Nonlinear Control Theory

Lecture 9

- Periodic Perturbations
- Averaging
- Singular Perturbations

Khalil Chapter (9, 10) 10.3-10.6, 11

Today: Two Time-scales

Averaging

$$\dot{x} \quad = \quad \epsilon f(t, x, \epsilon)$$

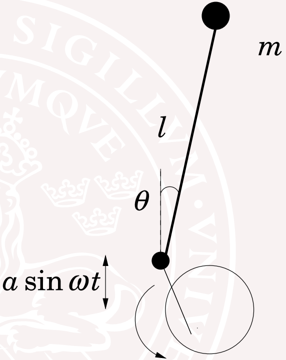
The state x moves slowly compared to f.

Singular perturbations

$$\begin{aligned} \dot{x} &= f(t, x, z, \epsilon) \\ \epsilon \dot{z} &= g(t, x, z, \epsilon) \end{aligned}$$

The state x moves slowly compared to z.

Example: Vibrating Pendulum I



$$x = l \sin(\theta),$$
$$y = l \cos(\theta) - a \sin(\omega t)$$

Newton's law in tangential direction

$$m(l\ddot{\theta} - a\omega^2 \sin\omega t \sin\theta)$$
$$= -mg\sin\theta - k(l\dot{\theta} + a\omega\cos\omega t\sin\theta)$$

(incl. viscous friction in joint)

Let $\epsilon = a/l, \tau = \omega t, \alpha = \omega_0 l/\omega a$, and $\beta = k/m\omega_0$

$$\begin{aligned}
x_1 &= \theta \\
x_2 &= \epsilon^{-1}(d\theta/d\tau) + \cos\tau\sin\theta \\
f_1(\tau,x) &= x_2 - \cos\tau\sin x_1 \\
f_2(\tau,x) &= -\alpha\beta x_2 - \alpha^2\sin x_1 \\
&\quad + x_2\cos\tau\cos x_1 - \cos^2\tau\sin x_1\cos x_1
\end{aligned}$$

the state equation is given by

$$\frac{dx}{d\tau} = \epsilon f(\tau,x)$$

Averaging Assumptions

Consider the system

$$\dot{x} = \epsilon f(t, x, \epsilon), \quad x(0) = x_0$$

where f and its derivatives up to second order are continuous and bounded.

Let x_{av} be defined by the equations

$$\begin{aligned} \dot{x}_{av} &= \epsilon f_{av}(x_{av}), \quad x_{av}(0) = x_0 \\ f_{av}(x) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T f(\tau, x, 0)d\tau \end{aligned}$$

Example: Vibrating Pendulum II

The averaged system

$$\begin{aligned}\dot{x} &= \epsilon f_{av}(x) \\ &= \epsilon \begin{bmatrix} x_2 \\ -\alpha\beta x_2 - \alpha^2 \sin x_1 - \frac{1}{4}\sin 2x_1 \end{bmatrix}\end{aligned}$$

has

$$\frac{\partial f_{av}}{\partial x}(\pi, 0) = \begin{bmatrix} 0 & 1 \\ \alpha^2 - 0.5 & -\alpha\beta \end{bmatrix}$$

which is Hurwitz for $0 < \alpha < 1/\sqrt{2}$, $\beta > 0$.

Can this be used for rigorous conclusions?

Periodic Averaging Theorem

Let f be periodic in t with period T.

Let $x = 0$ be an exponentially stable equilibrium of $\dot{x}_{av} = \epsilon f(x_{av})$.

If $|x_0|$ is sufficiently small, then

$$x(t,\epsilon) \quad = \quad x_{av}(t,\epsilon) + O(\epsilon) \text{ for all } t \in [0,\infty]$$

Furthermore, for sufficiently small $\epsilon > 0$, the equation $\dot{x} = \epsilon f(t,x,\epsilon)$ has a unique exponentially stable periodic solution of period T in an $O(\epsilon)$ neighborhood of $x = 0$.

General Averaging Theorem

Under certain conditions on the convergence of

$$f_{av}(x) \;\; = \;\; \lim_{T\to\infty} \frac{1}{T}\int_0^T f(\tau, x, 0)d\tau$$

there exists a $C > 0$ such that for sufficiently small $\epsilon > 0$

$$|x(t,\epsilon) - x_{av}(t,\epsilon)| \;\; < \;\; C\epsilon$$

for all $t \in [0, 1/\epsilon]$.

Example: Vibrating Pendulum III

The Jacobian of the averaged system is Hurwitz for $0 < \alpha < 1/\sqrt{2}$, $\beta > 0$.

For a/l sufficiently small and

$$\omega \quad > \quad \sqrt{2}\omega_0 l/a$$

the unstable pendulum equilibrium $(\theta, \dot{\theta}) = (\pi, 0)$ is therefore stabilized by the vibrations.

Periodic Perturbation Theorem

Consider

$$\dot{x} = f(x) + \epsilon g(t, x, \epsilon)$$

where f, g, $\partial f/\partial x$ and $\partial g/\partial x$ are continuous and bounded.

Let g be periodic in t with period T.

Let $x = 0$ be an exponentially stable equilibrium point for $\epsilon = 0$.

Then, for sufficiently small $\epsilon > 0$, there is a unique periodic solution

$$\bar{x}(t, \epsilon) = O(\epsilon)$$

which is exponentially stable.

Proof ideas of Periodic Perturbation Theorem

Let $\phi(t, x_0, \epsilon)$ be the solution of

$$\dot{x} = f(x) + \epsilon g(t, x, \epsilon), \quad x(0) = x_0$$

- Exponential stability of $x = 0$ for $\epsilon = 0$, plus bounds on the magnitude of g, shows existence of a bounded solution $\bar{x}$ for small $\epsilon > 0$.
- The implicit function theorem shows solvability of

  $$x = \phi(T, 0, x, \epsilon)$$

  for small ϵ. This gives periodicity of $\bar{x}$.
- Put $z = x - \bar{x}$. Exponential stability of $x = 0$ for $\epsilon = 0$ gives exponential stability of $z = 0$ for small $\epsilon > 0$.

Proof idea of Averaging Theorem

For small $\epsilon > 0$ define u and y by

$$\begin{aligned} u(t,x) &= \int_0^t [f(\tau,x,0) - f_{av}(x)]d\tau \\ x &= y + \epsilon u(t,y) \end{aligned}$$

Then

$$\dot{x} = \dot{y} + \epsilon \frac{\partial u(t,y)}{\partial t} + \epsilon \frac{\partial u(t,y)}{\partial y} \dot{y}$$

$$\begin{aligned} \left[I + \epsilon \frac{\partial u}{\partial y} \right] \dot{y} &= \epsilon f(t, y + \epsilon u, \epsilon) - \epsilon \frac{\partial u}{\partial t}(t,y) \\ &= \epsilon f_{av}(y) + \epsilon^2 p(t,y,\epsilon) \end{aligned}$$

With $s = \epsilon t$,

$$\frac{dy}{ds} = f_{av}(y) + \epsilon q\left(\frac{s}{\epsilon}, y, \epsilon\right)$$

which has a unique and exponentially stable periodic solution for small ϵ. This gives the desired result.

Application: Second Order Oscillators

For the second order system

$$\ddot{y} + \omega^2 y = \epsilon g(y, \dot{y}) \qquad (1)$$

introduce

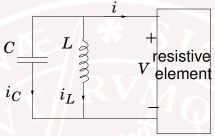
$$\begin{aligned} y &= r \sin\phi \\ \dot{y}/\omega &= r\cos\phi \\ f(\phi, r, \epsilon) &= \frac{g(r\sin\phi, \omega r\cos\phi)\cos\phi}{\omega^2 - (\epsilon/r) g(r\sin\phi, \omega r\cos\phi)\sin\phi} \\ f_{av}(r) &= \frac{1}{2\pi}\int_0^{2\pi} f(\phi, r, 0)d\phi \\ &= \frac{1}{2\pi\omega^2}\int_0^{2\pi} g(r\sin\phi, \omega r\cos\phi)\cos\phi d\phi \end{aligned}$$

Then (1) is equivalent to

$$\frac{dr}{d\phi} = \epsilon f(\phi, r, \epsilon)$$

and the periodic averaging theorem may be applied.

Illustration: Van der Pol Oscillator I



For an ordinary resistance we will get a damped oscillation.
For a negative resistance/admittance chosen as

$$i = h(V) \quad = \quad \underbrace{(-V + \frac{1}{3}V^3)}_{gives\ van\ der\ Pol\ eq.}$$

we get

$$i_C + i_L + i = 0, \qquad i = h(V)$$
$$\Rightarrow$$
$$CL\frac{d^2V}{dt^2} + V + Lh'(V)\frac{dV}{dt} = 0$$

Example: Van der Pol Oscillator I

The vacuum tube circuit equation (a k a the *van der Pol equation*)

$$\ddot{y} + y = \epsilon \dot{y}(1 - y^2)$$

gives

$$\begin{aligned} f_{av}(r) &= \frac{1}{2\pi} \int_0^{2\pi} r \cos\phi (1 - r^2 \sin^2 \phi) \cos\phi d\phi \\ &= \frac{1}{2} r - \frac{1}{8} r^3 \end{aligned}$$

The averaged system

$$\frac{dr}{d\phi} = \epsilon \left(\frac{1}{2} r - \frac{1}{8} r^3 \right)$$

has equilibria $r = 0$, $r = 2$ with

$$\left. \frac{d f_{av}}{dr} \right|_{r=2} = -1$$

so small ϵ give a stable limit cycle, which is close to circular with radius $r = 2$.

Singular Perturbations

Consider equations of the form

$$\begin{aligned} \dot{x} &= f(t, x, z, \epsilon), \quad x(0) = x_0 \\ \epsilon \dot{z} &= g(t, x, z, \epsilon) \quad z(0) = z_0 \end{aligned}$$

For small $\epsilon > 0$, the first equation describes the *slow dynamics*, while the second equation defines the *fast dynamics*.

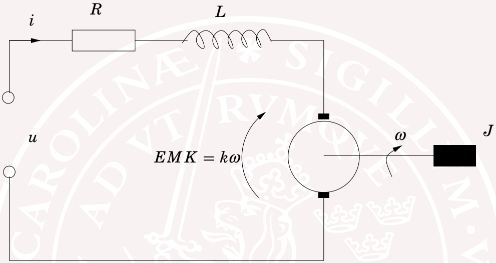
The main idea will be to approximate x with the solution of the *reduced problem*

$$\dot{\bar{x}} = f(t, x, h(t, \bar{x}), 0) \quad \bar{x}(0) = x_0$$

where $h(t, \bar{x})$ is defined by the equation

$$0 = g(t, x, h(t, x), 0)$$

Example: DC Motor I



$$J\frac{d\omega}{dt} = ki$$

$$L\frac{di}{dt} = -k\omega - Ri + u$$

With $x = \omega$, $z = i$ and $\epsilon = Lk^2/JR^2$ we get

$$\dot{x} = z$$

$$\epsilon\dot{z} = -x - z + u$$

Linear Singular Perturbation Theorem

Let the matrix A_{22} have nonzero eigenvalues $\gamma_1, \ldots, \gamma_m$ and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}$.

Then, $\forall \delta > 0 \; \exists \epsilon_0 > 0$ such that the eigenvalues $\alpha_1, \ldots, \alpha_{n+m}$ of the matrix

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21}/\epsilon & A_{22}/\epsilon \end{bmatrix}$$

satisfy the bounds

$$\begin{aligned} |\lambda_i - \alpha_i| &< \delta, \quad i = 1, \ldots, n \\ |\gamma_{i-n} - \epsilon\alpha_i| &< \delta, \quad i = n+1, \ldots, n+m \end{aligned}$$

for $0 < \epsilon < \epsilon_0$.

Proof

A_{22} is invertible, so it follows from the implicit function theorem that for sufficiently small ϵ the Riccati equation

$$\epsilon A_{11}P_\epsilon + A_{12} - \epsilon P_\epsilon A_{21} P_\epsilon - P_\epsilon A_{22} \quad = \quad 0$$

has a unique solution $P_\epsilon = A_{12}A_{22}^{-1} + O(\epsilon)$.

The desired result now follows from the similarity transformation

$$\begin{aligned}
&\begin{bmatrix} I & -\epsilon P_\epsilon \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21}/\epsilon & A_{22}/\epsilon \end{bmatrix} \begin{bmatrix} I & \epsilon P_\epsilon \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} I & -\epsilon P_\epsilon \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} + \epsilon A_{11} P_\epsilon \\ A_{21}/\epsilon & A_{22}/\epsilon + A_{21}P_\epsilon \end{bmatrix} \\
&= \begin{bmatrix} A_0 + O(\epsilon) & 0 \\ * & A_{22}/\epsilon + O(1) \end{bmatrix}
\end{aligned}$$

Example: DC Motor II

In the example

$$\begin{aligned}\dot{x} &= z \\ \epsilon \dot{z} &= -x - z + u\end{aligned}$$

we have

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$A_{11} - A_{12}A_{22}^{-1}A_{21} = -1$$

so stability of the DC motor model for small

$$\epsilon = \frac{Lk^2}{JR^2}$$

is verified.

See Khalil for example where reduced system is stable but fast dynamics unstable.

The Boundary-Layer System

For fixed (t, x) the *boundary layer system*

$$\frac{d\hat{y}}{d\tau} = g(t, x, \hat{y} + h(t, x), 0), \quad \hat{y}(0) = z_0 - h(0, x_0)$$

describes the fast dynamics, disregarding variations in the slow variables t, x.

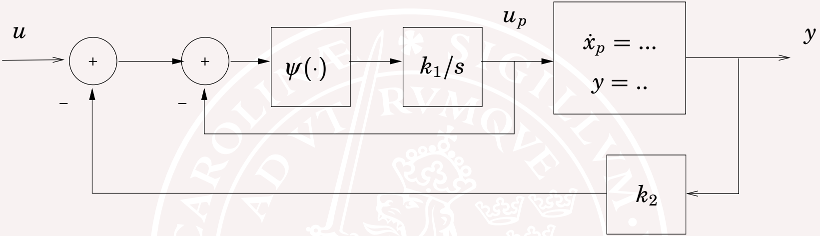
Tikhonov's Theorem

Consider a singular perturbation problem with $f, g, h, \partial g/\partial x \in C^1$. Assume that the reduced problem has a unique bounded solution $\bar{x}$ on $[0, T]$ and that the equilibrium $\hat{y} = 0$ of the boundary layer problem is exponentially stable uniformly in (t, x). Then

$$
\begin{aligned}
x(t, \epsilon) &= \bar{x}(t) + O(\epsilon) \\
z(t, \epsilon) &= h(t, \bar{x}(t)) + \hat{y}(t/\epsilon) + O(\epsilon)
\end{aligned}
$$

uniformly for $t \in [0, T]$.

Example: High Gain Feedback



Closed loop system

$$\begin{aligned} \dot{x}_p &= Ax_p + Bu_p \\ \frac{1}{k_1}\dot{u}_p &= \psi(u - u_p - k_2 C x_p) \end{aligned}$$

Reduced model

$$\dot{x}_p = (A - Bk_2C)x_p + Bu$$

Proof ideas of Tikhonov's Theorem

Replace f and g with F and G that are identical for $|x| < r$, but nicer for large x.

For small ϵ, $G(t, x, y, \epsilon)$ is close to $G(t, x, y, 0)$.

$$\begin{aligned} & y\text{-bound for } G(\cdot,\cdot,\cdot,0)\text{-equation} \\ \Rightarrow \quad & y\text{-bound for } G\text{-equation} \\ \Rightarrow \quad & x, y\text{-bound for } F, G\text{-equations} \end{aligned}$$

For small $\epsilon > 0$, the x, y-solutions of the F, G-equations will satisfy $|x| < r$. Hence, they also solve the f, g-equations

The Slow Manifold

For small $\epsilon > 0$, the system

$$\begin{aligned} \dot{x} &= f(x,z) \\ \epsilon\dot{z} &= g(x,z) \end{aligned}$$

has the invariant manifold

$$z = H(x,\epsilon)$$

It can often be computed approximately by Taylor expansion

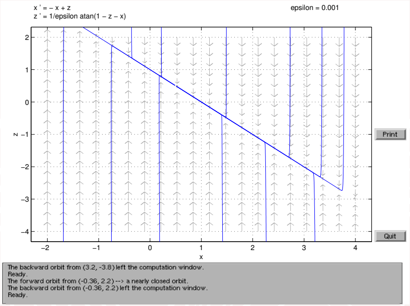
$$H(x,\epsilon) = H_0(x) + \epsilon H_1(x) + \epsilon^2 H_2(x) + \cdots$$

where H_0 satisfies

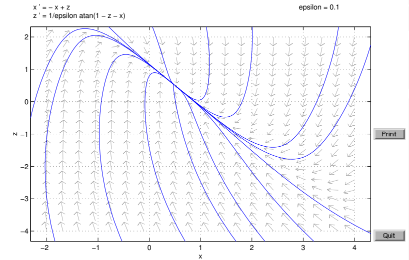
$$0 = g(x,H_0)$$

The Fast Manifold

$$\epsilon = 0.001$$



$$\epsilon = 0.1$$



Example: Van der Pol Oscillator III

Consider

$$\frac{d^2v}{ds^2} - \mu(1-v^2)\frac{dv}{ds} + v = 0$$

With

$$\begin{aligned} x &= -\frac{1}{\mu}\frac{dv}{ds} + v - \frac{1}{3}v^3 \\ z &= v \\ t &= s/\mu \\ \epsilon &= 1/\mu^2 \end{aligned}$$

we have the system

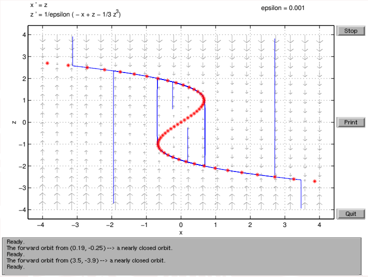
$$\begin{aligned} \dot{x} &= z \\ \epsilon\dot{z} &= -x + z - \frac{1}{3}z^3 \end{aligned}$$

with slow manifold

$$x = z - \frac{1}{3}z^3$$

Illustration: Van der Pol III

Phase plot for van der Pol example $\epsilon = 0.001$



$\epsilon = 0.1$