\frac{n}{b_1}\right]} = (1-s)^{\left[\frac{n}{b_1}\right]}\lambda^{b_1\left[\frac{n}{b_1}\right]}$, with $b_1 \leq b_2$, $0 \leq s \leq 1$.

Stability

Berman and Plemmons: Nonnegative Matrices in the Mathematical Sciences.

The Metzler-matrix property gives a scalable stability verification. The following are equivalent:

- A is Hurwitz
- $\exists D > 0$ diagonal : $AD + DA^T < 0$
- $\exists \xi \in \mathbb{R}^n_{>0} : A\xi \ll 0$
- $(-A)^{-1} \geqq 0$

Robustness: D-Stability and connectively stable

Farina and Rinaldi: Positive Linear Systems.

Theorem: D-stability

Every positive system is D-stable, i.e. if $\dot{x}(t) = Ax(t)$ is stable, then also $\dot{x}(t) = DAx(t)$, with diagonal $D > 0$ remains stable.

Theorem: Connectively stable

Every positive system is connectively stable, i.e. if A is Hurwitz, then also $A - \Delta A$ is Hurwitz, with $0 \leq \Delta a_{ij} \leq a_{ij}$.

Robustness: Structured Uncertainty

Son and Hinrichsen: Robust Stability of Positive Continuous Time Systems (2007).

A norm $\|\cdot\|$ on $\mathbb{K}^n$ is said to be monotonic, if it satisfies

$$|x| \leqq |y| \Rightarrow \|x\| \leq \|y\|.$$

Theorem: Real stability radius

Suppose $A \in \mathbb{R}^{n\times n}$ is a Hurwitz Metzler-matrix, $D \in \mathbb{R}_+^{n\times l}$, $E \in \mathbb{R}_+^{q\times n}$ and $\mathbb{C}^l, \mathbb{C}^q$ are equipped with monotonic norms. Then

$$r_{\mathbb{C}}(A; D, E) = r_{\mathbb{R}}(A; D, E) = r_{\mathbb{R}_+}(A; D, E) = \|EA^{-1}D\|_{\text{ind}}^{-1},$$

where $r_{\mathbb{K}}(A; D, E) = \inf\{\|\Delta\|_{\text{ind}}; \Delta \in \mathbb{K}^{l\times q}, \mu(A + D\Delta E) \geq 0\}$.

Control: Feedback

Tanaka and Langbort: The Bounded Real Lemma for Internally Positive Systems and H-Infinity Structured Static State Feedback (2011).

Similar, as before, a diagonal solution exists for the bounded real lemma:

$$\|G\|_\infty < \gamma \Leftrightarrow \exists P > 0 \text{ diagonal}: \begin{pmatrix} A^T P + PA + C^T C & PB \\ B^T P & -\gamma^2 I \end{pmatrix} < 0$$

Hence, considering the positive system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_1 w + D_2 u \end{aligned}$$

with feedback law $u = Kx$, we can minimize the gain from w to z under the constraint of positivity-preservation. This even allows to assign a structure to K.

Control: Feedback

Ebihara, Peaucelle, and Denis Arzelier: Optimal L_1-Controller Synthesis for Positive Systems and Its Robustness Properties (2012).

Moreover, if $B_1, D_1 \gg 0$, then the optimal solution $K^\star$ is robust under the variation over $B_1, D_1 \geqq 0$.

For the static output-feedback-case with $u = ky$ look at *Rami: Solvability of static output-feedback stabilization for LTI positive systems (2011).*

Questions:

- How to obtain the same results independent of a positive realization?
- When is an optimal solution one, that preserves positivity?
- What can be done with PID-control?

Realizability

Ohta, Maeda, and Kodama: Reachability, observability, and realizability of continuous-time positive systems (1984).

In the following we consider the SISO system (A, b, c).

Definition: Reachable cone

Let R be the set of all points that can be reached within finite time from the origin by nonnegative inputs, i.e.

$$R = \text{cone}(b, Ab, A^2 b, \dots)$$

Definition: Observable cone

Let O be the set of all states, that cause a nonnegative output for all $k \in \mathbb{N}$ if $u(t) = 0$, i.e.

$$O := \left\{x | c^T A^k x \geq 0, k = 0, 1, \dots\right\},$$

Theorem: Positive Realization

Let (A, b, c) be a minimal realization of transfer function $H(z)$. Then $H(z)$ has a positive realization if and only if there exists a polyhedral proper cone K such that

- $AK \subset K$
- $R \subset K \subset O$.

Remark: Once K is found, we can denote by K_M the matrix spanning the cone K. Then a positive realization (A_+, b_+, c_+) is given by

$$AK_M = K_M A_+, \quad b = K_M b_+, \quad c_+ = cK_M.$$

Obviously, R is A-invariant and because of the non-negativity of the impulse response, we have that $R \subset O$. However, R is not always polyhedral!
In fact there are systems with non-negative impulse response, which do not allow for a positive realization:

$$A = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad c^T = \begin{pmatrix} 0.5 \\ 0.5 \\ 1 \end{pmatrix}$$

with $\frac{\phi}{\pi} \in \mathbb{R} \setminus \mathbb{Q}$ and $h_k = 1 + \cos[(k-1)\phi]$.

Anderson et. al.: Nonnegative realizations of a linear system with nonnegative impulse response (1995).

Theorem: Realization of primitive transfer functions

If $H(z)$ has a unique dominant pole, then $H(z)$ has a finite positive realization.

Farina: On the existence of a positive realization (1996).

Theorem: Complete Algo. for pos. disc. SISO systems

Let $H^{(0)}(z)$ be a strictly proper transfer function of order n. Then $H^{(0)}(z)$ has a positive realization if and only if

1. $h^{(0)}(k)$ is non-negative
2. all $H^{(0,i_0,i_1,\dots,i_q)}(z)$ have a unique dominant pole.

Question:

- Is there an algorithm, that is
  1. Numerically more efficient
  2. Gives a minimal realization
  3. Allows us to choose non-minimal modes manually
  4. Works also in continuous time?
- Can it help to realize part of a non-positive system positively?

Model Reduction

Reis, and Virnik: Positivity preserving balanced truncation for descriptor systems

Find diagonal $P, Q \geq 0$ such that

$$AP + PA^T + BB^T \leq 0$$
$$A^T Q + QA + C^T C \leq 0$$

Advantage:

+ Generic approach: Works for cont. and disc. time MIMO descriptor systems.
+ Preserves the meaning of the states.

Disadvantages:

- Inapplicable for large scale systems: Solving LMIs is expensive.
- Dependence on a positive realization.
- Large errors and non-minimal approximations: Balancing is a permutation.

Grussler, Damm: A Symmetry Approach for Balanced Truncation of Positive Linear Systems.

Theorem: Positive balanced truncation to order 1

Let (A_1, B_1, C_1, D_1) be the reduced system of order one obtained by balanced truncation of a positive system. Then $(A_1, |B_1|, |C_1|, D_1)$ is a positive, asymptotically stable realization.

Theorem: Positivity of symmetric systems

Let (A, b, c) be an asymptotically stable SISO system with $A = A^T$ and $c = kb^T$, $k > 0$. Then the system possesses a positive realization, which can be computed by Lanczos algorithm.

Theorem: Sign-symmetry of balanced SISO-systems
[Fernando, and Nicholson, TAC 1983]

Let $G(s)$ be the transfer function of an arbitrary SISO-system. Then a balanced realization (A, b, c) of $G(s)$ is sign-symmetric, i.e.

$$|A| = |A^T| \quad \text{and} \quad |b| = |c^T|.$$

Questions:

- Can positivity help to reduce a system (not necessarily positivity-preserving)?
- How does it depend on the dominant poles?
- What about Hankel optimal approximation?

What I did not show you:

- Positive systems with time-delays,
- Switched positive systems,
- Positive systems on time scales,
- Funnel control of positive systems,
- 2D positive systems,
- Behavior approach for positive systems,
- Bi-linear positive systems,
- Discretization of PDEs with positivity
- Non-linear positive system
- Infinite dimensional positive systems

External Positivity

Definition: Externally positive system

A linear system (A, B, C, D) is called **externally** *positive* if and only if its forced output (w.r.t. to the zero initial state) is nonnegative for every nonnegative input.

Theorem: External Positivity

A linear system (A, B, C, D) is **externally** positive if and only its impulse response is non-negative.

Properties

From the non-negativity of the impulse response it follows:

- $G(0) = \int_0^\infty g(t)dt = \|G\|_\infty$
- The dominant pole is real.
- $\forall t \geq 0 : Ce^{-At}B \geq 0 \Longleftrightarrow \forall s \geq 0 : (-1)^k G^k(s) > 0$
- No real zero is larger than the dominant pole.
- There are no over- or undershoots in the step-response.

It seems externally positive systems are characterized on the real line and the $j\omega$-axis.

Questions:

- Can an iterative realization algorithm for positive systems give a reasonable reduced model for external positive systems?
- Can infinite dimensional theory of positive systems help?

Positive Dominance

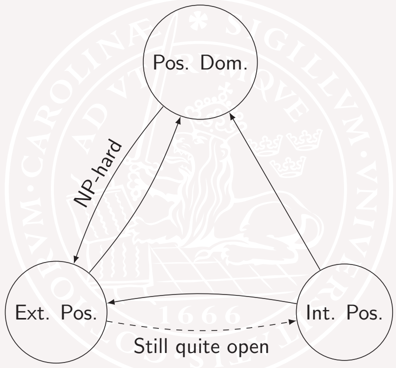
Definition: Positively dominated system

A system with transfer matrix $G(s)$ is called *positively dominated* if every matrix entry satisfies $|G_{jk}(i\omega)| \leq G_{jk}(0)$.

Properties:

- The set of positively dominated systems is convex (as for internal/external positivity).
- $(I - G)^{-1}$ is pos. dom. $\Leftrightarrow \exists \xi \in \mathbb{R}^n_+ : G(0)\xi \ll \xi$

Summary



Some new definitions of positivity

Definition 1: Monotone frequency response (Karl Johan's)

A system with transfer function $G(s)$ has a *monotone frequency response* if $|G(i\omega)|$ and $\arg(G(i\omega))$ are monotone in ω.

Definition 2: Complex monotonicity

A system with transfer function $G(s)$ is called *complex monotone* if $|G(x_1 + iy_1)| \geq |G(x_2 + iy_2)|$ whenever $0 \leq x_1 \leq x_2$ and $0 \leq y_1 \leq y_2$.