Exercise set 3
— Absolute stability, Kalman-Yakubovich-Popov Lemma, The Circle and Popov criteria

Reading assignment

Lecture notes, Khalil (3rd ed.) Chapters 6, 7.1.
Extra material on the K-Y-P Lemma (paper by Rantzer).

3.1 Comments on the text

This section of the book presents some of the core material of the course. The results have played a central role in control theory for a long time and have recently been vitalized by new progress, both in theory and in computational methods.

The concept *absolute stability* is introduced for nonlinear systems consisting of two parts, one linear time-invariant and one nonlinear. Detailed knowledge about the nonlinear part is not used, only inequality constraints.

The Kalman-Yakubovich-Popov Lemma shows that a transfer function inequality is equivalent to a condition on solvability of a linear matrix inequality (LMI) defined by the state space matrices. In the proof of the circle and Popov criteria, the LMI appears naturally in the attempt to construct a Lyapunov function. The K-Y-P Lemma therefore connects the existence of a certain Lyapunov function to a transfer function condition on the linear part. Khalil does not provide a complete proof, instead we refer to separate notes which are distributed this week.

Recently, the same lemma has often been used in the opposite direction, as frequency conditions on multivariable transfer functions are verified by translating them into an LMI condition, which can be solved by convex optimization. Some of the exercises below will illustrate this and the MATLAB Toolbox IQCbeta (here used as wrapper to LMI-lab) will be useful for the calculations.

Soon after the appearance of the Popov criterion, for example in the textbook by Aiserman and Gantmacher from 1965, it was pointed out that the Popov criterion holds also with negative slope $1/\eta$ on the Popov line. However, this fact is ignored by Khalil and several other western textbooks. Can you see why it must be true?

Exercise 3.1 = Kha. 7.3

Exercise 3.2 = Kha. 7.4

Exercise 3.3 = Kha. 7.1 (1),(2)

Exercise 3.4 Solve the previous exercise with the circle criterion replaced by the Popov criterion. **Exercise 3.5** In the first problem set we considered subproblem **(a)** below.

a. Find a quadratic simultaneous Lyapunov function (using *e.g.,* Löfgren's `yalmip` or Boyd's `cvx`) for the linear time-varying system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & -8 \\ 6 & -13 \end{bmatrix} x + \begin{bmatrix} 8 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} \delta_1(t) & 0 \\ 0 & \delta_2(t) \end{bmatrix} y \\ y &= \begin{bmatrix} 1 & -1 \\ -11 & 16 \end{bmatrix} x \end{aligned}$$

where $|\delta_k(t)| \le 1$, $k = 1, 2$.

b. Does Theorem 7.1 prove stability if the δ-matrix is replaced by a memoryless nonlinearity satisfying the sector condition

$$[\psi(t, y) + y]^T [\psi(t, y) - y] \le 0, \quad \forall t \ge 0, y \in \mathbf{R}^2$$

Exercise 3.6 Consider a $p \times p$ matrix function $M(s)$, which is analytic for $\mathrm{Re}\, s > 0$ and satisfies $M(s) = \overline{M(\bar{s})}$. The matrix function is called

output strictly passive (OSP) if $\exists \epsilon > 0$ such that for $\mathrm{Re}\, s > 0$

$$M(s) + M(s)^* \;\ge\; \epsilon M(s)^* M(s)$$

input strictly passive (ISP) if $\exists \epsilon > 0$ such that for $\mathrm{Re}\, s > 0$

$$M(s) + M(s)^* \;\ge\; \epsilon$$

positive real (PR) if for $\mathrm{Re}\, s > 0$

$$M(s) + M(s)^* \ge 0$$

strictly positive real (SPR) if $\exists \epsilon > 0$ such that $M(s - \epsilon)$ is PR.

a. For the scalar transfer function $M(s) = C(sI - A)^{-1}B + D$ with A Hurwitz, show that

$$\text{ISP} \;\Rightarrow\; \text{SPR} \;\Rightarrow\; \text{OSP} \;\Rightarrow\; \text{PR}$$

b. Prove a counterpart to Lemma 6.2 (Khalil, 3rd ed) with SPR replaced by OSP and (6.14-16) replaced by a convex LMI.