Nonlinear Control Theory 2006

- Nonlinear Phenomena and Stability theory

- Nonlinear phenomena [Khalil Ch 3.1]
  - existence and uniqueness
  - finite escape time
  - peaking
- Linear system theory revisited
- Second order systems [Khalil Ch 2.4, 2.6]
  - periodic solutions / limit cycles
- Stability theory [Khalil Ch. 4]
  - Lyapunov Theory revisited
  - exponential stability
  - quadratic stability
  - time-varying systems
  - invariant sets
  - center manifold theorem

Existence problems of solutions

Example: The differential equation

$$\frac{dx}{dt} = x^2, \qquad x(0) = x_0$$

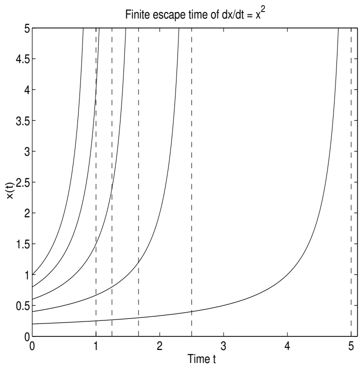
has the solution

$$x(t) = \frac{x_0}{1 - x_0 t}, \qquad 0 \le t < \frac{1}{x_0}$$

Finite escape time

$$t_f = \frac{1}{x_0}$$

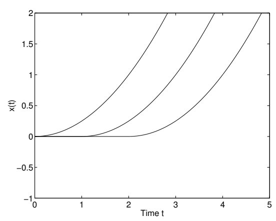
Finite Escape Time



Uniqueness Problems

Example: The equation $\dot{x} = \sqrt{x}$, $x(0) = 0$ has many solutions:

$$x(t) = \begin{cases} (t-C)^2/4 & t > C \\ 0 & t \le C \end{cases}$$



Compare with water tank:

Previous problem is like the water-tank problem in backward time

(Substitute $\tau = -t$ in differential equation).

$$dh/dt = -a\sqrt{h}, \qquad h : \text{height (water level)}$$

Change to backward-time: "If I see it empty, when was it full?")

Existence and Uniqueness

Theorem
Let Ω_R denote the ball

$$\Omega_R = \{z; \|z - a\| \le R\}$$

If f is Lipschitz-continuous:

$$\|f(z) - f(y)\| \le K\|z - y\|, \qquad \text{for all } z, y \in \Omega$$

then $\dot{x}(t) = f(x(t)), x(0) = a$ has a unique solution in

$$0 \le t < R/C_R,$$

where $C_R = \max_{\Omega_R} \|f(x)\|$

see [Khalil Ch. 3]

The peaking phenomenon

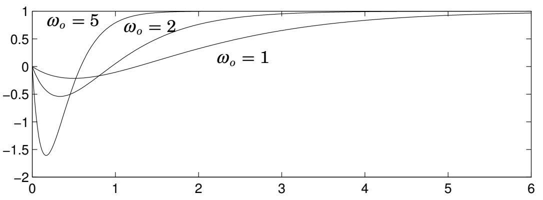
Example: Controlled linear system with right-half plane zero

Feedback can change location of poles but not location of zero (unstable pole-zero cancellation not allowed).

$$G_{cl}(s) = \frac{(-s+1)\omega_o^2}{s^2 + 2\omega_o s + \omega_o^2} \tag{1}$$

A step response will reveal a transient which grows in amplitude for faster closed loop poles $s = -\omega_o$, see Figure on next slide.

The peaking phenomenon –*cont.*



Step responses for the system in Eq. (1), $\omega_o = 1, 2$, and 5. Faster poles gives shorter settling times, but the transients grow significantly in amplitude, so called *peaking*.

The peaking phenomenon – *cont.*

Note!

- Linear case: Performance may be severely deteriorated by peaking, but stability still guaranteed.
- Nonlinear case: Instability and even finite escape time solutions may occur.

What bandwidth constraints does a non-minimum zero impose for linear systems? See e. g., [2, 1, 3]

The peaking phenomenon – *cont.*

We will come back to the peaking phenomenon for

- cascaded systems [Kokotovic & Sussman '91]
- "exact linearization"
- observers [observer backstepping]

Consider the linear system with *relative degree r*

$$G(s) = K \cdot \frac{b_0 + b_1 s + b_2 s^2 + \cdots + s^{n-r}}{a_0 + a_1 s + a_2 s^2 + \cdots + s^n}$$

(Note: numbering has different notation than "standard" AK)

and a minimal realization

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

with

$$\left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right) = \left(\begin{array}{ccccc|c} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} & K \\ \hline b_0 & \ldots & b_{n-r-1} & 1 & \ldots\ 0 & 0 \end{array}\right)$$

Introduce the coordinate transformation:

$$\begin{aligned} z_1 &= y = Cx & &= b_0 x_1 + b_1 x_2 + \cdots + x_{n-r+1} \\ z_2 &= \dot{y} = CAx & &= b_0 x_2 + b_1 x_3 + \cdots + x_{n-r+2} \\ z_r &= y^{(r-1)} = CA^{r-1}x & &= b_0 x_r + b_1 x_{r+1} + \cdots + x_n \end{aligned}$$

+ freedom to introduce additional $(n - r)$ states.

Simplest is to take

$$\begin{aligned} z_{r+1} &= x_1 \\ z_{r+2} &= x_2 \\ &\ldots \\ z_n &= x_{n-r} \end{aligned}$$

(Check that this is a valid coordinate change!)

Ends up with

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\ldots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= R\zeta + S\eta + Ku \\ \dot{\eta} &= P\zeta + Q\eta \\ y &= z_1 \end{aligned}$$

where

$$\zeta = \begin{pmatrix} z_1 & z_2 & \ldots & z_r \end{pmatrix}^T, \quad \eta = \begin{pmatrix} z_{r+1} & \ldots & z_n \end{pmatrix}^T,$$

The zero dynamics: "Remaining dynamics when you choose control input to keep output $y \equiv 0$ $\implies$

$$\boxed{\dot{\eta} = Q\eta}$$

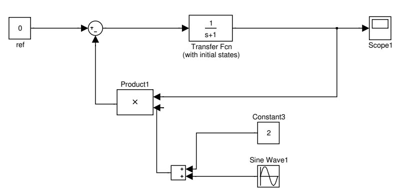
Exercise (HW): Show that

$$Q = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -b_0 & -b_1 & -b_2 & \ldots & -b_{n-r-1} \end{pmatrix}$$

Example: LTV-system

Consider the linear time-varying (LTV) system

$$\begin{aligned} \dot{x} &= -x + u \\ u &= u_{ref} - (2 + sin(t)) \cdot x \end{aligned}$$



What happens when

- $u_{ref} = 0$?
- $u_{ref} = 1$?

(equilibrium, stability, etc.)

Time-varying vs. autonomous systems

Example:

Transfer from first order time-varying system to second-order autonomous system by introducing the states

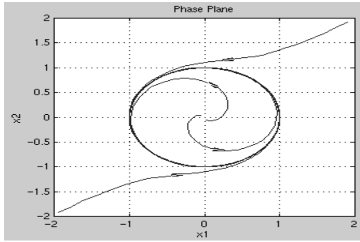
$$x_1 = x, \; x_2 = t \quad \text{(i.e. time)}$$

$$\begin{aligned} \dot{x}_1 &= -x_1 + u_{ref} - (2 + sin(x_2)) \cdot x_1 \\ \dot{x}_2 &= 1 \end{aligned}$$

Periodic Solutions: $x(t+T) = x(t)$

Example of an asymptotically stable periodic solution:

$$\begin{aligned} \dot{x}_1 &= x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{aligned} \tag{2}$$



Periodic solution: Polar coordinates.

Let

$$\begin{aligned} x_1 &= r\cos\theta \quad &\Rightarrow dx_1 = \cos\theta dr - r\sin\theta d\theta \\ x_2 &= r\sin\theta \quad &\Rightarrow dx_2 = \sin\theta dr + r\cos\theta d\theta \end{aligned}$$

$\Rightarrow$

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

Now

$$\begin{aligned} \dot{x}_1 &= r(1-r^2)\cos\theta - r\sin\theta \\ \dot{x}_2 &= r(1-r^2)\sin\theta + r\cos\theta \end{aligned}$$

which gives

$$\begin{aligned} \dot{r} &= r(1-r^2) \\ \dot{\theta} &= 1 \end{aligned}$$

Only $r = 1$ is a stable equilibrium!

A system has a **periodic solution** if for some $T > 0$

$$x(t+T) = x(t), \quad \forall t \geq 0$$

Note that a constant value for x(t) by convention not is regarded as periodic.

- When does a periodic solution exist?
- When is it locally (asymptotically) stable? When is it globally asymptotically stable?

2nd order systems - existence of periodic cycles

Poincaré-Bendixson Criterion

$$\dot{x} = f(x) \tag{3}$$

Consider the system (3) and let M be a closed bounded subset of the plane s.t.

(i) M contains no equilibrium, or only one equilibrium such that the eigenvalues of the Jacobian $[\frac{\partial f}{\partial x}]_{x=x_o}$ has $\in RHP$ (*unstable node* or *unstable focus*)

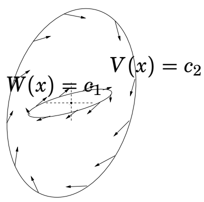
(ii) Every trajectory starting in M stays in M for all future time

Then, M contains a periodic orbit of (3)

Note: no uniqueness.

Checking condition (ii).

Find $V(x)$ s.t.



$$f(x) \cdot \nabla V(x) = \frac{\partial V}{\partial x_1}(x) \cdot f_1(x) + \frac{\partial V}{\partial x_2}(x) \cdot f_2(x) < 0$$

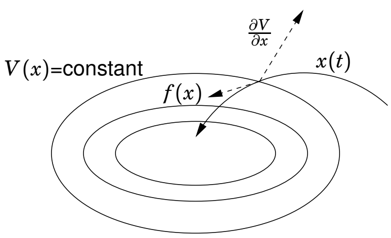
on $V(x) = c_2$ (vector field pointing *inwards*, solutions can't escape outside...)

Exclude vicinity of unstable focus by finding region $W(x) = c_1$ s.t.

$$f(x) \cdot \nabla W > 0$$

Remark: V,W s.t. "larger values of level sets outwards".

Geometric interpretation



Vector field points inwards (scalar product negative , angle > 90 deg)

Trajectories can only go to lower value of $V(x)$

In Lyapunov theory we want this to hold for all level sets, here only necessary for one level set.

Bendixson Criterion

If, on a simple connected region D of the plane, the expression

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

- is **not** identically zero
- does **not** change sign

then the system (3) **has no periodic orbits** lying entirely in D.

Bendixson Criterion — cont'd

Proof (sketch): On any closed orbit γ we have

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2) \Rightarrow dx_2/dx_1 = f_2/f_1\end{aligned}$$

and

$$\int_\gamma f_2(x_1, x_2)dx_1 - f_1(x_1, x_2)dx_2 = 0$$

Green's theorem gives

$$\int\int_S (\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2})dx_1 dx_2 = 0 \qquad (4)$$

where S is the interior area of the closed orbit γ

Now, if the expression is sign definite (> 0 or < 0) on D then we can NOT find any area S such that Eq. (4) holds.

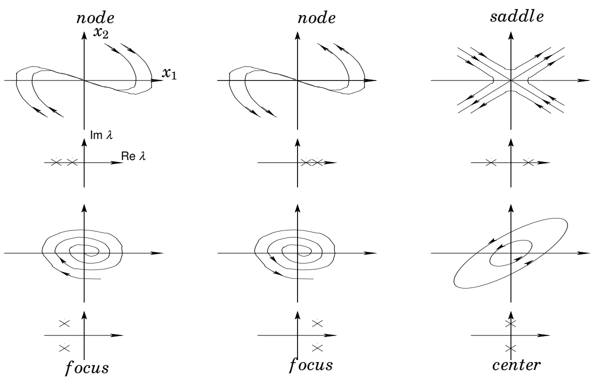
Poincare index

Useful for existence of limit cycles:

Poincare index:

- The index of a node, a focus or a center is $+1$
- The index of a saddle point is -1
- The index of a closed orbit is $+1$
- The index of a closed curve not encircling any equilibrium is 0
- The index of a closed curve equals the sum of indices of the equilibria inside it

Equilibrium Points for Linear Systems



Poincare index, cont'd

Corollary: Inside any periodic orbit γ, there must be at least one equilibrium point.

If the equilibria are hyperbolic (i.e., $Re(\lambda_J) \neq 0$), it must be that

$$N - S = 1$$

where

N=# nodes and foci,

S=# saddles.

Used to rule out existence of periodic orbits in a region

Example:[Khalil]

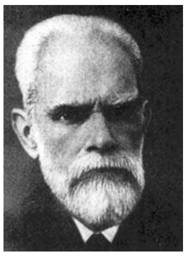
$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1 x_2\\ \dot{x}_2 &= x_1 + x_2 - 2x_1 x_2\end{aligned}$$

Equilibra: $\{(0, 0), (1, 1)\}$

$$\begin{aligned}\left[\frac{\partial f}{\partial x}\right]_{x=(0,0)} &= \begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix} \qquad \text{(saddle)}\\ \left[\frac{\partial f}{\partial x}\right]_{x=(1,1)} &= \begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix} \qquad \text{(stable focus)}\end{aligned}$$

Can be limit cycle around the single focus, but not a limit cycle around both equilibra.

Alexandr Mihailovich Lyapunov (1857–1918)



Master thesis "On the stability of ellipsoidal forms of equilibrium of rotating fluids," St. Petersburg University, 1884.

Doctoral thesis "The general problem of the stability of motion," 1892.

Lyapunov formalized the idea:

If the total energy is dissipated, the system must be stable.

Main benefit: By looking at an energy-like function (a so called Lyapunov function), we might conclude that a system is stable or asymptotically stable **without solving** the nonlinear differential equation.
Trades the difficulty of solving the differential equation to:

"How to find a Lyapunov function?"

Many cases covered in [5]

Stability Definitions

An equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

locally stable, if for every $R > 0$ there exists $r > 0$, such that

$$\|x(0)\| < r \quad \Rightarrow \quad \|x(t)\| < R, \quad t \geq 0$$

locally asymptotically stable, if locally stable and

$$\|x(0)\| < r \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = 0$$

globally asymptotically stable, if asymptotically stable for all $x(0) \in \mathbf{R}^n$.

Lyapunov Theorem for Local Stability

Theorem Let $\dot{x} = f(x)$, $f(0) = 0$, and $0 \in \Omega \subset \mathbf{R}^n$. Assume that $V : \Omega \to \mathbf{R}$ is a C^1 function. If

- $V(0) = 0$
- $V(x) > 0$, for all $x \in \Omega$, $x \neq 0$
- $\frac{d}{dt}V(x) \leq 0$ along all trajectories in Ω

then $x = 0$ is locally stable. Furthermore, if also

- $\frac{d}{dt}V(x) < 0$ for all $x \in \Omega$, $x \neq 0$

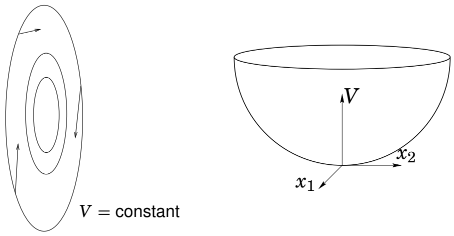
then $x = 0$ is locally asymptotically stable.

Proof: Read proof in [Khalil] or [Slotine].

Lyapunov Functions ($\approx$ Energy Functions)

A *Lyapunov function* fulfills $V(x_0) = 0$, $V(x) > 0$ for $x \in \Omega$, $x \neq x_0$, and

$$\dot{V}(x) = \frac{d}{dt}V(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}f(x) \leq 0$$



Lyapunov Theorem for Global Stability

Theorem Let $\dot{x} = f(x)$ and $f(0) = 0$. Assume that $V : \mathbf{R}^n \to \mathbf{R}$ is a C^1 function. If

- $V(0) = 0$
- $V(x) > 0$, for all $x \neq 0$
- $\dot{V}(x) < 0$ for all $x \neq 0$
- $V(x) \to \infty$ as $\|x\| \to \infty$ **radially unbounded**

then $x = 0$ is globally asymptotically stable.

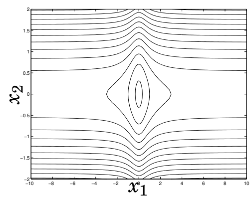
Note! Can be only one equilibrium.

Radial Unboundedness is Necessary

If the condition $V(x) \to \infty$ as $\|x\| \to \infty$ is not fulfilled, then global stability cannot be guaranteed.

Example Assume $V(x) = x_1^2/(1 + x_1^2) + x_2^2$ is a Lyapunov function for a system. Can have $\|x\| \to \infty$ even if $\dot{V}(x) < 0$.

Contour plot $V(x) = C$:



See [Khalil, p.123] and Exc. 4.8

Example – saturated control

Exercise - 5 min

Find a bounded control signal $u = \text{sat}(v)$, which **globally** stabilizes the system

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= u \\ u &= \text{sat}\,(v(x_1, x_2)) \end{aligned} \tag{5}$$

What is the problem with using the 'standard candidate'

$$V_1 = x_1^2/2 + x_2^2/2 \ ?$$

Hint: Use the Lyapunov function candidate

$$V_2 = \ln(1 + x_1^2) + \alpha x_2^2$$

for some appropriate value of α.

Lyapunov Function for Linear System

Theorem The eigenvalues λ_i of A satisfy Re $\lambda_i < 0$ if and only if: for every positive definite $Q = Q^T$ there exists a positive definite $P = P^T$ such that

$$PA + A^T P = -Q$$

Proof of $\exists Q, P \Rightarrow$ *Re* $\lambda_i(A) < 0$: Consider $\dot{x} = Ax$ and the Lyapunov function candidate $V(x) = x^T P x$.

$$\dot{V}(x) = x^T P\dot{x} + \dot{x}^T P x = x^T(PA + A^T P)x = -x^T Q x < 0, \quad \forall x \neq 0$$

$$\Rightarrow \quad \dot{x} = Ax \text{ asymptotically stable} \iff \text{Re}\,\lambda_i < 0$$

Proof of Re $\lambda_i(A) < 0 \Rightarrow \exists Q, P$: Choose $P = \int_0^\infty e^{A^T t} Q e^{At} dt$

Linear Systems – *cont.*

Discrete time linear system:

$$x(k+1) = \Phi x(k)$$

The following statements are equivalent

- $x = 0$ is asymptotically stable
- $|\lambda_i| < 1$ for all eigenvalues of Φ
- Given any $Q = Q^T > 0$ there exists $P = P^T > 0$, which is the unique solution of the (discrete Lyapunov equation)

$$\Phi^T P \Phi - P = -Q$$

Exponential Stability

The equilibrium point $x = 0$ of the system $\dot{x} = f(x)$ is said to be **exponentially stable** if there exist c, k, γ such that for every $t \geq t_0 \geq 0$, $\|x(t_0)\| \leq c$ one has

$$\|x(t)\| \quad \leq \quad k\|x(t_0)\|e^{-\gamma(t-t_0)}$$

It is **globally** exponentially stable if the condition holds for arbitrary initial states.

For linear systems asymptotic stability implies global exponential stability.

“Comparison functions– class $\mathcal{K}$”

The following two function classes are often used as lower or upper bounds on growth condition of Lyapunov function candidates and their derivatives.

Definition (Class $\mathcal{K}$ functions [4])

A continuous function $\alpha : [0, a) \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\lim_{r \to \infty} \alpha(r) = \infty$.

Common choice is $\alpha_i(||x||) = k_i ||x||^c$, $k, c > 0$

“Comparison functions– class $\mathcal{KL}$”

Definition (Class $\mathcal{KL}$ functions [4])

A continuous function $\beta : [0, a) \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{KL}$ if for each fixed s the mapping $\beta(r, s)$ is a class $\mathcal{K}$ function with respect to r, and for each fixed r the mapping $\beta(r, s)$ is decreasing with respect to s and $\lim_{s \to \infty} \beta(r, s) = 0$. The function $\beta(\cdot, \cdot)$ is said to belong to class $\mathcal{KL}_\infty$ if for each fixed s, $\beta(r, s)$ belongs to class $\mathcal{K}_\infty$ with respect to r.

For exponential stability $\beta(||x||, t) =$ .... (fill in)

Lyapunov Theorem for Exponential Stability

Let $V : \mathbf{R}^n \to \mathbf{R}$ be a continuously differentiable function and let $k_i > 0$, $c > 0$ be numbers such that

$$\begin{aligned} k_1|x|^c \quad \le \quad & V(x) \quad \le \quad k_2|x|^c \\ & \frac{\partial V}{\partial x} f(t, x) \quad \le \quad -k_3|x|^c \end{aligned}$$

for $t \ge 0$, $\|x\| \le r$. Then $x = 0$ is exponentially stable.

If r is arbitrary, then $x = 0$ is **globally** exponentially stable.

Proof

$$\dot{V} = \frac{\partial V}{\partial x} f(t, x) \le -k_3|x|^c \le -\frac{k_3}{k_2} V$$

$$V(x) \le V(x_0) e^{-(k_3/k_2)(t-t_0)} \le k_2 |x_0|^c e^{-(k_3/k_2)(t-t_0)}$$

$$|x(t)| \le \left(\frac{V}{k_1}\right)^{1/c} \le \left(\frac{k_2}{k_1}\right)^{1/c} |x_0| e^{-(k_3/k_2)(t-t_0)/c}$$

Quadratic Stability

Given $A, B, C, \Delta_1, \ldots, \Delta_m$, suppose there exists a $P > 0$ such that

$$0 > (A + B\Delta_i C)' P + P(A + B\Delta_i C) \qquad \text{for all } i$$

Then the system

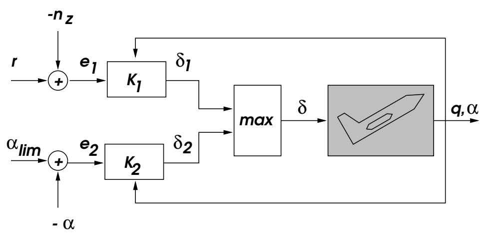
$$\dot{x} = [A + B\Delta(x, t)C]x$$

is globally exponentially stable for all functions Δ satisfying

$$\Delta(x, t) \in \mathrm{conv}\{\Delta_1, \ldots, \Delta_m\}$$

for all x and t

Aircraft Example



(Branicky, 1993)

Piecewise linear system

Consider the nonlinear differential equation

$$\dot{x} = \begin{cases} A_1 x & \text{if } x_1 < 0 \\ A_2 x & \text{if } x_1 \ge 0 \end{cases}$$

with $x = (x_1, x_2)$. If the inequalities

$$\begin{aligned} A_1^* P + P A_1 &< 0 \\ A_2^* P + P A_2 &< 0 \\ P &> 0 \end{aligned}$$

can be solved simultaneously for the matrix P, then stability is proved by the Lyapunov function $x^* P x$

Matlab Session

Copy `/home/kursolin/matlab/lmiinit.m` to the current directory or download and install the IQCbeta toolbox from `http://www.ee.mu.oz.au/staff/cykao/`

```
>> lmiinit
>> A1=[-5 -4;-1 -2];
>> A2=[-2 -1; 2 -2];
>> p=symmetric(2);
>> p>0;
>> A1'*p+p*A1<0;
>> A2'*p+p*A2<0;
>> lmi_mincx_tbx
>> P=value(p)

P =
    0.0749   -0.0257
   -0.0257    0.1580
```

Trajectory Stability Theorem

Let f be differentiable along the trajectory $\hat{x}(t)$ of the system

$$\dot{x} = f(x, t)$$

Then, under some regularity conditions on $\hat{x}(t)$, exponential stability of the linear system $\dot{x}(t) = A(t)x(t)$ with

$$A(t) = \frac{\partial f}{\partial x}(\hat{x}(t), t)$$

implies that

$$|x(t) - \hat{x}(t)|$$

decays exponentially for all x in a neighborhood of $\hat{x}$.

Time-varying systems

Note that autonomous systems only depends on $(t - t_0)$ while solutions for non-autonomous systems may depend on t_0 and t independently.

A second order autonomous system can never have "non-simply intersecting" trajectories (A limit cycle can never be a 'figure eight')

Stability definitions for time-varying systems

An equilibrium point $x = 0$ of $\dot{x} = f(x, t)$ is

locally stable at t_0, if for every $R > 0$ there exists $r = r(R, t_0) > 0$, such that

$$\|x(t_0)\| < r \quad \Rightarrow \quad \|x(t)\| < R, \quad t \geq t_0$$

locally asymptotically stable at time t_0, if locally stable and

$$\|x(t_0)\| < r(t_0) \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = 0$$

globally asymptotically stable, if asymptotically stable for all $x(t_0) \in \mathbf{R}^n$.

A system is said to be **uniformly stable** if r can be independently chosen with respect to t_0, i. e., $r = r(R)$.

Example of **non**-uniform convergence [Slotine, p.105/Khalil p.134]

Consider

$$\dot{x} = -x/(1 + t)$$

which has the solution

$$x(t) = \frac{1 + t_0}{1 + t}x(t_0) \quad \Rightarrow |x(t)| \leq |x(t_0)| \quad \forall t \geq t_0$$

The solution $x(t) \to 0$, but we can not get a 'decay rate estimate' independently of t_0.

Time-varying Lyapunov Functions

Let $V : \mathbf{R}^{n+1} \to \mathbf{R}$ be a continuously differentiable function and let $k_i > 0$, $c > 0$ be numbers such that

$$\begin{aligned} k_1|x|^c \leq V(t, x) &\leq k_2|x|^c \\ \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x)f(t, x) &\leq -k_3|x|^c \end{aligned}$$

for $t \geq 0$, $\|x\| \leq r$. Then $x = 0$ is exponentially stable.

If r is arbitrary, then $x = 0$ is **globally** exponentially stable.

Time-varying Linear Systems

The following conditions are equivalent

- The system $\dot{x}(t) = A(t)x(t)$ is exponentially stable
- There exists a symmetric matrix function $P(t) > 0$ such that

  $$-I \geq \dot{P}(t) + A(t)'P(t) + P(t)A(t)$$

  for all t.

Proof

Given the second condition, let $V(x, t) = x'P(t)x$. Then

$$\dot{V}(x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}Ax = x'(\dot{P} + A'P + PA)x < -|x|^2$$

so exponential stability follows the Lyapunov theorem.

Conversely, given exponential stability, let $\Phi(t, s)$ be the transition matrix for the system. Then the matrix $P(t) = \int_t^\infty \Phi(t, s)'\Phi(t, s)ds$ is well-defined and satisfies

$$-I = \dot{P}(t) + A(t)'P(t) + P(t)A(t)$$

Lyapunov's first theorem revisited

Suppose the time-varying system

$$\dot{x} = f(x, t)$$

has an equilibrium $x = 0$, where $\partial^2 f/\partial x^2$ is continuous and uniformly bounded as a function of t.

Then the equilibrium is exponentially stable provided that this is true for the linearization $\dot{x}(t) = A(t)x(t)$ where

$$A(t) = \frac{\partial f}{\partial x}(0, t)$$

Proof

The system can be written

$$\dot{x}(t) = f(x,t) = A(t)x(t) + o(x,t)$$

where $|o(x,t)|/|x| \to 0$ uniformly as $|x| \to 0$. Choose $P(t) > 0$ with

$$\dot{P}(t) + A(t)'P(t) + P(t)A(t) \le -I$$

and let $V(x) = x'Px$. Then

$$\frac{\partial V}{\partial x} f(x) = x'(\dot{P} + A'P + PA)x + 2x'P(t)o(x,t) < -|x|^2/2$$

in a neighborhood of $x = 0$. Hence Lyapunov's theorem proves exponential stability.

Proof of Trajectory Stability Theorem

Let $z(t) = x(t) - \hat{x}(t)$. Then $z = 0$ is an equilibrium and the system

$$\dot{z}(t) = f(z + \hat{x}) - f(\hat{x})$$

The desired implication follows by the time-varying version of Lyapunov's first theorem.

Lyapunov's Linearization Method revisited

Recall from Lecture 2 (undergraduate course):

Theorem Consider

$$\dot{x} = f(x)$$

Assume that $x = 0$ is an equilibrium point and that

$$\dot{x} = Ax + g(x)$$

is a linearization.

(1) If Re $\lambda_i(A) < 0$ for all i, then $x = 0$ is locally asymptotically stable.

(2) If there exists i such that $\lambda_i(A) > 0$, then $x = 0$ is unstable.

Proof of (1) in Lyapunov's Linearization Method

Lyapunov function candidate $V(x) = x^T Px$. $V(0) = 0$, $V(x) > 0$ for $x \neq 0$, and

$$\begin{aligned}\dot{V}(x) &= x^T P f(x) + f^T(x) P x \\ &= x^T P[Ax + g(x)] + [x^T A + g^T(x)]Px \\ &= x^T(PA + A^T P)x + 2x^T Pg(x) = -x^T Qx + 2x^T Pg(x)\end{aligned}$$

$$x^T Qx \ge \lambda_{\min}(Q)\|x\|^2$$

and for all $\gamma > 0$ there exists $r > 0$ such that

$$\|g(x)\| < \gamma\|x\|, \qquad \forall \|x\| < r$$

Thus, choosing γ sufficiently small gives

$$\dot{V}(x) \le -(\lambda_{\min}(Q) - 2\gamma\lambda_{\max}(P))\|x\|^2 < 0$$

First glimpse of the Center Manifold Theorem

What can we do if the linearization $A = \frac{\partial f}{\partial x}_{|x=x_o}$ has zeros on the imaginary axis at the equilibrium $x = x_o$?

The linearized system will have a *center point* at , but we cant say about the nonlinear system without further investigations (can be center point, stable focus or unstable focus at $x = x_o$).

Center Manifold Theorem Assume $z = 0$ is an equilibrium point. For every $k \ge 2$ there exists a C^k mapping φ such that $\phi(0) = 0$ and $d\phi(0) = 0$ and the surface

$$z_2 = \phi(z_1)$$

is invariant under the dynamics above.

Proof Idea: Construct a contraction with the center manifold as fix-point.

(To be continued)

First glimpse of the Center Manifold Theorem

Partition

$$A = \frac{\partial f}{\partial x}_{|x=x_o}$$

and assume

$$\begin{aligned}\dot{z}_1 &= A^0 z_1 + f^0(z_1, z_2) \\ \dot{z}_2 &= A^- z_2 + f^-(z_1, z_2)\end{aligned}$$

A^-: asymptotically stable

A^0: eigenvalues on imaginary axis

f^0 and f^- second order and higher terms.

Usage

1) Determine $z_2 = \phi(z_1)$, at least approximately

2) The local stability for the entire system can be proved to be the same as for the dynamics restricted to a center manifold:

$$\dot{z}_1 = A^0 z_1 + f^0(z_1, \phi(z_1))$$

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