Lecture 3 – Nonlinear Control

Center Manifold Theorem
Robustness analysis and quadratic inequalities

Material

- Khalil, Ch 8. Center manifold theorem
- lecture notes
- A. Megretski and A. Rantzer, System Analysis via Integral Quadratic Constraints, IEEE Transactions on Automatic Control, 47:6, 1997
- U. Jönsson, Lecture Notes on Integral Quadratic Constraints

- User's guide to μ—toolbox, Matlab

The Center Manifold Theorem

[Khalil ch 8]

What can we do if the linearization $A = \frac{\partial f}{\partial x}$ has zeros on the imaginary axis?

Center Manifold Theory

Assume that a system (possibly via a state space transformation $[x] \to [y^T, z^T]^T$) can be written as

$$
\begin{aligned}
\dot{y} &= A^0 y + f^0(y,z) \\
\dot{z} &= A^- z + f^-(y,z)
\end{aligned}
$$

A^-: asymptotically stable

A^0: eigenvalues on imaginary axis

f^0 and f^- second order and higher terms.

Center Manifold Theorem

Assume $[y^T, z^T]^T = 0$ is an equilibrium point. For every $k \geq 2$ there exists a $\delta_k > 0$ and C^k mapping h such that $h(0) = 0$ and $h'(0) = 0$ and the surface

$$z = h(y) \quad \|y\| \leq \delta_k$$

is invariant under the dynamics above.

Proof Outline

For any continuously differentiable function h_k, globally bounded together with its first partial derivative and with $h_k(0) = 0$, $h'(0) = 0$, let h_{k+1} be defined by the equations

$$\begin{aligned}\dot{y} &= A^0 y + f^0(y, h_k(y)) \\ \dot{z} &= A^- z + f^-(y, h_k(y))\end{aligned}$$

$$h_{k+1}(y) = z$$

Under suitable assumptions, it can be verified that this defines h_{k+1} uniquely. Furthermore, the sequence $\{h_i\}$ is contractive in the norm $\sup_y h_i(y)$ and the limit h satisfies the conditions for a center manifold.

Usage

1) Determine $z = h(y)$, at least approximately.
(E.g., do a series expansion and identify coefficients...)

2) The **local** stability for the entire system can be proved to be the same as for the dynamics restricted to a center manifold:

$$\dot{y} = A^0 y + f^0(y, h(y))$$

Usage — *cont'd*

In the case of using series expansion of $h(y) = c_2 y^2 + c_3 y^3 + ...$, you would need to continue (w.r.t the order of the terms) until you have been able to determined the local behavior. (Low order terms dominate locally).

Identify the coefficients from
the boundary condition [Khalil (8.8, 8.11)]

$$\frac{\partial h}{\partial y}(y)[A^0 y + f^0(y, h(y))] - A^- h(y) - f^-(y, h(y)) = 0$$

Example

$$
\begin{aligned}
\dot{y} &= z \\
\dot{z} &= -z + ay^2 + byz
\end{aligned}
$$

Here $A^0 = 0$ and $A^- = -1$. $z = h(y)$ gives

$$-h + ay^2 + byh - h'h = 0$$

hence

$$h(y) = ay^2 + O(|y|^3)$$

Substituting into the dynamics we get

$$\dot{y} = ay^2 + O(|y|^3)$$

so $x = (0, 0)$ is unstable for $a \neq 0$.

Non-uniqueness

The center manifold need not be unique

Example

$$\begin{aligned}\dot{y} &= -y^3 \\ \dot{z} &= -z\end{aligned}$$

$z = h(y)$ gives

$$h'y^3 = z = h(y)$$

which has the solutions

$$h(y) = Ce^{-1/(2y^2)}$$

for all constants C.

Robustness analysis and quadratic inequalities

- μ-analysis
- S-procedure
- Multipliers
- Integral Quadratic Constraints
- Performance analysis

Preview — Example

A linear system of equations

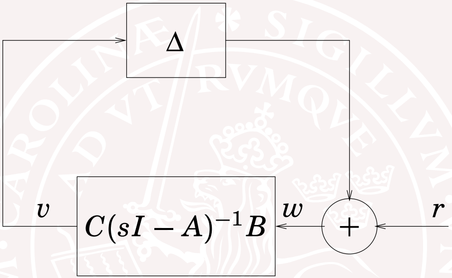
$$\begin{cases} x = y \\ y = 1.1 - 0.1x \end{cases} \quad \Rightarrow \quad x = y = 1$$

Equations with uncertainty

$$\begin{cases} (x-y)^2 < \epsilon_1 x^2 \\ (y + 0.1x - 1.1)^2 < \epsilon_2 \end{cases} \quad \Rightarrow \quad (x-1)^2 + (y-1)^2 < \epsilon_3$$

Given ϵ_1 and ϵ_2, how do we find a valid ϵ_3?

Example



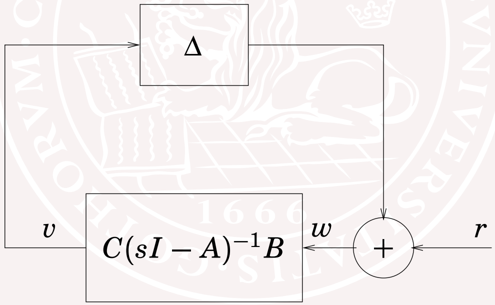
Question: For what values of Δ is the system stable?
Note: May be large differences if we consider complex or real uncertainties Δ.

"A formula for Computation of the Real Stability Radius", L. Qiu, B. Bernhardsson, A. Rantzer, E.J. Davison, and P.M. Young. Automatica, pp. 879–890, vol 31(6), 1995.

Parametric Uncertainty in Linear Systems

Let $\mathcal{D} \subset \mathbf{R}^{n\times n}$ contain zero. The system $\dot{x} = (A + B\Delta C)x$ is then exponentially stable for all $\Delta \in \mathcal{D}$ if and only if

- A is stable
- $\det\left[I - \Delta C(i\omega I - A)^{-1}B\right] \neq 0$ for $\omega \in \mathbf{R}$, $\Delta \in \mathcal{D}$



Use quadratic inequalities at each frequency!

$$w = \left[I - \Delta C(i\omega I - A)^{-1} B\right]^{-1} r$$

$$\begin{cases} w = \Delta v + r \\ v = C(i\omega I - A)^{-1} B w \end{cases}$$

For example, if

$$\mathcal{D} = \left\{ \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} : \quad \delta_k \in [-1, 1] \right\}$$

Then a bound of the form $|w|^2 < \gamma^2 |r|^2$ can be obtained using

$$\begin{matrix} |w_1 - r_1|^2 < |v_1|^2 \\ |w_2 - r_2|^2 < |v_2|^2 \end{matrix} \qquad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = C(i\omega I - A)^{-1} B \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

This verifies that $\det \left[I - \Delta C(i\omega I - A)^{-1} B\right] \neq 0$.

Structured Singular Values

Given $M \in \mathbf{C}^{n\times n}$ and a perturbation set

$$\mathcal{D} = \{\mathrm{diag}[\delta_1 I_{r_1}, \ldots, \delta_m I_{r_m}, \Delta_1, \ldots, \Delta_p] : \delta_k \in \mathbf{R}, \Delta_l \in \mathbf{C}^{m_l\times m_l}\}$$

the **structured singular value** $\mu_{\mathcal{D}}(M)$ is defined by

$$\mu_{\mathcal{D}}(M) = \sup\{\bar{\sigma}(\Delta)^{-1} : \Delta \in \mathcal{D},\, \det(I - M\Delta) = 0\}$$

See Matlab's $\mu - toolbox$

Reformulated Definition

The following two conditions are equivalent

(i) $0 \neq \det[I - \Delta M(i\omega)]$ for all $\Delta \in \mathcal{D}$ and $\omega \in \mathbf{R}$

(ii) $\mu_{\mathcal{D}}(M(i\omega)) < 1$ for $\omega \in \mathbf{R}$

Bounds on μ

If $\mathcal{D}$ consists of full complex matrices, then $\mu_{\mathcal{D}}(M) = \bar{\sigma}(M)$.

where $\bar{\sigma}(M)$ is the largest singular value of M = the larges eigenvalue of the matrix M^*M.

If $\mathcal{D}$ consists of perturbations of the form $\Delta = \delta I$ with $\delta \in [-1, 1]$, then $\mu_{\mathcal{D}}(M)$ is equal to the magnitude $\rho_{\mathbf{R}}(M)$ of the largest real eigenvalue of M ("the spectral radius"). In general

$$\rho_{\mathbf{R}}(M) \leq \mu_{\mathcal{D}}(M) \leq \bar{\sigma}(M)$$

Computation of μ

Define

$$\mathcal{U}_{\mathcal{D}} = \{U \in \mathcal{D} : U'U = I\}$$
$$\mathcal{D}_{\mathcal{D}} = \{D = D' \in \mathbf{C}^n : D\Delta = \Delta D \text{ for all } \Delta \in \mathcal{D}\}$$
$$\mathcal{G}_{\mathcal{D}} = \{G = G' \in \mathbf{C}^n : G\Delta = \Delta' G \text{ for all } \Delta \in \mathcal{D}\}$$

Then

$$\sup_{U \in \mathcal{U}_{\mathcal{D}}} \rho_{\mathbf{R}}(UM) \leq \mu_{\mathcal{D}}(M) \leq \inf_{\substack{D \in \mathcal{D}_{\mathcal{D}} \\ G \in \mathcal{G}_{\mathcal{D}}}} \widehat{\mu}(D, G) \leq \inf_{D \in \mathcal{D}_{\mathcal{D}}} \bar{\sigma}(DMD^{-1})$$

where

$$\widehat{\mu}(D, G) = \inf\{\mu > 0 : M'D'DM + j(GM - M'G) < \mu^2 D'D\}$$

S-procedure

Let M_0, M_1, $\ldots M_p$ be quadratic functions of $z \in R^n$

$$M_i = z^T T_i z + 2u_i z + v_i, \quad i = 0, \ldots, p$$

where $T_i = T_i^T$.

Consider the following condition on M_0, M_1, $\ldots M_p$:

$$M_0(z) \leq 0 \quad \text{for all } z \underline{\text{ such that }} \quad M_i(z) \geq 0, \quad i = 1, \ldots, p \tag{1}$$

Consider the following condition on M_0, M_1, $\dots M_p$:

$M_0(z_*) \leq 0$ <u>for all z_* such that</u> $M_i(z_*) \geq 0, \quad i = 1, \dots, p$ (1)

Obviously,

if there exists $\tau_1 \geq 0, \dots \tau_p \geq 0$ such that for all z

$$M_0(z) + \sum_{i=1}^{p} \tau_i M_i(z) \leq 0 \tag{2}$$

then (1) holds.

Nontrivial fact, that when $p = 1$, (1) implies (2), provided that there exist some z_0 such that $M_0(z_0) < 0$.

S-procedure for quadratic inequalities

The inequality

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^T M_0 \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \leq 0$$

follows from the inequalities

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^T M_1 \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \geq 0 \qquad \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^T M_2 \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \geq 0$$

if there exist $\tau_1, \tau_2 \geq 0$ such that

$$M_0 + \tau_1 M_1 + \tau_2 M_2 \leq 0$$

Numerical algorithms in available (e.g. in Matlab), see also [Boyd *et al*]

S-procedure in general

The inequality

$$\sigma_0(h) \leq 0$$

follows from the inequalities

$$\sigma_1(h) \geq 0, \ldots, \sigma_n(h) \geq 0$$

if there exist $\tau_1, \ldots, \tau_n \geq 0$ such that

$$\sigma_0(h) + \sum_k \tau_k \sigma_k(h) \leq 0 \qquad \forall h$$

S-procedure losslessness by Megretski/Treil

Let $\sigma_0, \sigma_1, \ldots, \sigma_n$ be time-invariant quadratic forms on $\mathbf{L}_2^m$.
Suppose that there exists z_* such that

$$\sigma_1(z_*) > 0, \ldots, \sigma_m(z_*) > 0$$

Then the following statements are equivalent

- $\sigma_0(z) \leq 0$ for all z such that $\sigma_1(z) \geq 0, \ldots, \sigma_n(z) \geq 0$
- There exist $\tau_1, \ldots, \tau_n \geq 0$ such that

$$\sigma_0(z) + \sum_k \tau_k \sigma_k(z) \leq 0 \qquad \forall z$$

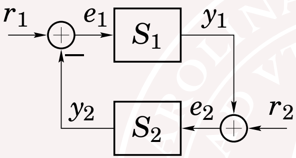
Integral Quadratic Constraint

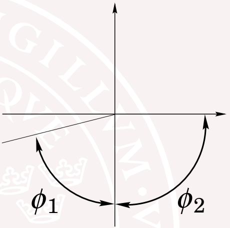
An IQC expresses information of a subsystem. Should be convenient to use for analysis of a larger system.

Unifies

- Multiplier (Zames-Falb)
- Passivity
- Absolute stability
- μ

Passivity Theorem is a “Small Phase Theorem”





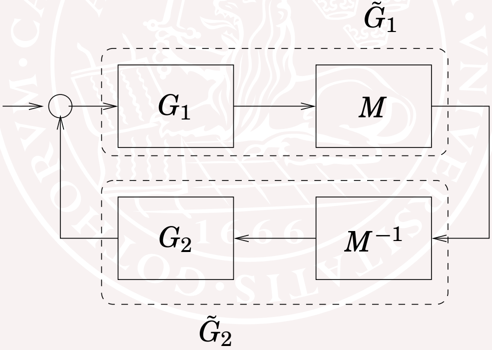
A passive operator can also be viewed as a sector condition $[0, \infty)$.

Compare circle criterion: Nyquist curve should avoid "circle" $(-\frac{1}{\alpha}, -\frac{1}{\beta}) \to$ whole LHP as $\alpha \to 0$ and $\beta \to \infty$.

Multipliers

Cut the loops in smart ways or introduce multipliers (Bounded operators M and M^{-1}).

Same idea as loop transformations: should be easier to prove stability for transformed system.



Multipliers, cont'd

Example: Negative feedback of linear system

$$G(s) = C(sI - A)^{-1}B$$

with nonlinearity with positivity property $\varphi(y) \cdot y \geq 0$.

Circle criterion assures exponential stability if

$$\mathrm{Re}\{G(i\omega)\} > 0 \quad \omega \in R$$

Compare (strict) passivity conditions

Multipliers, cont'd

Zames-Falb (1968):

Circle criterion can be improved if there are additional assumptions on the nonlinearity as *e.g.,* monotonicity or bounds on slope.

If $\frac{d\varphi(y)}{dy} \geq 0$, Z-F introduced extra freedom with

$$H \in RL_\infty \text{ and } ||H||_{L_1} \leq 1$$

such that absolute stability is assured if

$$\mathrm{Re}\{G(i\omega)^{-1} \cdot (1 + H(i\omega))\} > 0, \quad \omega \in R \ \{0\}$$

Compare with Popov-criterion conditions

Integral Quadratic Constraint

The causal bounded operator Δ on $\mathbf{L}_2^m$ is said to satisfy the IQC defined by the matrix function $\Pi(i\omega)$ if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix} d\omega \quad \geq \quad 0$$

for all $v \in \mathbf{L}_2$.

Trivial for $\Pi > 0$, but almost all interesting cases have non-positive Π.

Integral Quadratic Constraint

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for all $v \in \mathbf{L}_2$.

Trivial for $\Pi > 0$, but almost all interesting cases have non-positive Π.

Example — Gain and Passivity

Suppose the gain of Δ is at most one. Then

$$0 \leq \int_0^\infty (|v|^2 - |\Delta v|^2) dt = \int_{-\infty}^\infty \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix} d\omega$$

Suppose instead that Δ is passive. Then

$$0 \leq \int_0^\infty v(t)(\Delta v)(t) dt = \int_{-\infty}^\infty \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix}^* \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix} d\omega$$

Note: Scaling in Parseval's formula neglected here (does not affect sign of IQC).

Exercise

Show that a nonlinearity satisfying the sector condition

$$\alpha y^2 \leq \varphi(t, y)y \leq \beta y^2$$

satisfies the IQC, $\varphi \in IQC(\Pi)$ given by

$$\Pi(j\omega) = \Pi = \begin{bmatrix} -2\alpha\beta & \alpha + \beta \\ \alpha + \beta & -2 \end{bmatrix}$$

Note: Satisfies a quadratic inequality (for every frequency) $\Longrightarrow$ satisfies integral quadratic inequality

IQC's for Coulomb Friction

$$\begin{cases} f(t) = -1 & \text{if } v(t) < 0 \\ f(t) \in [-1, 1] & \text{if } v(t) = 0 \\ f(t) = 1 & \text{if } v(t) > 0 \end{cases}$$

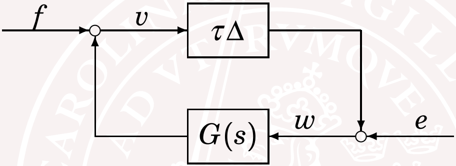
Zames/Falb's property

$$0 \leq \int_0^\infty v(t)[f(t) + (h * f)(t)]dt, \qquad \int_{-\infty}^\infty |h(t)|dt \leq 1$$

$$0 \leq \int_{-\infty}^\infty \begin{bmatrix} \widehat{v} \\ \widehat{f} \end{bmatrix}^* \begin{bmatrix} 0 & 1 + H(i\omega) \\ 1 + H(-i\omega) & 0 \end{bmatrix} \begin{bmatrix} \widehat{v} \\ \widehat{f} \end{bmatrix} d\omega$$

Δ structure	$\Pi(i\omega)$	Condition
Δ passive	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$	
$\\|\Delta(i\omega)\\| \leq 1$	$\begin{bmatrix} x(i\omega)I & 0 \\ 0 & -x(i\omega)I \end{bmatrix}$	$x(i\omega) \geq 0$
$\delta \in [-1, 1]$	$\begin{bmatrix} X(i\omega) & Y(i\omega) \\ Y(i\omega)^* & -X(i\omega) \end{bmatrix}$	$X = X^* \geq 0$ $Y = -Y^*$
$\delta(t) \in [-1, 1]$	$\begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix}$	
$(\Delta v)(t) = \mathrm{sgn}(v(t))$	$\begin{bmatrix} 0 & 1 + H(i\omega) \\ 1 + H(i\omega)^* & 0 \end{bmatrix}$	$\\|H\\|_{\mathbf{L_1}} \leq 1$

Well-posed Interconnection

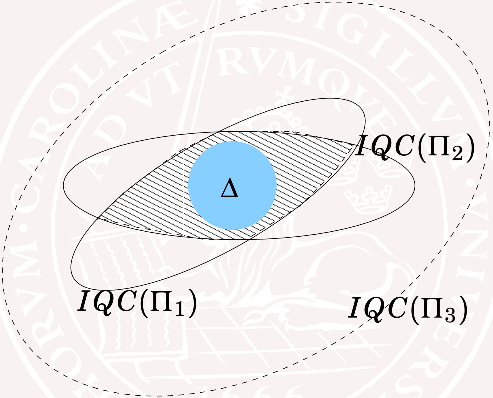


The feedback interconnection

$$\begin{cases} v &= Gw + f \\ w &= \Delta(v) + e \end{cases}$$

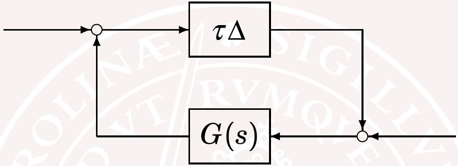
is said to well-posed if the map $(v, w) \mapsto (e, f)$ has a causal inverse. It is called BIBO stable if the inverse is also bounded.

Use as many IQCs as possible to characterize the nonlinearity/uncertainty.



In this case $IQC(\Pi_3)$ does not help to restrict the complete set that satisfy the IQCs.

IQC Stability Theorem



Let $G(s)$ be stable and proper and let Δ be causal.

For all $\tau \in [0, 1]$, suppose the loop is well posed and $\tau\Delta$ satisfies the IQC defined by $\Pi(i\omega)$. If

$$\begin{bmatrix} G(i\omega) \\ I \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} G(i\omega) \\ I \end{bmatrix} < 0 \quad \text{for } \omega \in [0, \infty]$$

then the feedback system is BIBO stable.

Computations via LMI's

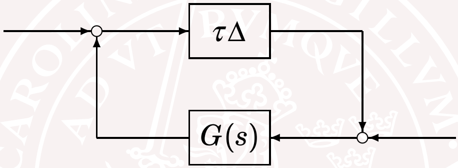
$$\begin{bmatrix} G(i\omega) \\ I \end{bmatrix}^* \sum_k \tau_k \Pi(i\omega) \begin{bmatrix} G(i\omega) \\ I \end{bmatrix} < 0 \quad \text{for } \omega \in [0,\infty]$$

$$\begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix}^* (M + \sum_k \tau_k M_k) \begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix} < 0 \quad \text{for } \omega \in [0,\infty]$$

$$\sum_{i=1}^{n} \tau_k M_k + \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} < 0.$$

Solve for $\tau_1, \ldots, \tau_n \geq 0$ and P.

Relation to Passivity and Gain Theorems



A stability theorem based on gain is recovered with $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

A passivity based stability theorem is recovered with $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.

Special Case — μ Analysis

Note that $\Delta = \text{diag}\{\delta_1, \dots, \delta_m\}$, with $|\delta_k| \le 1$ satisfies the IQC defined by

$$\Pi(i\omega) = \begin{bmatrix} X(i\omega) & 0 \\ 0 & -X(i\omega) \end{bmatrix}$$

where $X(i\omega) = \text{diag}\{x_1(i\omega), \dots, x_m(i\omega)\} > 0$.

Feedback loop stability follows if there exists $X(i\omega) > 0$ with

$$G(i\omega)^* X(i\omega) G(i\omega) < X(i\omega) \qquad \omega \in [0, \infty]$$

or equivalently, with $D(i\omega)^* D(i\omega) = X(i\omega)$

$$\sup_{\omega} \|D(i\omega) G(i\omega) D(i\omega)^{-1}\| < 1$$

Combination of Uncertain and Nonlinear Blocks

The operator $\Delta(v_1, v_2) = (\delta v_1, \phi(v_2))$ where

$$\delta \in [-1, 1]$$
$$\alpha \le \phi(v_2)/v_2 \le \beta$$

satisfies all IQC's defined by matrix functions of the form

$$\Pi(i\omega) = \begin{bmatrix} X(i\omega) & 0 & Y(i\omega) & 0 \\ 0 & -2\alpha\beta & 0 & \alpha+\beta \\ Y(i\omega)^* & 0 & -X(i\omega) & 0 \\ 0 & \alpha+\beta & 0 & -2 \end{bmatrix}$$

where $X(i\omega) = X(i\omega)^*$ and $Y(i\omega) = Y(i\omega)^*$.

Proof idea of IQC Theorem

Combination of the IQC for Δ with the inequality for G gives existence of $c_0 > 0$ such that

$$\|v\| \leq c_0 \|v - \tau G\Delta(v)\| \qquad\qquad v \in \mathbf{L}_2, \tau \in [0,1]$$

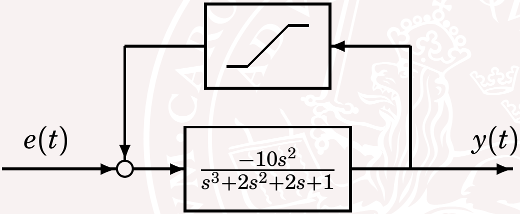
If $(I - \tau G\Delta)^{-1}$ is bounded for some $\tau \in [0,1]$ then the above inequality gives boundedness of $(I - \nu G\Delta)^{-1}$ for all ν with

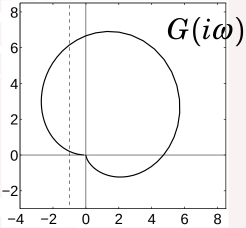
$$c_0 \|G\Delta\| \cdot |\tau - \nu| < 1$$

Hence, boundedness for $\tau = 0$ gives boundedness for $\tau < (c_0\|G\Delta\|)^{-1}$. This, in turn, gives boundedness for $\tau < 2(c_0\|G\Delta\|)^{-1}$ and so on. Finally the whole interval $[0,1]$ is covered.

A toolbox for IQC analysis

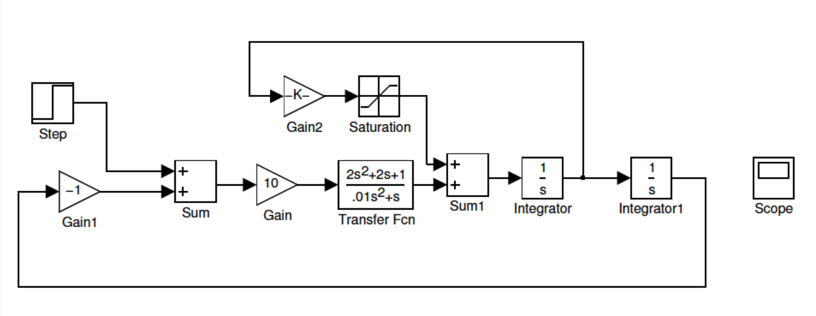
Copy `/home/kursolin/matlab/lmiinit.m` to the current directory or download and install the IQCbeta toolbox from `http://www.ee.mu.oz.au/staff/cykao/`



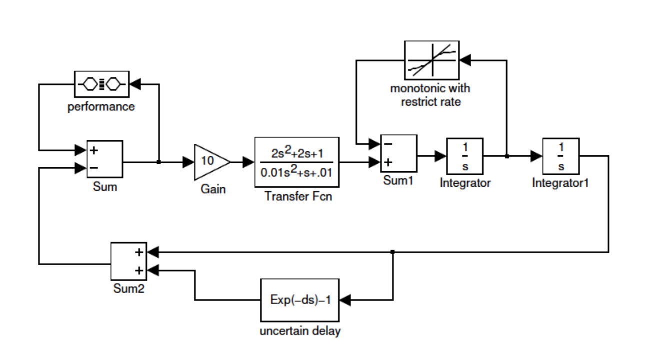



```
>> abst_init_iqc;
>> G = tf([10 0 0],[1 2 2 1]);
>> e = signal
>> w = signal
>> y = -G*(e+w)
>> w==iqc_monotonic(y)
>> iqc_gain_tbx(e,y)
```

A simulation model



An analysis model defined graphically

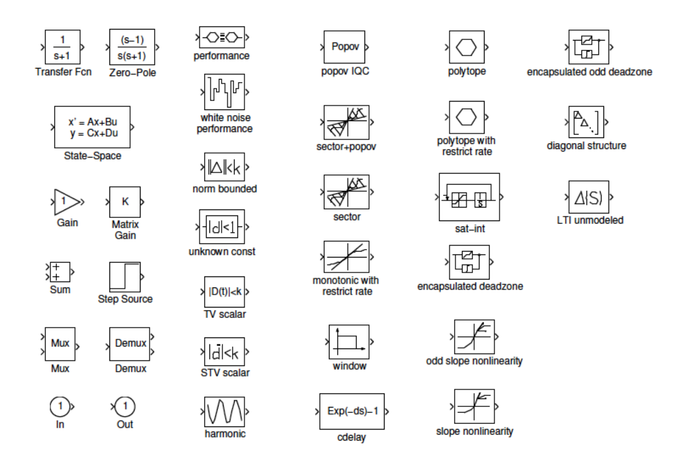


The text version (i.e., NOT the gui) is strongly recommended by the IQCbeta author(s) at present version!!

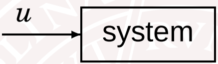
```
z iqc_gui('fricSYSTEM')

extracting information from fricSYSTEM ...

    scalar inputs:  5
    states:         10
    simple q-forms: 7

    LMI #1    size = 1    states: 0
    LMI #2    size = 1    states: 0
    LMI #3    size = 1    states: 0
    LMI #4    size = 1    states: 0
    LMI #5    size = 1    states: 0

  Solving with 62 decision variables ...

  ans =    4.7139
```

The text version (i.e., NOT the gui) is strongly recommended by the IQCbeta author(s) at present version!!

A library of analysis objects



Bounds on Auto Correlation



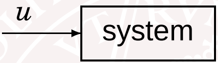
The auto correlation bound

$$\int_{-\infty}^{\infty} u(t)^* u(t-T)dt \quad \leq \quad \alpha \int_{-\infty}^{\infty} u(t)^* u(t)dt,$$

corresponds to

$$\Psi(i\omega) = 2\alpha - e^{i\omega T} - e^{-i\omega T}.$$

Dominant Harmonics



For small $\epsilon > 0$, the constraint

$$\int_0^\infty |\widehat{u}(i\omega)|^2 d\omega \quad \leq \quad (1+\epsilon)\int_a^b |\widehat{u}(i\omega)|^2 d\omega$$

means that the energy of u is concentrated to the interval $[a, b]$.

Incremental Gain and Passivity

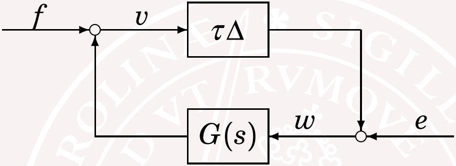
A causal nonlinear operator Δ on $\mathbf{L}_2^m$ is said to have **incremental gain** less than γ if

$$\|\Delta(v_1) - \Delta(v_2)\| \leq \gamma \|v_1 - v_2\| \qquad\qquad v_1, v_2 \in \mathbf{L}_2$$

It is called **incrementally passive** if

$$0 \leq \int_0^T [\Delta(v_1) - \Delta(v_2)][v_1 - v_2]dt \qquad T > 0, v_1, v_2 \in \mathbf{L}_2$$

Incremental Stability



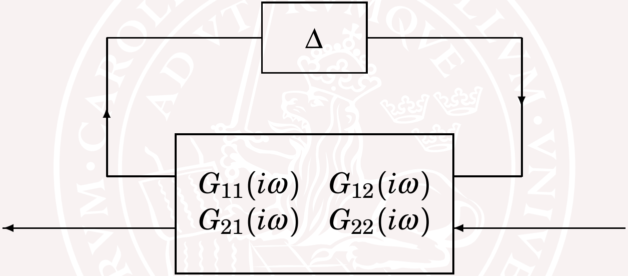
The feedback interconnection

$$\begin{cases} v &= Gw + f \\ w &= \Delta(v) + e \end{cases}$$

is called **incrementally stable** if there is a constant C such that any two solutions (e_1, f_1, v_1, w_1), (e_2, f_2, v_2, w_2) satisfies

$$\|v_1 - v_2\| + \|w_1 - w_2\| \leq C\|e_1 - e_2\| + C\|f_1 - f_2\|$$

Robust Performance



A Converse Small Gain Theorem

The static case
A matrix M satisfies $\bar{\sigma}(M) < 1$ if and only if $0 \neq \det(I - \Delta M)$ for all matrices Δ with $\bar{\sigma}(\Delta) \leq 1$.

The dynamic case
A stable transfer matrix $G(s)$ satisfies $\|G\|_\infty < 1$ if and only if $[I - \Delta(s)G(s)]^{-1}$ is stable for every stable $\Delta(s)$ with $\|\Delta\|_\infty < 1$.

Proof in the Static Case

Suppose that $\det(I - \Delta M) = 0$ and $\bar{\sigma}(\Delta) \leq 1$. Then there exists $x \neq 0$ such that $x - \Delta M x = 0$ and

$$|x| = |\Delta M x| \leq |M x|$$

so $\bar{\sigma}(M) \geq 1$.

On the other hand, if $\bar{\sigma}(M) \geq 1$, there exists $x \neq 0$ with $|M x| \geq |x|$. Let

$$\Delta = \frac{x M' x'}{|M x|^2}$$

Then $\bar{\sigma}(\Delta) \leq 1$ and $(I - \Delta M)x = 0$, so $\det(I - \Delta M) = 0$.

Robust Performance Theorem

Suppose that $\bar{\sigma}(M_{11}) \leq 1$ and that $\mathcal{D}$ is a connected set of matrices with $0 \in \mathcal{D}$. Let $\mathcal{D}_1 = \{\text{diag}(\Delta_1, \Delta) : \bar{\sigma}(\Delta_1) \leq 1, \Delta \in \mathcal{D}\}$. Then the following conditions are equivalent.

(i) $\bar{\sigma}(M_{11} + M_{12}[I - \Delta M_{22}]^{-1}\Delta M_{21}) < 1$ for $\Delta \in \mathcal{D}$

(ii) $0 \neq \det\left(I - \begin{bmatrix}\Delta_1 & 0 \\ 0 & \Delta\end{bmatrix}\begin{bmatrix}M_{11} & M_{12} \\ M_{21} & M_{22}\end{bmatrix}\right)$ for $\begin{cases}\Delta \in \mathcal{D} \\ \bar{\sigma}(\Delta_1) \leq 1\end{cases}$

(iii) $\mu_{\mathcal{D}_1}\left(\begin{bmatrix}M_{11} & M_{12} \\ M_{21} & M_{22}\end{bmatrix}\right) \leq 1$ for $\omega \in \mathbf{R}$

Proof

The conditions (ii) and (iii) are equivalent by the definition of μ. Moreover, we showed on the previous slide that (i) fails if and only if there exists Δ_1 with $\bar{\sigma}(\Delta_1) \leq 1$ and $x_1 \neq 0$ such that

$$0 = \{I - \Delta_1(M_{11} + M_{12}[I - \Delta M_{22}]^{-1}\Delta M_{21})\}x_1$$

Introduce $x_2 = [I - \Delta M_{22}]^{-1}\Delta M_{21}x_1$. Then

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This is possible if and only if (ii) fails, so the equivalence of (i) and (ii) is proved.

Example

Compute

$$\max_{|\delta_k|\le 1} \sup_{\omega} \frac{\delta_1}{(i\omega)^2 + (2+\delta_2)i\omega + 2 + \delta_1\delta_2}$$

This is the worst case gain of the system

$$\begin{cases} \ddot{y} = -(2+\delta_2)\dot{y} - (2+\delta_1\delta_2)y + \delta_1 u = -2\dot{y} - 2y - \delta_1 v_1 - \delta_2 v_2 \\ v_1 = -\delta_2 v_3 + u, \qquad v_2 = \dot{y}, \qquad v_3 = y \end{cases}$$

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \\ v_1 \\ v_2 \\ v_3 \\ y \end{bmatrix} \left[\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & -2 & 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \begin{bmatrix} y \\ \dot{y} \\ \delta_1 v_1 \\ \delta_2 v_2 \\ \delta_2 v_3 \\ u \end{bmatrix}$$

Performance Analysis via S-procedure

The performance criterion

$$\sigma_0(h) \le 0 \qquad \forall h \in \mathcal{N}$$

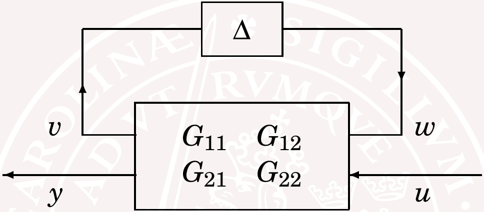
follows from the IQC's

$$\sigma_1(h) \ge 0, \ldots, \sigma_n(h) \ge 0 \qquad \forall h \in \mathcal{N}$$

if there exist $\tau_1, \ldots, \tau_n \ge 0$ such that

$$\sigma_0(h) + \sum_k \tau_k \sigma_k(h) \le 0 \qquad \forall h \in \mathbf{L}_2^n$$

Performance Bounds from IQC's



Suppose that Δ satisfies the IQC defined by Π. Then the gain bound $\|y\| \leq \gamma\|u\|$ holds provided that the system is stable and

$$0 \geq \begin{bmatrix} G(i\omega) \\ I \end{bmatrix}^* \begin{bmatrix} \Pi_{11} & 0 & \Pi_{12} & 0 \\ 0 & I & 0 & 0 \\ \Pi_{21} & 0 & \Pi_{22} & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} G(i\omega) \\ I \end{bmatrix} \qquad \omega \in \mathbf{R}$$