Synthesis, Nonlinear design

- Introduction
- Relative degree & zero-dynamics (rev.)
- Exact Linearization (intro)
- Control Lyapunov functions
- Lyapunov redesign
- Nonlinear damping
- Backstepping
  - Control Lyapunov functions (CLFs)
  - passivity
  - robust/adaptive

Ch 13.1-13.2, 14.1-14.3 Nonlinear Systems, Khalil

The Joy of Feedback, P V Kokotovic

Why nonlinear design methods?

- Linear design degraded by nonlinearities (e.g. saturations)
- Linearization not controllable (e.g. pocket parking)
- Long state transitions (e.g. satellite orbits)
- Inherently nonlinear...

Relative degree

" A system's relative degree: How many times you need to take the derivative of the output signal before the input shows up"

Note: A nonlinear system may have state-dependent relative degree.

Example: The ball and beam process (see process homepage for more information).

If nothing else stated we assume a fixed relative degree in the sequel.

For a nonlinear system with *relative degree* d

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

we have

$$\begin{aligned} \dot{y} &= \frac{d}{dt}h(x) = \frac{\partial h(x)}{\partial x}\dot{x} = \frac{\partial h}{\partial x}f(x) + \frac{\partial h}{\partial x}g(x)u \\ &= L_f h(x) + \underbrace{L_g h(x)}_{=0 \ if \ d>1} u \\ &\ \vdots \\ y^{(k)} &= L_f^k h(x) \qquad \text{if } k < d \\ &\ \vdots \\ y^{(d)} &= L_f^d h(x) + L_g L_f^{(d-1)} h(x) u \end{aligned} \tag{2}$$

Using the same kind of coordinate transformations as for the feedback linearizable systems above, we can introduce new state space variables, ξ, where the first d coordinates are chosen as

$$\begin{cases} \xi_1 &= h(x) \\ \xi_2 &= L_f h(x) \\ \vdots \\ \xi_d &= L_f^{(d-1)} h(x) \end{cases} \tag{3}$$

Under some conditions on involutivity, the Frobenius theorem guarantees the existence of another $(n-d)$ functions to provide a local state transformation of full rank. Such a coordinate change transforms the system to the *normal form*

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ &\ \vdots \\ \dot{\xi}_{d-1} &= \xi_d \\ \dot{\xi}_d &= L_f^d h(\xi,z) + L_g L_f^{d-1} h(\xi,z) u \\ \dot{z} &= \psi(\xi,z) \\ y &= \xi_1 \end{aligned} \tag{4}$$

where $\dot{z} = \psi(\xi,z)$ represent the zero dynamics of order $n-d$ [Byrnes+Isidori 1991].

Example (Zero dynamics for linear systems)

Consider the linear system

$$y = \frac{s-1}{s^2+2s+1}u \tag{5}$$

with the following state-space description

$$\begin{cases} \dot{x}_1 &= -2x_1 + x_2 \quad +u \\ \dot{x}_2 &= -x_1 \qquad\quad -u \\ y &= x_1 \end{cases} \tag{6}$$

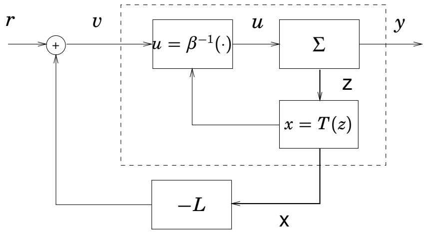
We have the relative degree =1
Find the zero-dynamics, by assigning $y \equiv 0$.

$$\begin{aligned} & y \equiv 0 \Rightarrow x_1 \equiv 0 \ \Rightarrow \ \dot{x}_1 \equiv 0 \ \Rightarrow x_2 + u = 0 \\ & \Rightarrow \dot{x}_2 = -u = x_2 \end{aligned} \tag{7}$$

The remaining dynamics is an unstable system corresponding to the zero $s=1$ in the transfer function (**??**).

Exact (feedback) Linearization

Idea: Transform the nonlinear system into a linear system by means of feedback and/or a change of variables. After this, a stabilizing state feedback is designed.



Inner *feedback linearization* and outer *linear* feedback control

State transformation

More difficult example, where we need a state transformation

$$\dot{x}_1 = a\sin(x_2)$$
$$\dot{x}_2 = -x_1^2 + u$$

Can not cancel $a\sin(x_2)$. Introduce

$$z_1 = x_1$$
$$z_2 = a\sin x_2$$

so that

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = (-z_1^2 + u)a\cos x_2$$

Then feedback linearization is (locally) possible by

$$u = z_1^2 + v/(a\cos(z_2))$$

When to cancel nonlinearities?

$$\dot{x}_1 = -x_1^3 + u_1$$
$$\dot{x}_2 = x_2^3 + u_2 \qquad (8)$$

Nonrobust and/or not necessary.
However, note the difference between tracking or regulation!!

Will see later how "optimal criteria" will give hints.

Achieving passivity by feedback (*Feedback passivation*)
Need to have

- relative degree one
- weakly minimum phase

NOTE! (Nonlinear) relative degree and zero-dynamics *invariant* under feedback!
Two major challenges:

- avoid non-robust cancellations
- make it constructive by finding matching input-output pairs

For general nonlinear systems feedback linearization comprises

- state transformation
- inversion of nonlinearities
- linear feedback

Simple example

$$\ddot{x} = \frac{g}{l}\sin(x) + \cos(x)u$$

Put

$$u = \frac{1}{\cos(x)}(-\frac{g}{l}\sin(x) + v)$$

gives (locally)

$$\ddot{x} = v$$

Design linear controller $v = -l_1 x + -l_2\dot{x}$, etc

Feedback linearization ("nonlinear version of pole-zero cancellation")

Feedback linearization can be interpreted as a nonlinear version of pole-zero cancellations which can not be used if the zero-dynamics are unstable, i. e., for *nonminimum-phase system*.

Linear systems: See paper [Middleton (1999) Automatica 35(5), "Slow stable open-loop poles: to cancel or not to cancel"]

"Matching" uncertainties

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{n-1} &= x_d \\
\dot{x}_n &= L_f^d h(x,z) + L_g L_f^{d-1} h(x,z) u \\
\dot{z} &= \psi(x,z) \\
y &= x_1
\end{aligned} \qquad (9)$$

Integrator chain and nonlinearities (+ zero-dynamics)
Note that uncertainties due to parameters etc. are "collected in"

$$L_f^d h(x,z) + L_g L_f^{d-1} h(x,z) u$$

Exact Linearization

- Often useful in simple cases
- Important intuition may be lost
- Nonlinear version of "pole-zero cancellations"
- Related to "Lie brackets" and "flatness"

From analysis to synthesis

Lyapunov criterion Search for (V, u) such that

$$\frac{\partial V}{\partial x}[f + gu] < 0$$

IQC criterion Search for $Q(s)$ and $\tau_1, \dots, \tau_m$ such that

$$\begin{bmatrix} [T_1 + T_2QT_3](i\omega) \\ I \end{bmatrix}^* \left[\sum_k \tau_k \Pi_k(i\omega)\right] \begin{bmatrix} [T_1 + T_2QT_3](i\omega) \\ I \end{bmatrix} < 0$$

for $\omega \in [0, \infty]$

In both cases, the problem is non-convex and hard.
Heuristic idea: Iterate between the arguments

Control Lyapunov Function (CLF)

A positive definite radially unbounded C^1 function V is called a CLF for the system $\dot{x} = f(x, u)$ if for each $x \neq 0$, there exists u such that

$$\frac{\partial V}{\partial x}(x) f(x, u) < 0 \qquad \text{(Notation: } L_f V(x) < 0\text{)}$$

When $f(x, u) = f(x) + g(x)u$, V is a CLF if and only if

$$L_f V(x) < 0 \text{ for all } x \neq 0 \text{ such that } |L_g V(x)| = 0$$

Sontag's formula

If V is a CLF for the system $\dot{x} = f(x) + g(x)u$, then a continuous asymptotically stabilizing feedback is defined by

$$u(x) := \begin{cases} 0 & \text{if } L_g V(x) = 0 \\ -\frac{L_f V + \sqrt{(L_f V)^2 + ((L_g V)(L_g V)^T)^2}}{(L_g V)(L_g V)^T}[L_g V]^T & \text{if } L_g V(x) \neq 0 \end{cases}$$

Note: Can cancel factor $L_g V \neq 0$ if scalar.

$$u(x) := \begin{cases} 0 & \text{if } L_g V(x) = 0 \\ -\frac{L_f V + \sqrt{(L_f V)^2 + (L_g V)^4}}{L_g V}(x) & \text{if } L_g V(x) \neq 0 \end{cases}$$

motivation: Feedback Linearization

One of the drawbacks with feedback linearization is that exact cancellation of nonlinear terms may not be possible due to e. g., parameter uncertainties.

A suggested solution:

- stabilization via feedback linearization around a nominal model
- consider known bounds on the uncertainties to provide an additional term for stabilization (*Lyapunov redesign*)

Convexity for state feedback

Problem Suppose $\alpha \leq \phi(v)/v \leq \beta$. Given the system

$$\dot{x} = f_u(x) := Ax + E\phi(Fx) + Bu$$

find $u = -Lx$ and $V(x) = x^T P x$ such that $\frac{\partial V}{\partial x} f_u(x) < 0$

Solution Solve for P, L

$$(A + \alpha EF - BL)^T P + P(A + \alpha EF - BL) < 0$$
$$(A + \beta EF - BL)^T P + P(A + \beta EF - BL) < 0$$

or equivalently convex in $(Q, K) = (P^{-1}, LP^{-1})$

$$(AQ + \alpha EFQ - BK)^T + (AQ + \alpha EFQ - BK) < 0$$
$$(AQ + \beta EFQ - BK)^T + (AQ + \beta EFQ - BK) < 0$$

Example

Check if $V(x, y) = [x^2 + (y + x^2)^2]^2/2$ is a CLF for the system

$$\begin{cases} \dot{x} = xy \\ \dot{y} = -y + u \end{cases}$$

$$L_f V(x, y) = x^2 y + (y + x^2)(-y + 2x^2 y)$$
$$L_g V(x, y) = 2(y + x^2)[x^2 + (y + x^2)^2]$$

$$L_g V(x, y) = 0 \quad \Rightarrow \quad y = -x^2 \quad \Rightarrow \quad L_f V(x, y) = -x^4 < 0 \quad \text{if } (x, y)$$

Backstepping idea

Problem

Given a CLF for the system

$$\dot{x} = f(x, u)$$

find one for the extended system

$$\dot{x} = f(x, y)$$
$$\dot{y} = h(x, y) + u$$

Idea

Use y to control the first system. Use u for the second.

Note potential for recursivity

Lyapunov Redesign

Consider the nominal system

$$\dot{x} = f(x, t) + G(x, t)u$$

with the known control law

$$u = \psi(x, t)$$

so that the system is uniformly asymptotically stable.

Assume that a Lyapunov function $V(x, t)$ is known s.t.

$$\alpha_1(||x||) \leq V(x, t) \leq \alpha_2(||x||)$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}[f(t, x) + G\psi] \leq -\alpha_3(||x||)$$

Lyapunov Redesign — *cont.*

Perturbed system

$$\dot{x} = f(x,t) + G(x,t)[u + \delta] \qquad (10)$$

disturbance $\delta = \delta(t,x,u)$

Assume the disturbance satisfies the bound

$$||\delta(t,x,\psi + v)|| \leq \rho(x,t) + \kappa_0 ||v||$$

If we know ρ and κ_0 how do we design *additional control* v such that $u = \psi(x,t) + v$ stabilizes (**??**)?

The matching condition: perturbation enters at same place as control signal u.

Apply $u = \psi(x,t) + v$

$$\dot{x} = f(x,t) + G(x,t)\psi + G(x,t)[v + \delta(t,x,\psi + v)] \qquad (11)$$

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}[f(t,x) + G\psi] + \frac{\partial V}{\partial x}G[v + \delta] \leq -\alpha_3(||x||) + \frac{\partial V}{\partial x}G[v +$$

Introduce $w = [\frac{\partial V}{\partial x}G]$

$$\dot{V} \leq -\alpha_3(||x||) + w^T v + w^T \delta$$

Choose v such that $w^T v + w^T \delta \leq 0$:

Two alternatives presented in Khalil ($|| \cdot ||_2$-norm / $|| \cdot ||_\infty$-norm)

Note: v appears at same place as δ due to the matching condition

Lyapunov Redesign — *cont.*

$$w^T v + w^T \delta \leq w^T v + ||w^T||_2 ||\delta||_2$$
$$w^T v + w^T \delta \leq w^T v + ||w^T||_1 ||\delta||_\infty$$

Alternative 1:

If
$$||\delta(t,x,\psi + v)||_2 \leq \rho(x,t) + \kappa_0 ||v||_2, \quad 0 \leq \kappa_0 < 1$$

take
$$v = -\eta(t,x)\frac{w}{||w||_2}$$

where $\eta \geq \rho/(1-\kappa_0)$

Alternative 2:

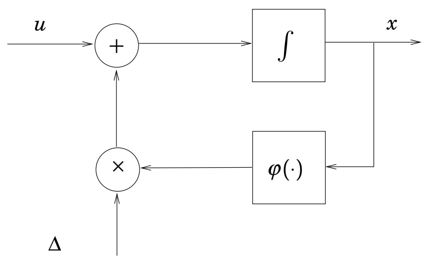
If
$$||\delta(t,x,\psi + v)||_\infty \leq \rho(x,t) + \kappa_0 ||v||_\infty, \quad 0 \leq \kappa_0 < 1$$

take
$$v = -\eta(t,x)\,\mathrm{sgn}\,w$$

where $\eta \geq \rho/(1-\kappa_0)$
Restriction on $\kappa_0 < 1$ but not on growth of ρ.
Alt 1 and alt 2 coincide for single-input systems.

Note: control laws are discontinues fcn of x (risk of *chattering*)

Example: Matched uncertainty



$$\dot{x} = u + \varphi(x)\Delta(t)$$

Example *cont.*

Example:

Exponentially decaying disturbance $\Delta(t) = \Delta(0)e^{-kt}$
linear feedback $u = -cx, \quad c > 0$
$\varphi(x) = x^2$

$$\dot{x} = -cx + \Delta(0)e^{-kt}x^2$$

Similar to peaking problem in the first lecture: Finite escape of solution to infinity if $\Delta(0)x(0) > c + k$

We want to guarantee that x(t) stay bounded for all initial values $x(0)$ and all bounded disturbances $\Delta(t)$

Nonlinear damping

Modify the control law in the previous example as:

$$u = -cx - s(x)x$$

where
$$-s(x)x$$
will be denoted *nonlinear damping.*

Use the Lyapunov function candidate $V = \dfrac{x^2}{2}$

$$\dot{V} = xu + x\varphi(x)\Delta$$
$$= -cx^2 - x^2 s(x) + x\varphi(x)\Delta$$

How to proceed?

Choose
$$s(x) = \kappa\varphi^2(x)$$
to complete the squares!

$$\dot{V} = -cx^2 - x^2 s(x) + x\varphi(x)\Delta$$
$$= -cx^2 - \kappa\left[x\varphi - \frac{\Delta}{2\kappa}\right]^2 + \kappa \cdot \frac{\Delta^2}{4\kappa^2} \leq -cx^2 + \frac{\Delta^2}{4\kappa}$$

Note! $\dot{V}$ is negative whenever

$$|x(t)| \geq \frac{\Delta}{2\sqrt{\kappa c}}$$

Can show that $x(t)$ converges to the set

$$R = \left\{ x : |x(t)| \leq \frac{\Delta}{2\sqrt{\kappa c}} \right\}$$

i. e., x(t) stays bounded for all bounded disturbances Δ

Remark: The nonlinear damping $-\kappa x \varphi^2(x)$ renders the system Input-To-State Stable (ISS) with respect to the disturbance.

Backstepping idea

Problem

Given a CLF for the system

$$\dot{x} = f(x, u)$$

find one for the extended system

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= h(x, y) + u \end{aligned}$$

Idea

Use y to control the first system. Use u for the second.

Note: potential for recursivity

Backstepping Example

For the system

$$\begin{cases} \dot{x} = x^2 + y \\ \dot{y} = u \end{cases}$$

we can choose $V_x(x) = x^2$ and $\phi(x) = -x^2 - x$ to get the control law

$$\begin{aligned} u &= \phi'(x) f(x, y) - h(x, y) + \phi(x) - y \\ &= -(2x + 1)(x^2 + y) - x^2 - x - y \end{aligned}$$

with Lyapunov function

$$\begin{aligned} V(x, y) &= V_x(x) + [y - \phi(x)]^2/2 \\ &= x^2 + (y + x^2 + x)^2/2 \end{aligned}$$

We can't expect to realize $x_2 = \alpha(x_1)$ exactly, but we can always try to get
the error $\to 0$.

Introduce the error states

$$\begin{cases} z_1 = x_1 \\ z_2 = x_2 - \alpha_1(x_1) \end{cases} \qquad (13)$$

where $\alpha_1(x_1) = -x_1 - x_1^2$

$$\begin{aligned} \Rightarrow \dot{z}_1 &= \dot{x}_1 = z_1^2 + \overbrace{z_2 + \alpha_1(z_1)}^{x_2} = \\ &= z_1^2 + z_2 - z_1^2 - z_1 = -z_1 + z_2 \\ \dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 = u(x) - \overbrace{\dot{\alpha}_1}^{known} \\ \dot{\alpha}_1 &= \frac{d}{dt}(-z_1^2 - z_1) = -z_1 \dot{z}_1 - \dot{z}_1 \\ &= -z_1(-z_1 + z_2) - (-z_1 + z_2) = \\ &= z_1^2 - z_1 z_2 - z_2 - z_1 \end{aligned}$$

Young's inequality

Let $p > 1$, $q > 1$ s.t. $(p-1)(q-1) = 1$,
then for all $\epsilon > 0$ and all $(x, y) \in |R^2$

$$xy < \frac{\epsilon^p}{p} |x|^p + \frac{1}{q \epsilon^q} |y|^q$$

Standard case: $(p = q = 2, \epsilon^2/2 = \kappa)$

$$xy < \kappa |x|^2 + \frac{1}{4\kappa} |y|^2$$

Our example:

$$x \varphi(x) \Delta(t) < \kappa x^2 \varphi^2(x) + \frac{\Delta^2(t)}{4\kappa}$$

Backstepping

Let V_x be a CLF for the system $\dot{x} = f(x) + g(x)\bar{y}$ with corresponding asymptotically stabilizing control law $\bar{y} = \phi(x)$.
Then $V(x, y) = V_x(x) + [y - \phi(x)]^2/2$ is a CLF for the system'

$$\begin{aligned} \dot{x} &= f(x) + g(x) y \\ \dot{y} &= h(x, y) + u \end{aligned}$$

with corresponding control law

$$u = \frac{\partial \phi}{\partial x}[f(x) + g(x)y] - \frac{\partial V_x}{\partial x} g(x) - h(x, y) + \phi(x) - y$$

Proof.

$$\begin{aligned} \dot{V} &= (\partial V_x/\partial x)(f + gy) + (y - \phi)\,[h + u - (\partial \phi/\partial x) \cdot (f + gy)] \\ &= (\partial V_x/\partial x)(f + g\phi) + (y - \phi)[(\partial V_x/\partial x) g - (\partial \phi/\partial x) \cdot (f + gy) + h \\ &= (\partial V_x/\partial x)(f + g\phi) - (y - \phi)^2 < 0 \end{aligned}$$

Example again (step by step)

$$\begin{cases} \dot{x}_1 = {x_1}^2 + x_2 \\ \dot{x}_2 = u(x) \end{cases} \qquad (12)$$

Find u(x) which stabilizes (**??**).

Idea : Try first to stabilize the x_1-system with x_2 and then stabilize the whole system with u.

We know that if $x_2 = -x_1 - x_1^2$
then $x_1 \to 0$ asymptotically (exponentially)
as $t \to \infty$.

Start with a Lyapunov for the first subsystem (z_1-dynamics):

$$\begin{aligned} V_1 &= \frac{1}{2} z_1^2 \geq 0 \\ \dot{V}_1 &= z_1 \dot{z}_1 = -z_1^2 + z_1 z_2 \end{aligned}$$

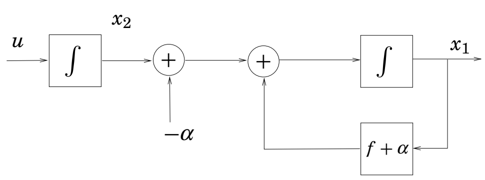
<u>Note :</u>
If $z_2 = 0$ we would achieve $V_1 = -z_1^2 \leq 0$
with $\alpha_1(x_1)$

Now look at the augmented Lyapunov fcn for the error system

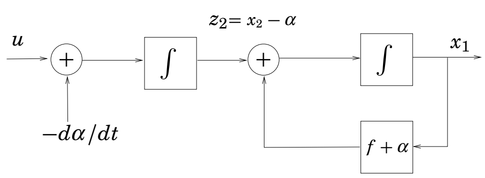
$$
\begin{aligned}
V_2 &= V_1 + \frac{1}{2}z_2^2 \geq 0 \\
\dot{V}_2 &= \dot{V}_1 + z_2\dot{z}_2 = \\
&= -z_1^2 + z_1z_2 + z_2(u - z_1^2 + z_1z_2) \\
&= -z_1^2 + z_2\underbrace{(u - z_1^2 + z_1z_2 + z_2 + z_1)}_{choose\ =\ -z_2} \\
&= -z_1^2 - z_2^2 \leq 0
\end{aligned}
$$

so if $u = z_1^2 - z_1z_2 - z_2 - z_1$
$\Rightarrow (z_1, z_2) \to 0$ asymptotically (exponentially)
$\Rightarrow (x_1, x_2) \to 0$ asymptotically

As $z_1 = x_1$ and $z_2 = x_2 - \alpha_1 = x_2 + x_1^2 + x_1$,
we can express u as a (nonlinear) state feedback function of x_1 and x_2.



Move the control “backwards” through the integrator



Note the change of coordinates!

Lyapunov function : $V_1 = \frac{1}{2}{z_1}^2 + \frac{1}{2}\tilde{\theta}^2$ where $\tilde{\theta} = (\hat{\theta} - \theta)$ is the parameter error

(Back-) Step 1:

$$
\dot{z}_1(t) = \overbrace{z_2(t) + \alpha_1(z_1, \hat{\theta})}^{x_2} + \theta\gamma(z_1(t))
$$

$$
\begin{aligned}
\dot{V}_1 &= z_1\dot{z}_1 + \tilde{\theta}\dot{\hat{\theta}} = z_1(z_2 + \alpha_1 + \theta\gamma) + \tilde{\theta}\dot{\hat{\theta}} = \\
&= z_1[\ z_2 + \underbrace{\alpha_1 + \hat{\theta}\gamma}_{-z_1}\ ] + \tilde{\theta}(\dot{\hat{\theta}} - \underbrace{z_1\gamma}_{\tau_1})
\end{aligned}
$$

Choose $\alpha_1 = -z_1 - \hat{\theta}\gamma$

$$
\Rightarrow \dot{V}_1 = -z_1^2 + z_1z_2 + \tilde{\theta}(\dot{\hat{\theta}} - \tau_1)
$$

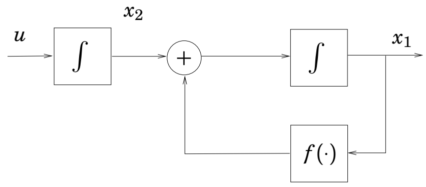
Augmented Lyapunov function :

$$
V_2 = V_1 + \frac{1}{2}{z_2}^2
$$

$$
\begin{aligned}
\dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 = \\
&= \overbrace{z_3 + \alpha_2}^{x_3} - \frac{\partial\alpha_1}{\partial z_1}(x_2 + \theta\gamma) - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} \\
\dot{V}_2 &= \dot{V}_1 + z_2\dot{z}_2 = \ \ldots = \\
&= -z_1^2 + z_2[\ \underbrace{z_3 + \alpha_2 + z_1 + \frac{\partial\alpha_1}{\partial z_1}(z_1 - z_2) - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}}_{-z_2}\ ] + \\
&+ \ \tilde{\theta}[\ \dot{\hat{\theta}} - \underbrace{(\tau_1 + z_2\frac{\partial\alpha_1}{\partial z_1}\gamma)}_{\tau_2}\ ]
\end{aligned}
$$

Choose $\alpha_2 = -z_2 - z_1 - \frac{\partial\alpha_1}{\partial z_1}(z_1 - z_2) + \frac{\partial\alpha_1}{\partial\hat{\theta}}\overset{\Downarrow}{\tau_2}$

Backward propagation of desired control signal



If we could use x_2 as control signal, we would like to assign it to $\alpha(x_1)$ to stabilize the x_1-dynamics.

Adaptive Backstepping

System :

$$
\begin{cases}
\dot{x}_1 = x_2 + \theta\gamma(x_1) \\
\dot{x}_2 = x_3 \\
\dot{x}_3 = u(t)
\end{cases}
\tag{14}
$$

where γ is a known function of x_1 and
θ is an unknown parameter
Introduce new (error) coordinates

$$
\begin{cases}
z_1(t) = x_1(t) \\
z_2(t) = x_2(t) - \alpha_1(z_1, \hat{\theta})
\end{cases}
\tag{15}
$$

where α_1 is used as a control to stabilize the z_1- system w.r.t a certain Lyapunov-function.

Note: If we used $\dot{\hat{\theta}} = \tau_1$ as update law
and if $z_2 = 0$ then $\dot{V}_1 = -z_1^2 \leq 0$
Step 2: Introduce $z_3 = x_3 - \alpha_2(z_1, z_2, \hat{\theta})$ and use α_2 as control to stabilize the (z_1, z_2)-system

Note : If $z_3 = 0$ and we used $\dot{\hat{\theta}} = \tau_2$ as update law we would get $\dot{V}_2 = -z_1^2 - z_2^2 \leq 0$

Resulting subsystem

$$
\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}}_{Hurwitz} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -\gamma \\ \frac{\partial\alpha_1}{\partial z_1}\gamma \end{bmatrix} \tilde{\theta} + \begin{bmatrix} 0 \\ z_3 - \frac{\partial\alpha_1}{\partial\hat{\theta}}(\dot{\hat{\theta}} - \tau_2) \end{bmatrix}
$$

$$
\tau_2 = \begin{bmatrix} \gamma & -\frac{\partial\alpha_1}{\partial x_1}\gamma \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}
$$

$\dot{V}_2 = -z_1^2 - z_2^2 + z_3z_2 + \tilde{\theta}(\dot{\hat{\theta}} - \tau_2) - z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}(\dot{\hat{\theta}} - \tau_2)$

Step 3 :

$$
\begin{aligned}
\dot{z}_3 &= \dot{x}_3 - \dot{\alpha}_2 \\
&= u - \frac{\partial\alpha_2}{\partial z_1}\dot{z}_1 - \frac{\partial\alpha_2}{\partial z_2}\dot{z}_2 - \frac{\partial\alpha_2}{\partial\hat{\theta}}\dot{\hat{\theta}} = \ \ldots = \\
&= puh...
\end{aligned}
$$

We now want to choose $u = u(z_1, z_2, \hat{\theta})$ such that the whole system will be stabilized w.r.t V_3

Just to simplify the expressions, introduce $\overline{u}$, and choose

$$\begin{aligned} u &= -z_2 - z_3 + \frac{\partial \alpha_2}{\partial z_1}(z_2 + \alpha_2 + \hat{\theta}\gamma) + \frac{\partial \alpha_2}{\partial z_2}x_3 + \frac{\partial \alpha_2}{\partial \hat{\theta}}\dot{\hat{\theta}} + \overline{u} \\ \overline{u} &= ? \end{aligned}$$

Augmented Lyapunov function :

$$\begin{aligned} V_3 &= V_2 + \frac{1}{2}{z_3}^2 = \frac{1}{2}\| z \|^2 + \frac{1}{2}\tilde{\theta}^2 \\ \dot{V}_3 &= -z_1^2 - z_2^2 - z_3^2 + z_3\overline{u} + \tilde{\theta}[\dot{\hat{\theta}} - \underbrace{(\tau_2 - z_3\frac{\partial \alpha_2}{\partial z_1}\gamma)}_{\tau_3}] \\ &- z_2\frac{\partial \alpha_1}{\partial \hat{\theta}}(\dot{\hat{\theta}} - \tau_2) \end{aligned}$$

$\Rightarrow \dot{\hat{\theta}} = (\tau_3) = \tau_2 - z_3\frac{\partial \alpha_2}{\partial z_1}\gamma$

$\dot{V}_3 = -\| z \|^2 + z_3(\overline{u} + z_2\sigma)$

Choose $\overline{u} = -z_2\sigma$

Finally :

$\dot{V}_3 = -\| z \|^2 \Rightarrow$ GS of $z = 0, \hat{\theta} = \theta$ and $x \to 0$ (by La Salle's Theorem)

Closed-loop system :

$$\dot{z} = \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 1+\sigma \\ 0 & -1-\sigma & -1 \end{bmatrix}}_{skew-symmetric\ -\ I} z + \begin{bmatrix} -\gamma \\ \frac{\partial \alpha_1}{\partial z_1}\gamma \\ \frac{\partial \alpha_2}{\partial z_1}\gamma \end{bmatrix} \tilde{\theta}$$

$$-\tau_3 = \begin{bmatrix} \gamma & -\frac{\partial \alpha_1}{\partial z_1}\gamma & -\frac{\partial \alpha_2}{\partial z_1}\gamma \end{bmatrix} z$$

Backstepping applies to systems in *strict-feedback form*

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + x_3 \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \dots x_{n-1}, x_n) + u \end{aligned}$$

Compare with

Strict-feedforward systems

$$\begin{aligned} \dot{x}_1 &= x_2 + f_1(x_2, x_3, \dots, x_n, u) \\ \dot{x}_2 &= x_3 + f_2(x_3, \dots, x_n, u) \\ &\vdots \\ \dot{x}_{n-1} &= x_n + f_{n-1}(x_n, u) \\ \dot{x}_n &= u \end{aligned}$$

We are almost there :

$$\dot{z} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix} z + \begin{bmatrix} -\gamma \\ \frac{\partial \alpha_1}{\partial z_1}\gamma \\ \frac{\partial \alpha_2}{\partial z_1}\gamma \end{bmatrix} \tilde{\theta} + \begin{bmatrix} 0 \\ \frac{\partial \alpha_1}{\partial \hat{\theta}}(\tau_2 - \dot{\hat{\theta}}) \\ \overline{u} \end{bmatrix}$$

$$\dot{\hat{\theta}} = \begin{bmatrix} \gamma & -\frac{\partial \alpha_1}{\partial z_1}\gamma & -\frac{\partial \alpha_2}{\partial z_1}\gamma \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$\dot{V}_3 = -\| z \|^2 + z_3\overline{u} + z_2\frac{\partial \alpha_1}{\partial \hat{\theta}}(\tau_2 - \dot{\hat{\theta}})$

Crucial :

$$\frac{\partial \alpha_1}{\partial \hat{\theta}}(\tau_2 - \dot{\hat{\theta}}) = z_3 \underbrace{\frac{\partial \alpha_1}{\partial \hat{\theta}}\frac{\partial \alpha_2}{\partial z_1}\gamma}_{known \overset{\text{def}}{=} \sigma}$$

$$\dot{z} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 1+\sigma \\ 0 & -1 & -1 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \overline{u} \end{bmatrix} + \begin{bmatrix} -\gamma \\ \frac{\partial \alpha_1}{\partial z_1}\gamma \\ \frac{\partial \alpha_2}{\partial z_1}\gamma \end{bmatrix} \tilde{\theta}$$

Observer backstepping

Observer backstepping is based on the following steps:

1. A (nonlinear) observer is designed which provides (exponentially) convergent estimates.
2. Backstepping is applied to a system where the states have been replaces by their estimates.
   The observation errors are regarded as (bounded) disturbances and handled by *nonlinear damping.*