Lecture 4 – Nonlinear Control

Nonlinear Controllability

Material

Notes

Handout from Nonlinear Control Theory, Torkel Glad (Linköping)

Nonlinear System

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}$$

Important special affine case:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

f : drift term

g : input term

Basic Result: Linearization at (x_0, u_0)

$$\dot{x} = f(x) + g(x)u$$

Theorem: Suppose $f(x_0) + g(x_0)u_0 = 0$. If

$$\begin{aligned} \dot{z} &= Az + Bv \\ A &= \frac{\partial f}{\partial x}(x_0) + \frac{\partial g}{\partial x}(x_0)u_0 \\ B &= g(x_0) \end{aligned}$$

is controllable, then for all $T, \epsilon > 0$ the set

$$X_{T,\epsilon} = \{x(T);\ \ |u - u_0| < \epsilon\}$$

contains a neighborhood of x_0.

Rolling Penny

$$
\begin{aligned}
\dot{x} &= u_1 \cos(\theta) \\
\dot{y} &= u_1 \sin(\theta) \\
\dot{\varphi} &= u_1 \\
\dot{\theta} &= u_2
\end{aligned}
$$

penny

The linearization is not controllable (check)

Can the penny be moved sideways in small time (keeping the head up)?

Rolling Penny

Yes it can. But it is not obvious.

Non-holonomic constraints $a(z)\dot{z} = 0$

$$\begin{pmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \cos\theta & \sin\theta & -1 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\varphi} \\ \dot{\theta} \end{pmatrix} = 0$$

Holonomic constraints $h(z) = 0 \implies h_z \dot{z} = 0$.

Main new object: Lie Bracket of vector fields

Consider two vector fields $\dot{x} = f(x)$ and $\dot{x} = g(x)$

Lie-bracket. New vector field

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

Why is it interesting?

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2$$

Controllability: If the Liebracket "tree" has full rank, then the system is controllable

Example

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 \pm x_2 u_1 \end{aligned}$$

This means $g_1 = \begin{pmatrix} 1 \\ 0 \\ \pm x_2 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 0 \\ 1 \\ x_1 \end{pmatrix}$

$$[g_1, g_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \pm x_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \pm 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ x_1 \end{pmatrix}$$

Example

Hence at $x = 0$ we have

$$g_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad [g_1, g_2] = \begin{pmatrix} 0 \\ 0 \\ 1 - \pm 1 \end{pmatrix}$$

With the minus-sign the three vector fields span R^3, and we have controllability.

With the plus-sign the system is not controllable, in fact it can be seen that $x_1^2 + x_2^2 - 2x_3$ is an invariant.

Some more notation

$$X^t(p) = \text{ solution to } \dot{x} = X(x), x(0) = p$$

X^t is smooth. $X^0 = \text{ id}$

$$L_X(g) = X(g) = \sum_{i=1}^{n} X_i \frac{\partial g}{\partial x_i} = \lim_{h \to 0} \frac{g(X^h(p)) - g(p)}{h}$$

$L_{\alpha X + \beta Y} = \alpha L_X + \beta L_Y, \;\; \alpha, \beta \in R$

Example

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{aligned}
$$

$$
\begin{aligned}
\dot{y} &= \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}(f + gu) = L_{f+gu}h \\
&= L_f h + u L_g h \\
y^{(k)} &= (L_{f+gu})^k h
\end{aligned}
$$

Lie-Brackets

$$[X,Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$$

$$X \sim \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}; \qquad Y \sim \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$$

$$[X,Y] = \frac{\partial Y}{\partial x} X - \frac{\partial X}{\partial x} Y$$

Another example

$$
\begin{aligned}
X &= \begin{pmatrix} \cos\phi \\ r \end{pmatrix} \sim \cos\phi \frac{\partial}{\partial r} + r \frac{\partial}{\partial \phi} \\
Y &= \begin{pmatrix} r \\ 1 \end{pmatrix} \sim r \frac{\partial}{\partial r} + \frac{\partial}{\partial \phi} \\
[X, Y] &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\phi \\ r \end{pmatrix} - \begin{pmatrix} 0 & -\sin\phi \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\phi - \sin\phi \\ -r \end{pmatrix} \sim (\cos\phi - \sin\phi) \frac{\partial}{\partial r} - r \frac{\partial}{\partial \phi}
\end{aligned}
$$

Lie-Brackets

Why are Lie-brackets so fundamental?

$\dot{x} = g_1 u_1 + g_2 u_2$

$$
\begin{aligned}
(u_1(t), u_2(t)) &= \begin{cases} (1,0) & t \in [0,h) \\ (0,1) & t \in [h,2h) \\ (-1,0) & t \in [2h,3h) \\ (0,-1) & t \in [3h,4h) \end{cases} \\
x(4h) &= x_0 + h^2 [g_1, g_2] + O(h^3)
\end{aligned}
$$

Trotters Product Formula

$$\Phi^t_{[X,Y]} = \lim_{n\to\infty} \left(\Phi^{\sqrt{\frac{t}{n}}}_{-Y} \Phi^{\sqrt{\frac{t}{n}}}_{-X} \Phi^{\sqrt{\frac{t}{n}}}_{Y} \Phi^{\sqrt{\frac{t}{n}}}_{X} \right)^n$$

Proof sketch

$$\left(1 + \frac{tf}{n} + o\left(\frac{tf}{n}\right)\right)^n \to e^{tf}$$

Some Lie-Bracket Formulas

$$
\begin{aligned}
&[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X \\
&[X, Y] = -[Y, X] \\
&[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0 \\
&L_X Y = [X, Y] = \lim_{h \to 0} \frac{1}{h}[X_*^{-h} Y - Y]
\end{aligned}
$$

$$X_*^{-h} Y = \sum_{n=0}^{\infty} \mathrm{ad}_X^n Y \frac{h^n}{n!} = Y + h[X, Y] + \frac{h^2}{2}[X, [X, Y]] \ldots$$

related to

$$e^A e^B = e^C; \qquad C = A + B + \frac{1}{2}[A, B] + \ldots$$

Park Your Car Using Lie-Brackets!

$$\begin{aligned}
(x, y) &: \quad \text{position} \\
\phi &: \quad \text{direction of car} \\
\theta &: \quad \text{direction of wheels} \\
(x, y, \phi, \theta) &\in \quad R^2 \times S^1 \times [\theta_{\text{min}}, \theta_{\text{max}}]
\end{aligned}$$

Parking cont'd

$$g_1 = \text{Steer} \quad := \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$g_2 = \text{Drive} \quad := \quad \begin{pmatrix} \cos(\phi + \theta) \\ \sin(\phi + \theta) \\ \sin(\theta) \\ 0 \end{pmatrix}$$

$$[\mathsf{Steer}, \mathsf{Drive}] = \dots \qquad = \begin{pmatrix} -\sin(\phi+\theta) \\ \cos(\phi+\theta) \\ \cos(\theta) \\ 0 \end{pmatrix}$$

$$:= \mathsf{Wriggle}$$

Define

$$\text{Slide} := \begin{pmatrix} -\sin(\phi+\theta) \\ \cos(\phi+\theta) \\ 0 \\ 0 \end{pmatrix}$$

We have

$$\text{Slide}^t(x, y, \phi, \theta) = (x - t\sin(\phi), x + t\cos(\phi), \phi, \theta)$$

An easy calculation (exercise) shows that

$$[\text{Wriggle}, \text{Drive}] = \text{Slide}$$

Fundamental Parking Theorem

You can get out of any parking lot that is larger than the car. Use the following control: Wriggle, Drive, –Wriggle (this requires a cool head), –Drive (repeat).

Proof: Trotters Product Formula

Linear Systems

$$\dot{x} = Ax + Bu = f(x) + g(x)u$$

$$\begin{aligned}
[f, g] &= [Ax, B] = 0 - AB \\
[g, [f, g]] &= 0 \\
[f, [f, g]] &= [Ax, -AB] = A^2 B \\
&\vdots
\end{aligned}$$

$$\mathrm{Ad}_f^k g = \underbrace{[f, [f, \ldots, [f, g]]]}_{k \text{ Lie-brackets}} = (-1)^k A^k B$$

Controllability Theorem

$$\dot{x} = \sum_i g_i(x) u_i$$

$$C \quad = \quad \text{smallest Lie subalg. containing } \{g_1, \ldots, g_m\}$$

Controllability:

$$\dim C = n \implies \text{can reach open set}$$

With drift term $f(x)$ the theorem is slightly different