Based on Ch 18.5 [Glad & Ljung]

Find the state feedback law

$$u = k(x)$$

which solves minimization problem

$$\begin{aligned}\min_u \int_{t_0}^{t_f} (L(x,\, u)dt + \varphi(t_f,\, x(t_f))\\ \dot{x} = f(x(t),\, u(t))\\ u \in \mathcal{U}, \quad t_0 \leq t \leq t_f\\ x(t_0) = x_0, \quad \psi(t_f,\, x(t_f)) = 0\end{aligned}$$

Assume that u^* and x^* solves this optimization problem.

Define $V(t_0,\, x_0)$ as the optimal return function

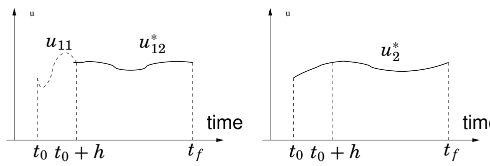
$$V(t_0,\, x_0) = \int_{t_0}^{t_f} (L(x^*,\, u^*)dt + \varphi(t_f,\, x^*(t_f))$$

if we start in $(t_0,\, x(t_0) = x_0)$

Remark: Need to satisfy ...

Assume that we for

- $t \in [t_0, t_0 + h]$ use any control $u(t)$
- $t \in [t_0 + h, t_f]$ use optimal control $u(t)^*$



The 0ptimization criterion becomes

$$\int_{t_0}^{t_0+h} (L(x(r),\, u(r))dr + V(t_0 + h, x(t_0 + h))$$

If optimal control from t_0: $V(t_0,\, x(t_0)) \implies$

$$V(t_0,\, x(t_0)) \leq \int_{t_0}^{t_0+h} (L(x(r),\, u(r))dr + V(t_0 + h, x(t_0 + h))$$

Theorem: If the *optimal return value* V is differentiable it satisfies

$$-\frac{\partial V}{\partial t} = \min_{u \in \mathcal{U}} \left(\frac{\partial V}{\partial x} f(x, u) + L(x, y) \right) \qquad (2)$$

Proof: The chain rule gives

$$\frac{d}{dt} V(t,\, x(t)) = V_t + V_x f$$

and from Eq.(1) gives

$$-\frac{\partial V}{\partial t} \leq \frac{\partial V}{\partial x} f(x, u) + L(x, y)$$

with equality for optimal control u^*.

Eq.(2) is called the *Hamilton-Jacobi equation* (HJ) for a finite t_f the *Hamilton-Jacobi-Bellman equation* (HJB) for $t_f = \infty$.

Nonlinear Control

Lecture 7

- Optimal and inverse(!) optimal design
- Saturated control and feedforwarding

Property of V:

If V is differentiable along a solution $x(t)$, then

$$\frac{d}{dt} V(t,\, x(t)) + L(x(t),\, u(t)) \geq 0 \qquad (1)$$

with equality for x^* and u^*.

which gives

$$\frac{V(t_0 + h, x(t_0 + h)) - V(t_0,\, x(t_0))}{h} + \frac{1}{h} \int_{t_0}^{t_0+h} (L(x(r),\, u(r))dr \geq 0$$

which in the limit $h \to 0^+$ gives

$$\frac{d}{dt} V(t,\, x(t)) + L(x(t),\, u(t)) \geq 0$$

Remarks: Severe restriction to assume V differentiable (*e.g.*, bang-bang solutions for minimal time problems give "corners" in V but results can be extended to this case as well.

- State feedback law $u = k(t,\, x) = \arg\min_{u \in \mathcal{U}} \left(\frac{\partial V}{\partial x} f(x, u) + L(x, y) \right)$
- Necessary conditions while Pontryagin gives sufficient.

Outline

- HJB
- Inverse optimal control
- Stabilization with Saturations
- Integrator forwarding
- Relations between the concepts
- Conclusions

Optimality

Two main alternatives

- Pontryagin's Maximum Principle (Necessary cond)
- Hamilton-Jacobi-Bellman (Dyn prog.) (Sufficient cond)

Consider the system

$$\dot{x} = f(x) + g(x)u$$

Find $u = u^*$ such that

(i) u achieves asymptotic stability of the origin $x = 0$
(ii) u minimizes the cost functional

$$\int_0^\infty (l(x) + u^T R(x)u)dt \tag{3}$$

where $l(x) \geq 0$ and $R(x) \geq 0 \forall x$.

For a given optimal feedback $u(x)^*$ the value of V depends on the initial state $x(0)$: $V(x(0))$ or simply $V(x)$ (and start time according to previous slides).

Theorem (Optimality and Stability)
Suppose there exist a C^1-function $V(x) \geq 0$ which satisfies the *Hamilton-Jacobi-Bellman equation*

$$\begin{aligned} &l(x) + L_f V(x) - \frac{1}{4} L_g V(x) R^{-1} (L_g V(x))^T = 0 \\ &V(0) = 0 \end{aligned} \tag{4}$$

such that the feedback control

$$u^*(x) = -\frac{1}{2} R^{-1} (L_g V(x))^T$$

achieves asymptotic stability of the origin $x = 0$.

Then $u^*(x)$ is the *optimal stabilizing control* which minimizes the cost (3).

Example:
Linear system

$$\dot{x} = Ax + Bu$$

Cost Function

$$V = \int_0^\infty (x^T C^T Cx + u^T Ru)dt, \qquad R > 0$$

Riccati-equation

$$PA + AP^T - PBR^{-1}B^T P + C^T C = 0 \tag{5}$$

If (A,B) controllable and (A,C) observable, then (5) has a unique solution $P = P^T > 0$ such that the optimal cost is $V = x^T Px$ and

$$u^*(x) = -R^{-1}B^T Px$$

is the optimal stabilizing control

5-min exercise:
Consider the system

$$\dot{x} = x^2 + u$$

and the cost functional

$$V = \int_0^\infty (x^2 + u^2)dt$$

What is the optimal stabilizing control?

HJB:

$$x^2 + \frac{\partial V}{\partial x} x^2 - \frac{1}{4}\left(\frac{\partial V}{\partial x}\right)^2 = 0, \qquad V(x) = 0$$

$$\begin{aligned} \frac{\partial V}{\partial x} &= 2x^2 \pm \sqrt{4x^4 + 4x^2} \\ &= 2x^2 + 2x\sqrt{x^2 + 1} \end{aligned} \tag{6}$$

$$V(x) = \frac{2}{3}x^3 + \frac{2}{3}(x^2+1)^{3/2} + C, \qquad C = -2/3 \text{ so that } V(0) = 0 \tag{7}$$

$$u^*(x) = -\frac{1}{2}\frac{\partial V}{\partial x} = -x^2 - x\sqrt{x^2+1}$$

Remark: We have chosen the positive solution in (6) as $V(x) \geq 0$

Remark: If (A,B) stabilizable and (A,C) detectable then P is positive semi-definite.
Example (non-detectability in cost)
System

$$\dot{x} = x + u$$

Cost functional

$$V = \int_0^\infty u^2 dt$$

Riccati-eq

$$2P - P^2 = 0, \qquad P = 0 \text{ or } P = 2$$

Corresponding HJB

$$x\frac{\partial V}{\partial x} - \frac{1}{4}(\frac{\partial V}{\partial x})^2 = 0, \qquad V(0) = 0$$

$$V = 0 \text{ or } V = 2x^2$$

Inverse optimality

A stabilizing control law $u(x)$ solves an *inverse* optimal problem for the system

$$\dot{x} = f(x) + g(x)u$$

if it can be written as

$$u(x) = -k(x)/2 = -\frac{1}{2}R^{-1}(x)(L_gV(x))^T, \qquad R(x) > 0$$

where $V(x) \geq 0$ and

$$\dot{V} = L_fV + L_gV = \underbrace{L_fV - \frac{1}{2}L_gVk(x)}_{-l(x)} \leq 0$$

Then $V(x)$ is the solution of the HJB-eqn

$$l(x) + L_fV - \frac{1}{4}(L_gV)R^{-1}(L_gV)^T = 0$$

Damping Control / Jurdjevic-Quinn

Consider the system

$$\dot{x} = f(x) + g(x)u$$

Assume that the *drift part* of the system is stable, *i.e.,*

$$\dot{x} = f(x), \quad f(0) = 0$$

and that we know a function $V(x)$ such that $L_fV \leq 0$ for all x.
How to make it asymptotically stable (robustly)?

Connection to passivity:
The system

$$\dot{x} = f(x) + g(x)u$$
$$y = (L_gV)^T(x)$$

is passive with $V(x)$ as storage function if $L_fV \leq 0$ as

$$\dot{V} = L_fV + L_gVu \leq y^Tu$$

The feedback law $u = -\kappa y$ guarantees GAS if the system is ZSD (zero state detectable).

Note: May be a conservative choice as it does not fully exploit the possibility to choose $V(x)$ for the whole system (only $\dot{x} = f(x)$).

Feedforward systems

Particular form of cascaded systems

1991 **A. Teel**
... Sussman, Sontag, Yang
... Saberi, Lin

1996 Mazenc, Praly
1996 Sepulchre, Jankovic, Kokotovic

The underlying idea of formulating an *inverse* optimal problem is to get some help to avoid non-robust cancellations and gain some stability margins.
Example: Non-robust cancellation
Consider the system

$$\dot{x} = x^2 + u$$

and the control law

$$u_n = -x^2 - x \qquad \Rightarrow \qquad \dot{x} = -x$$

However, if there is some small perturbation gain $u = (1+\epsilon)u_n$, we get

$$\dot{x} = -(1+\epsilon)x - \epsilon x^2$$

This system may has finite escape time solutions.

How does u^* from previous example behave?

To add more damping to the system to render it asymptotically stable the following suggestion was made by Jurdjevic-Quinn (1978)

$$\dot{V} = L_fV + L_gVu \leq L_gVu$$

Choose

$$u = -\kappa \cdot (L_gV)^T$$

It also solves the global optimization problem for the cost functional

$$V(x) = \int_0^\infty (l(x) + \frac{2}{\kappa}u^Tu)dt$$

for the state cost function

$$l(x) = -L_fV + \frac{\kappa}{2}(L_gV)(L_gV)^T \geq 0$$

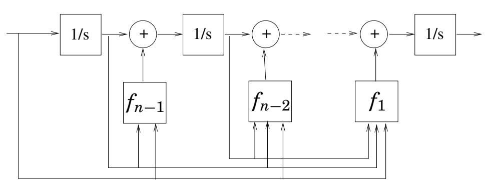
Systems with saturations of control signal

Problem: System runs in “open loop” when in saturation

- Anti-windup designs from FRT075
- Consider Lyapunov function candidates of type $V = log(1+x^2)$ (see Lecture 1)
- Saturated controls [Sussmann, Yang And Sontag]
- Cascaded saturations [Teel *et al*]

Strict-feedforward systems

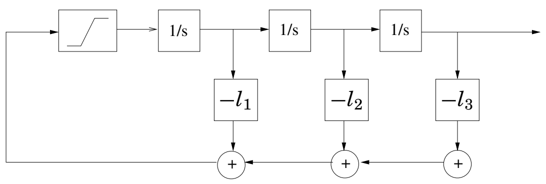
$$\dot{x}_1 = x_2 + f_1(x_2, x_3, \ldots, x_n, u)$$
$$\dot{x}_2 = x_3 + f_2(x_3, \ldots, x_n, u)$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n + f_{n-1}(x_n, u)$$
$$\dot{x}_n = u$$



Compare with e.g.
Strict-feedback systems

$$
\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_1)\\
\dot{x}_2 &= x_3 + f_2(x_1, x_2)\\
&\vdots\\
\dot{x}_n &= x_n + f_n(x_1, x_2, \dots x_{n-1}) + u
\end{aligned}
$$

Sussman and Yang (1991) :
There does not exist any (simple) saturated feedback-law which stabilizes an integrator chain of order ≥ 3 globally.



Teel's idea:
using nested saturations

$$u = -\sigma_n(h_n(x) + \sigma_{n-1}(h_{n-1}(x) + \cdots + \sigma_1(h_1(x))\dots)$$

Sketch of proof: (n=3, $L_i = M_i$)
Consider a state transformation $y = Tx$ which transforms the integrator chain into

$$\dot{y} = Ay + Bu$$

where

$$A = \begin{pmatrix} 0 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1\\ 1\\ 1 \end{pmatrix}$$

The control law

$$u = -\sigma_3(y_3 + \sigma_2(y_2 + \sigma_1(y_1)))$$

will give the closed loop system

$$
\begin{aligned}
\dot{y}_1 &= y_2 + y_3 & &-\sigma_3(y_3 + \sigma_2(y_2 + \sigma_1(y_1)))\\
\dot{y}_2 &= y_3 & &-\sigma_3(y_3 + \sigma_2(y_2 + \sigma_1(y_1)))\\
\dot{y}_3 &= & &-\sigma_3(y_3 + \sigma_2(y_2 + \sigma_1(y_1)))
\end{aligned}
$$

$$
\begin{aligned}
\dot{y}_2 &= y_3 - (y_3 + \sigma_2(y_2 + \sigma_1(y_1)))\\
&= -\sigma_2(y_2 + \sigma_1(y_1)))
\end{aligned}
$$

Same kind of argument shows us that after finite time, the closed loop will look like

$$
\begin{aligned}
\dot{y}_1 &= -y_1\\
\dot{y}_2 &= -y_1 - y_2\\
\dot{y}_3 &= -y_1 - y_2 - y_3
\end{aligned}
$$

i.e. after a finite time, the dynamics are exponentially stable

Remark:
Although we have found a globally stabilizing, bounded, control law, u, the internal states may have huge overshoots !!

Strict-feedforward systems are, in general, **not** feedback linearizable!
(*i.e.,* neither exact linearization nor backstepping is applicable for stabilization)

Restriction: Does not cover systems of the type

$$
\begin{aligned}
&\dots\\
&\dot{x}_k = -x_k^2 + \dots\\
&\dots
\end{aligned}
$$

i.e. don't have to worry about

finite escape-time

Definition: σ is a *linear saturation* for (L, M) if

- σ is continuous and nondecreasing
- $\sigma(s) = s$ when $|s| \leq L$
- $|\sigma(s)| \leq M$, $\forall s \in R$

Theorem (Teel):
For an integrator chain of any order and for any set $\{(L_i, M_i)\}$ where $L_i \leq M_i$ and $M_i < \frac{1}{2}L_{i+1}$, there exists $\{h_i\}$ for all linear saturations $\{\sigma_i\}$ such that the bounded control

$$u = -\sigma_n(h_n(x) + \sigma_{n-1}(h_{n-1}(x) + \cdots + \sigma_1(h_1(x))\dots)$$

results in global asymptotic stability for the closed loop system.

How does y_3 evolve ?
Let $V_3 = y_3^2 \Rightarrow$

$$\dot{V}_3 = -2y_3\sigma_3(y_3 + \sigma_2(y_2 + \sigma_1(y_1)))$$

As $|\sigma_2(.)| \leq M_2 < \frac{1}{2}L_3$,
$\dot{V}_3 < 0$ for all $|y_3| > \frac{1}{2}L_3$

$\Rightarrow |y_3|$ will decrease.

In finite time $|y_3|$ will be $< \frac{1}{2}L_3$ and σ_3 will now operate in the linear region.
(Note: no finite escape for the other states.)

Integrator forwarding

strict-feedforward systems

$$
\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_2, x_3, \dots, x_n, u)\\
&\vdots\\
\dot{x}_{n-1} &= x_n + f_{n-1}(x_n, u)\\
\dot{x}_n &= u
\end{aligned}
$$

Due to the lack of feedback connections, solutions always exists and are of the form

$$
\begin{aligned}
x_n(t) &= x_n(0) + \int_0^t u(s)ds\\
x_{n-1}(t) &= x_{n-1}(0) + \int_0^t (x_n(s) + f_{n-1}(x_n(s), u(s)))ds\\
&\vdots
\end{aligned}
$$

1. Begin with stabilizing the system $\dot{x}_n = u_n$
   Use e.g. $V_n = x_n^2$ and $u_n = -x_n$
2. Augment the control law
   $u_{n-1}(x_{n-1}, x_n) = u_n(x_n) + v_{n-1}$
   such that u_{n-1} stabilizes the cascade

$$\begin{aligned} \dot{x}_{n-1} &= x_n + f_{n-1}(x_n, u) \\ \dot{x}_n &= u_{n-1} \end{aligned}$$

...

k. Augment the control law
   $u_k(x_k, x_{k+1}) = u_n(x_{k+1}) + v_k$
   such that u_k stabilizes the cascade

$$\begin{aligned} \dot{x}_k &= x_{k+1} + f_k(\dots) \\ \dot{X}_{k+1} &= F_{k+1}(\dots, u_k) \end{aligned}$$

How is the cascade (in step k) stabilized?

We have a cascade of one GAS/LES system and a ISS-system with a linear growth-condition.

There exists a Lyapunov function for the (sub-) system

$$V_k = V_{k+1} + \frac{1}{2}x_k^2 + \int_0^\infty x_k(s) f_k(X_{k+1}(s))ds$$

It can be shown that $\dot{V}_k|_{u_k=-L_g V_k} < 0$ and finally u_1 minimizes a cost functional of the form

$$J = \int_0^\infty (l(x) + u^2)ds$$

The cross-term can only be exactly evaluated for very simple systems. In other cases it has to be numerically evaluated or approximated by i.e. Taylor series

Connection to Teel's results:
To avoid computations of the integrals we can use nested low-gain (saturated) control.
Also showed to be GAS/LES for the integrator chain, but LAS/LES for the general strict-feedforward system.

(Compare with high-gain design in backstepping)

Can use a feedback passivation design for a system if

1. A relative degree condition satisfied
2. The system is weakly minimum phase

Backstepping is a recursive way of finding a relative degree one output.

Integrator forwarding allows us to stabilize weakly non-minimum phase systems.

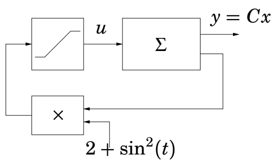
Conclusions

- Global/semiglobal stabilization of strict-feedforward system (No exact linearization possible)
- Tracking results reported
- Relaxes weakly minimum phase-condition
- Integration forwarding - "necessary" to simplify controller

Motivation: Simple example

Consider the following simple feedback system

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u = Ax + Bu \qquad (\Sigma) \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x = Cx \\ u = sat(x_2 \cdot (2 + \sin^2(t))) \end{cases}$$



Example cont'd

- linear subsystem unstable
- input saturation $\quad\Rightarrow$ *At best local stability.*

——————— *Tools* ———————

Locally valid Quadratic Contraint (QC) *(sector condition)*

$$0 \le (\kappa_2 \cdot x_2 - u)(u - \kappa_1 \cdot x_2) = \begin{bmatrix} x_1 & x_2 & u \end{bmatrix} \begin{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 0 & 2 \end{pmatrix} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} \text{ for some } |x_2| < c$$

$$\begin{aligned} \kappa_1 &= 1 && \text{Lower bound :} \\ & && \text{'linear feedback stability cond.'} \\ & && u = \kappa x_2, \kappa \in (1, \infty) \\ \kappa_2 &= 3 && \text{Upper bound :} \\ & && \text{sector of nonlinearity} \end{aligned}$$

Preliminaries

State feedback

$$\begin{cases} \dot{x} = Ax + Bu = Ax + B\phi(x) \\ y = Cx \\ u = \phi(x) \end{cases}$$

Observer feedback

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \\ \dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \\ u = \phi(\hat{x}) \end{cases}$$

Asymptotically stable for state feedback $u = \phi(x)$

Re-write with *error dynamics* ($e = \hat{x} - x$)

$$\begin{cases} \dot{e} = (A - LC)e \\ \dot{x} = Ax + B\phi(x + e) + LCe \\ u = \phi(\hat{x}) \end{cases}$$