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Lecture Notes on Integral Quadratic Constraints

by

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Lectures on Input-Output Stability and Integral Quadratic Constraints

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1 Introduction

The basic system under study in the course is pictured in the block-diagram in Figure 1. Here G is a stable linear system, Δ is an uncertainty, d is a disturbance input, and z is the output. We will discuss

1. How to verify stability (of lower loop) for various uncertainty classes
   (a) uncertain dynamics
   (b) parametric uncertainty
   (c) time-varying parameters
   (d) various nonlinearities
   (e) structured uncertainty, for example, a combination of the above.
2. How to investigate the performance of the closed loop
   (a) energy gain $d \to z$
   (b) energy to peak gain $d \to z$
   (c) exploit spectral characteristics of the disturbance d
3. The whole story from theory to software!!

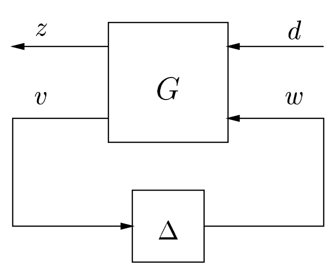

Figure 1: Basic system under consideration.

We will focus on a relatively new method for robust stability analysis, namely the framework of Integral Quadratic Constraints (IQC). The IQC framework did not appear from nowhere. In fact, it has its roots in at least three strong research fields: The input-output theory developed by Zames, Sandberg, Willems and many others [41, 42, 43, 29, 28, 31, 4, 26], the absolute stability theory with extraordinary contributions from Yakubovich and Popov [32, 33, 34, 35, 36, 37, 38, 24], and finally the robust control field with contributions from, for example, Doyle, Safonov, Zames, and many others [5, 22, 1, 6, 44, 27]. The relationship is indicated in Figure 2

It was A. Megretski, originally from Yakubovich group at S.t Petersburg state university, who first started to merge the western input/output tradition with the absolute stability theory of Soviet Union into unified framework. Some of the early work was in fact published as technical reports at KTH, where Megretski was a post-doc in 1992, see [14, 15, 17, 16, 20]. Further generalization was done in collaboration with A. Rantzer (alumni from KTH) and we will use their paper [19, 25] as the basis for an important part of these lectures.

We should also note that Yakubovich, who have contributed to many of the main ideas behind IQC framework, is a frequent visitor at KTH. Indeed, Yakubovich introduced the notion of IQCs in stability theory [36, 38, 40], he pioneered the use of the S-procedure in systems analysis [37, 39], and he developed the Kalman-Yakubovich-Popov Lemma [32], which will be used later in the course when we discuss computational robust control. Still, there are some conceptual as well as technical differences in the use of IQCs in these lecture notes compared to [36, 38, 40]. For example, our development will be developed for an operator representation of the system, and our well-posedness assumption is different from the minimal stability assumption in [36, 38, 40]. These distinctions will not be addressed in the course. The preliminary outline of the course is the following:

1. Introduce an abstract framework so that many different cases can be treated with one theory. This involves
   (a) a discussion of function spaces and operators
   (b) introduce the concepts of extended space, causality, and well-posedness of systems.

   Good references for this material can be found in [4, 31].
2. The small gain theorem and the passivity theorem.
3. Integral quadratic constraints
   (a) definition and examples
   (b) the IQC stability theorem
   (c) examples

   We base the discussion on [19, 17]. The first can be obtained at http://www.lib.kth.se/ (Go to *E-tidskrifter i fulltext* and then *IEE/IEEE se IEL Online*.)
4. The S-procedure. Here we discuss results in [20, 39].
5. Uncertain system models
   (a) structured uncertainty
   (b) linear fractional transformations
6. Performance analysis and signal characterizations
7. A useful formulation of the Kalman-Yakubovich-Popov lemma.
8. Optimization of IQCs and the IQCbeta toolbox.

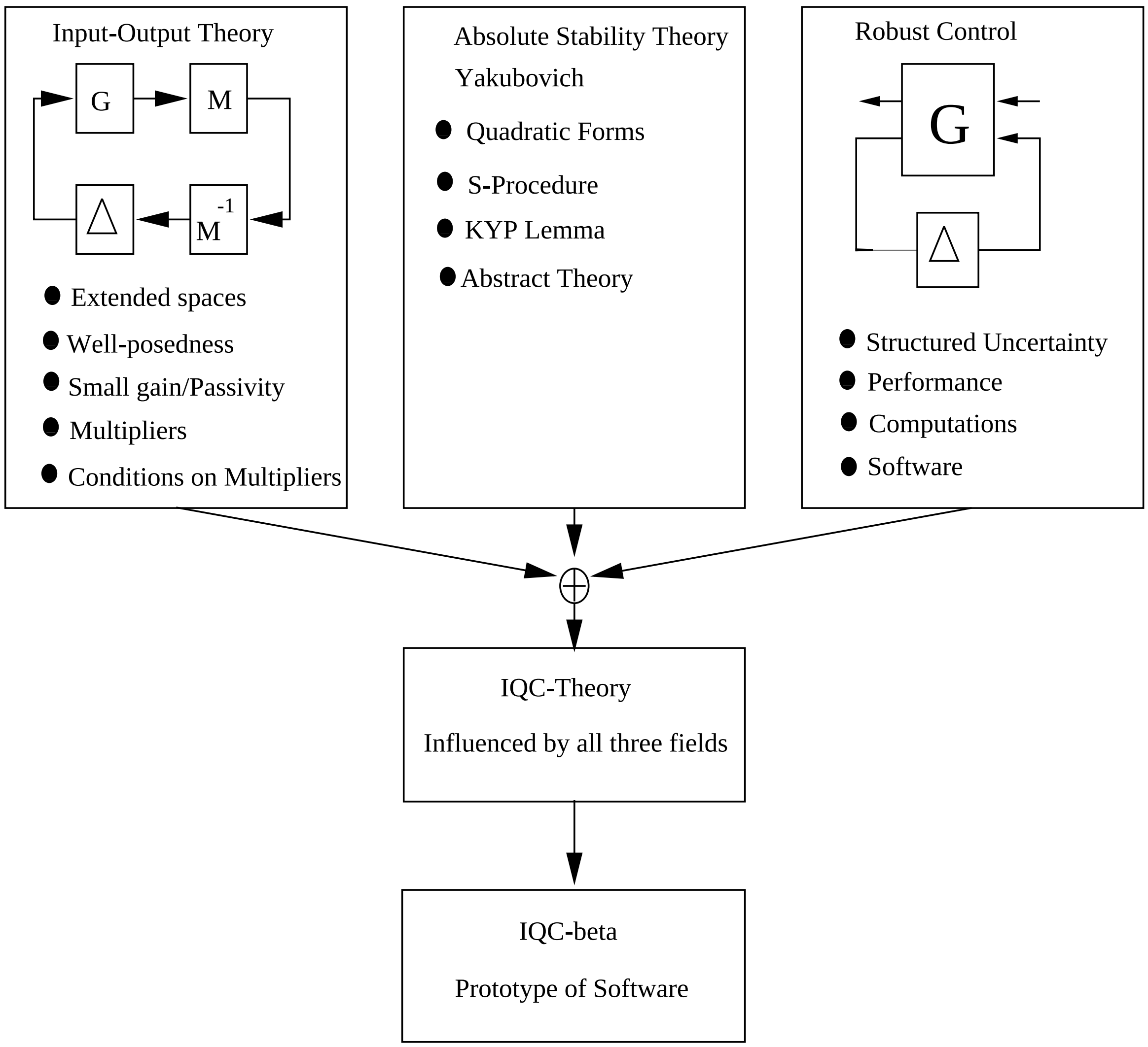

Figure 2: The IQC-theory that will be discussed in this course is essentially a unification of ideas from three now classical and very important research fields: 1) The input-output theory that was developed in the west in 1960-1970; 2) The abstract stability theory that was developed in the Soviet Union during 1960-1975, and finally 3) the robust control field in 1980-1990.

2 Function Spaces and Operators

In the input-output theory for stability analysis we represent the systems as operators and their input and output signals as function from appropriate vector spaces. It is remarkable that only the most basic concepts from operator theory are needed to develop a rich and useful stability theory.

2.1 Normed Vector Spaces

A normed vector space $\mathcal{L}$ is a linear vector space equipped with a norm. We will consider vector spaces consisting of functions that map an infinite "time axis" $\mathcal{T}$ into another vector space $\mathcal{V}$. We assume $\mathcal{T} \subset \mathbf{R}$. Examples are the integers $Z = \{\ldots, -2, -1, 0, 1, 2\ldots\}$, $Z_+ = \{0, 1, 2\ldots\}$, or the real numbers $\mathbf{R} = (-\infty, \infty)$ or $\mathbf{R}_+ = [0, \infty)$. $\mathcal{V}$ will always be $\mathbf{R}^n$ for a suitable dimension n. This means that we only consider vector spaces over the real scalar field in the lecture notes.

Every pair of functions $f, g \in \mathcal{L}$ satisfies the properties (linear vector space properties)

$$\begin{aligned}(f+g)(t) &= f(t) + g(t)\\ (\alpha f)(t) &= \alpha f(t)\end{aligned}$$

where $\alpha \in \mathbf{R}$.

The norm on $\mathcal{L}$ is a function $\|\cdot\| : \mathcal{L} \to \mathbf{R}_+$ (i.e. a nonnegative functional) that satisfies the properties

(i) $\|f\| = 0 \Leftrightarrow f \equiv 0$,

(ii) $\|\alpha f\| = |\alpha| \cdot \|f\|$,

(iii) $\|f + g\| \leq \|f\| + \|g\|$.

Every $f \in \mathcal{L}$ is supposed to have finite norm, i.e. $\|f\| < \infty$. The norm measures the size of the signal.

The most frequently appearing function spaces in control applications are the l_p and $\mathbf{L}_p$ spaces, $p \geq 1$. The first consists of discrete time functions, i.e. they map from Z or Z_+ into $\mathbf{R}$. The functions in these discrete time spaces can be represented as infinite sequences of numbers

$$\begin{array}{lll}(\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots), & f_i \in \mathbf{R} & (Z)\\ (f_0, f_1, f_2, \ldots), & f_i \in \mathbf{R} & (Z_+)\end{array}$$

where f_i represents the function value at time i. We will use notations as $l_p(Z_+)$ or $l_p(Z)$ if we explicitly want to specify the time axis.

The norms are defined as follows

$$\begin{array}{ll}\|f\|_p = \left(\sum_{i=1}^{\infty} |f_i|^p\right)^{1/p} & l_p(Z_+),\ p = 1, 2, 3..\\ \|f\|_\infty = \sup_{i \geq 0} |f_i| & l_\infty(Z_+)\end{array}$$

The norms for the cases with bi-infinite time axis are defined correspondingly.

The continuous time spaces, $\mathbf{L}_p$, consists of functions defined on the real axis. We use notation as $\mathbf{L}_p(-\infty, \infty)$ and $\mathbf{L}_p[0, \infty)$ to explicitly define what time axis is used. For our means it is enough to know that the vector spaces $L_p[0,\infty)$ consists of functions $f : \mathbf{R}_+ \to \mathbf{R}$ with norms

$$\|f\|_p = \left(\int_0^\infty |f|^p \, dt\right)^{1/p} \qquad \mathbf{L}_p[0,\infty),\ p = 1, 2, ...$$

$$\|f\|_\infty = \text{ess sup}_{t \in \mathbf{R}_+} |f(t)| \qquad \mathbf{L}_\infty[0,\infty)$$

The norms for the cases with bi-infinite time axis are defined correspondingly.

We often need to use vector valued functions. We use the notation $\mathbf{L}_p^m[0,\infty)$ to denote the functions $f : \mathbf{R}_+ \to \mathbf{R}^m$ with norm defined as above where now the spatial norm is the Euclidean norm $|f| = (f^T f)^{1/2}$.

Remark 1. All the normed vector spaces mentioned above are also complete, i.e., their Cauchy sequences converge. Such normed vector spaces are called *Banach spaces*. We will not exploit the completeness property.

2.2 Inner product Spaces

We often have additional structure on our vector space $\mathcal{L}$ in terms of an inner product. The inner product is a bilinear functional $\langle \cdot, \cdot \rangle : \mathcal{L} \times \mathcal{L} \to \mathbf{R}$ (a sesquilinear functional in complex inner product spaces) satisfying the following properties (where $f, g \in \mathcal{L}$ and $\alpha \in \mathbf{R}$)

(*i*) $\langle f, g \rangle = \langle g, f \rangle$

(*ii*) $\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$

(*iii*) $\langle f_1 + f_2, g \rangle = \langle f_1, g \rangle + \langle f_2, g \rangle$

Vector spaces with an inner product are called *inner product spaces* and the norm on these spaces can be defined in terms of the inner product as

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

There are several useful inequalities that hold for inner products. The following are particularly useful

$$\begin{aligned} &\langle f, g \rangle \le \|f\| \cdot \|g\| \quad \text{(Cauchy Schwartz)} \\ &\pm 2 \langle f, g \rangle \le \|f\|^2 + \|g\|^2 \\ &\|f + g\|^2 \le 2(\|f\|^2 + \|g\|^2) \end{aligned}$$

The last inequality holds for any normed vector space.

Notation: All inner product spaces considered below are complete, i.e., their Cauchy sequences converge. Complete inner product spaces are called *Hilbert spaces*. **We will denote Hilbert spaces by** $\mathcal{H}$ in order to distinguish their special structure from the normed vector spaces $\mathcal{L}$.

Remark 2. We will only use the completeness in order to ensure existence of an adjoint operator in the Hilbert space in a later section. Most results hold for any inner product space, but we will not distinguish the two cases.

The Hilbert spaces $l_2^m(Z_+)$ and $L_2^m[0,\infty)$ have inner products defined as

$$\langle f, g \rangle = \sum_{i=0}^{\infty} f_i^T g_i = \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{f}(j\omega)^* \widehat{g}(j\omega) \, d\omega \qquad l_2^m(Z_+) \tag{1}$$

$$\langle f, g \rangle = \int_0^\infty f(t)^T g(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(j\omega)^* \widehat{g}(j\omega) \, d\omega \qquad \mathbf{L}_2^m[0,\infty) \tag{2}$$

where the connection with the frequency domain integrals follows from the Plancherel formula. Here $\widehat{f}$ and $\widehat{g}$ denote the Fourier transforms of f and g, defined as

$$\widehat{f}(j\omega) = \lim_{N\to\infty} \sum_{k=0}^{N} f_k e^{-j\omega k}, \quad \omega \in [-\pi, \pi]$$

$$\widehat{f}(j\omega) = \lim_{T\to\infty} \int_0^T f(t) e^{-j\omega t} dt, \ \omega \in \mathbf{R}$$

for the discrete and continuous time respectively. The above relations are defined in an analogous way for the bi-infinite case.

2.3 Operators

An operator H is a mapping from one normed space into another. We will only consider the case when both spaces are the same, i.e. $H : \mathcal{L} \to \mathcal{L}$. This means that $H(f) \in \mathcal{L}$ for all $f \in \mathcal{L}$. We can think of the operators as mathematical objects that represent our system.

Any pair, H_1, H_2, of operators on $\mathcal{L}$ satisfy the following properties

(*i*) The composition $H_1 H_2$ is also an operator on $\mathcal{L}$ defined by $(H_1 H_2)(f) = H_1(H_2(f))$

(*ii*) The sum $\alpha H_1 + \beta H_2$ for any $\alpha, \beta \in \mathbf{R}$ is an operator on $\mathcal{L}$ defined by $(\alpha H_1 + \beta H_2)(f) = \alpha H_1(f) + \beta H_2(f)$

An operator is *linear* if

$$H(\alpha f + \beta g) = \alpha H(f) + \beta H(g)$$

We often use the shorthand notation $G(f) = Gf$ for the mapping of a linear operator G.

We will always assume that our operators satisfy $H(0) = 0$. This is often not a restriction and it will simplify the future development[1] An operator $H : \mathcal{L} \to \mathcal{L}$ is called *bounded* if the following "gain" is finite[2]

$$\|H\| = \sup_{\substack{f \in \mathcal{L} \\ f \neq 0}} \frac{\|H(f)\|}{\|f\|}$$

It satisfies the important submultiplicativity rule

$$\|H_1 H_2\| \leq \|H_1\| \cdot \|H_2\|$$

Examples of operators

Most of the systems we consider have a linear time invariant (LTI) part that is described in terms of a transfer function G with poles strictly in the left half plane. If the system is finite dimensional then the transfer function has realizations on the form $G(s) = C(sI - A)^{-1}B + D$. All continuous time LTI systems defines operators on $\mathbf{L}_1^m[0,\infty), \mathbf{L}_2^m[0,\infty)$ and $\mathbf{L}_\infty^m[0,\infty)$ in terms of convolutions. Let $g(t) = \mathcal{L}^{-1}\{G\}$ be the weighting function corresponding to $G(s)$ (here $\mathcal{L}^{-1}$ denotes the inverse Laplace transform). Then G is defined by the convolution

$$(Gf)(t) = (g * f)(t) = \int_0^t g(t-\tau) f(\tau)\, d\tau$$

[1] The assumption $H(0) = 0$ implies that the initial condition of operators with dynamics (such as operators defined in terms of a state space equation) is assumed to be zero. Instead the transient due to the initial condition is assumed to be part of the input signal.

[2] This is the induced norm in the case of linear operators.

It is well known from the linear systems course that $G(s)$ must have all poles strictly in the left half plane in order to be an operator on any of $\mathbf{L}_p^m[0,\infty)$ $p \geq 1$. At this point it may look as if we have the same operator independently of which of these spaces we consider. This is not the case since the induced norms (gains) are different and the norm is an important measure of how the signal through the system is amplified.

Remark 3. To see that a transfer function with poles in the right half plane cannot be bounded on $\mathbf{L}_p^m[0,\infty)$ ($p = 1, 2, \infty$ (or any other p)) we consider an example. Let $G(s) = 1/(s-1)$ and let

$$u(t) = \begin{cases} 1, & t \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

We get

$$(Gu)(t) = \int_0^t e^{t-\tau} u(\tau) d\tau = \begin{cases} e^t - 1, & t \in [0,1] \\ e^t(1-e^{-1}), & t > 1 \end{cases}$$

which has unbounded norm in any of the $\mathbf{L}_p^m[0,\infty)$-spaces.

For example, if G is an operator on $\mathbf{L}_2^m[0,\infty)$ then the norm gives an exact measure of the worst case energy gain in the system and it is given by

$$\|G\| = \sup_{\omega \in \mathbf{R}} \sigma_{\max}(G(j\omega))$$

On the other hand, if G instead is viewed as an operator on $\mathbf{L}_\infty[0,\infty)$ (SISO for simplicity) then the norm is a exact measure of the worst case increase of the peak-value of the signals and it is given by (the proof of this is a Homework problem)

$$\|G\| = \int_0^\infty |g(t)|\, dt$$

It is interesting to note that if G has poles in the right half plane then it is not an operator on $\mathbf{L}_2^m[0,\infty)$ but an operator from either of $\mathbf{L}_2^m[0,\infty)$ or $\mathbf{L}_2^m(-\infty,\infty)$ into $\mathbf{L}_2^m(-\infty,\infty)$. The operator is now defined in terms of a bi-infinite integral

$$(Gf)(t) = \int_{-\infty}^\infty g(t-\tau) f(\tau)\, d\tau$$

but the norm is unchanged. We will discuss this in more detail later when we have discussed the concept of causality.

Next follows two examples of nonlinear operators.

Example 1. Consider a nonlinear function $\varphi : \mathbf{R} \to \mathbf{R}$ with the property that $|\varphi(x)| \leq k|x|$ for some positive constant k. The nonlinearity defines a bounded operator on any of $\mathbf{L}_p[0,\infty)$, since

$$\int_0^\infty |\varphi(f(t))|^p dt \leq k^p \int_0^\infty |f(t)|^p dt$$
$$\text{ess sup}_{t\in[0,\infty)} |\varphi(f(t))| \leq k \cdot \text{ess sup}_{t\in[0,\infty)} |f(t)|$$

which implies that $\|\varphi\| \leq k$. The operator φ is often called *memoryless nonlinearity* or *static nonlinearity* since its output at time t only depends on the input at time t.

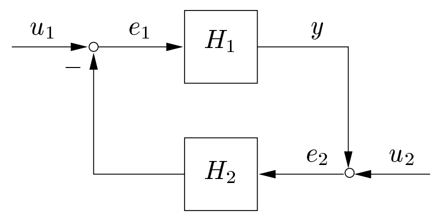

Figure 3: Block diagram for the system (4).

Example 2. Consider the nonlinear dynamic operator defined by the input output relation

$$y = H(u) \quad \Leftrightarrow \quad \begin{cases} \dot{x} = f(x) + g(x)u, \quad x(0) = 0 \\ y = h(x) \end{cases}$$

where f, g, h are nonlinear functions of suitable dimension and such that $f(0) = 0$, and $h(0) = 0$.

Assume there exists a continuously differentiable positive semi-definite function[3] V with $V(0) = 0$ such that

$$\frac{dV(x)}{dx}(f(x) + g(x)u) \leq \gamma^2 |u|^2 - |h(x)|^2 \tag{3}$$

for all $(x, u) \in \mathbf{R}^n \times \mathbf{R}^m$. Then the system is $\mathbf{L}_2$-bounded with gain less that γ. To see this let us integrate (3). This gives

$$V(x(t)) \leq \gamma^2 \int_0^t |u|^2 d\tau - \int_0^t |h(x)|^2 d\tau.$$

where we used $V(0) = 0$. If $u \in \mathbf{L}_2[0, \infty)$ then we see that $h(x) \in \mathbf{L}_2$, since otherwise the right hand side tends to $-\infty$ as $t \to \infty$, which contradicts the positive semi-definiteness of V. It then follows that

$$\int_0^\infty |h(x)|^2 d\tau \leq \gamma^2 \int_0^\infty |u|^2 d\tau,$$

which proves the gain bound.

3 The System under consideration

We will consider stability of the system

$$\begin{aligned} e_1 &= u_1 - H_2(e_2) \\ e_2 &= u_2 + H_1(e_1) \end{aligned} \tag{4}$$

which is also illustrated in Figure 3. There are many important issues that must be resolved before we can derive a reasonable stability theory for this system. For example,

- In many applications we want to consider inputs u_1 and u_2 that are unbounded in the norm we want to consider. For example, $f(t) = \sin(t)$ is not in $\mathbf{L}_2[0, \infty)$ but it is in $\mathbf{L}_\infty[0, \infty)$. Does this mean that it is impossible to exploit the additional structure of the inner product when analyzing systems with sinusoidal inputs?

[3] V is positive semi-definite if $V(x) \geq 0$ for all x.

- Even if the input u_1 and u_2 are in some appropriate normed vector space $\mathcal{L}$ there is no way we can ensure a priori that the signals in the loop are bounded (has finite norm). This would almost be the same as assuming stability before it is proven.

- Even if both H_1 and H_2 are reasonable models of a physical systems it need not mean that the closed loop makes sense. Such systems are ill-posed and we will soon give some examples of ill-posed systems.

- Physical systems are always causal in the sense that the systems response at a particular time instant is only dependent on the history of the input signal and not the future of it. The concept of causality need to be formalized.

Example 3. Consider the feedback interconnection of $H_1(s) = 1/(s+1)$ and the nonlinearity $H_2(x) = -x - x^2$. Let the injected signals be $u_1(t) = \theta(t)$ and $u_2 = 0$ (where θ is the unit step function). The closed loop system is described by the differential equation

$$\dot{x} = x^2 + 1, \quad t \geq 0$$

The solution $\arctan(x) = t$ for $t \geq 0$ or equivalently $x(t) = \tan(t)\theta(t)$, $t \geq 0$ goes to infinity as $t \to \pi/2$. Hence the system has finite escape time and we will consider it to be ill-posed.

The next two examples are taken from [31].

Example 4. Let $H_1(s) = 1, H_2(s) = e^{-sT} - 1$ and $u_2 \equiv 0$. In this case we get the closed loop system operator $(I + H_1(s)H_2(s))^{-1}H_1(s) = e^{sT}$, and thus $y(t) = u_1(t+T)$. Hence, the system is not causal.

Example 5. Consider the case when $H_1 = 1$, $H_2 = k$ and $u_2 = 0$. If $k = -1$, then the return ratio $(I + H_1H_2)$ is not invertible and the system is clearly ill-posed. For all other cases of k we get $(I + H_1H_2)^{-1}H_1 = 1/(1+k)$. However, even now it is questionable whether the system is well-posed or not in the case $|k| > 1$. For example, if the system is a model of two interconnected physical systems then there will always be some small delay in the loop. In this case it can be shown that the step response for the physical system is unstable, i.e., $y(t) \to \infty$ as $t \to \infty$. This is in conflict with the expected solution from the model $y(t) = 1/(1+k)\theta(t)$. Hence, for some applications this system should be regarded as ill-posed.

Example 6. In systems with discontinuous nonlinearities there may appear chattering. For example, we may have a relay that switch infinitely fast between its two output values. Such a signal is not sufficiently regular to be integrable and it does not belong to any of the function spaces above. There is a theory that deals with such problems but it is beyond the scope of this course.

As we have seen, many strange things can happen in a closed loop system and the methods we will develop are not able to detect some of the problems in the examples above. In fact, all the methods to be presented rely on an assumption that the loop signals e_1 and e_2 exist and are sufficiently regular over any finite time interval. This excludes the first example from consideration. Another deficiency of the forthcoming results is that they generally cannot detect if the loop signals depends causally on the inputs or not. In order to make reasonable assumptions on system (4) we will introduce extended spaces, the notion of causality, and well-posedness. In short well-posedness is just an assumption on the mathematical model (4) to make sense as a model of a physical system.

Extended spaces

An extension of a normed vector space consists of signals that may not be bounded in the norm of the vector space but where any truncation to a finite time intervals is bounded. This leads us to the introduction of extended spaces. We will consider extended spaces only for time-axes $\mathcal{T} \subset \mathbf{R}_+$. The reason is that we only consider causal systems starting at time zero. To formalize the definition of extended space we introduce the truncation operator P_T defined as follows. Let $f : \mathcal{T} \to \mathcal{V}$. Then

$$(P_T f)(t) = \begin{cases} f(t), & t \le T \quad (t, T \in \mathcal{T}) \\ 0, & t > T \end{cases}$$

Notation: We will often use the notation $f_T = P_T f$.

Definition 1. The extended space $\mathcal{L}_e$ is then defined as

$$\mathcal{L}_e = \{ f : \mathcal{T} \to \mathcal{V} : \|f_T\| < \infty, \ \forall T \ge 0 \}$$

where $\|\cdot\|$ is the norm on $\mathcal{L}$. We will assume that the norm $\|\cdot\|$ is such that

- For every $f \in \mathcal{L}_e$ we have $\|f_{T_1}\| \le \|f_{T_2}\|$ for all $T_2 \ge T_1$.
- For all $f \in \mathcal{L}$ we have $\|f_T\| \to \|f\|$ as $T \to \infty$.

These above conditions hold for the spaces $l_{pe}(Z_+)$ and $\mathbf{L}_{pe}[0, \infty)$, $p = 1, 2, 3, \ldots, \infty$ that will be considered in our applications.

Example 7. We have

1. $\sin(t) \in \mathbf{L}_{pe}[0, \infty)$
2. $e^t \in \mathbf{L}_{pe}[0, \infty)$
3. $2^k \in l_{pe}(Z_+)$

Causality of operators on extended spaces

An operator $H : \mathcal{L}_e \to \mathcal{L}_e$ (or $H : \mathcal{L} \to \mathcal{L}$) is said to be causal (nonanticipative) if

$$P_T H P_T = P_T H, \quad \text{for all } T \in \mathcal{T}.$$

This means that the value at a certain time instant does not depend on future values of the argument. To see this we just note that the definition means that $H(f_T)(t) = H(f)(t)$ when $t \le T$. In other words, it does not matter if we truncate the future of the input signal when considering the output at a certain time instant. In other words the system is not a "crystal ball".

An operator[4] $H : \mathcal{L} \to \mathcal{L}$ is said to be noncausal if it is not causal. The purest form of noncausality is anticausality. H is said to be anticausal if $(I - P_T)H = (I - P_T)H(I - P_T)$, for all $T \ge 0$. This means that the value at a certain time does not depend on past values of the argument. Figure 4 illustrates the concepts of causality and anti-causality.

[4]We will only consider noncausal operators on bi-infinite spaces as *analysis filters* in IQC analysis. That's the reason we do not discuss noncausality in connection with extended spaces.

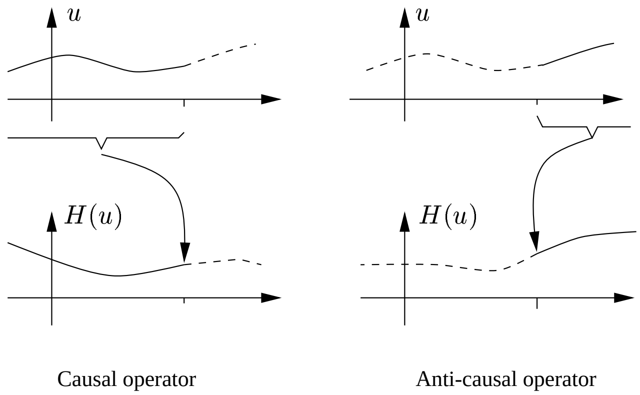

Figure 4: The left hand side illustrates the operation of an causal operator. Only the past of the input affect the output at a certain time instant. The right hand side illustrates an anti-causal operator.

Boundedness of a Causal Operator:

A causal operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is bounded if the gain defined as[5]

$$\|H\| = \sup_{\substack{f \in \mathcal{L} \\ f \neq 0}} \frac{\|H(f)\|}{\|f\|} \tag{5}$$

is finite. Note that the gain is defined in terms of functions in $\mathcal{L}$ and not the corresponding extended space. However, the definition in (5) implies boundedness on $\mathcal{L}_e$, since

$$\|P_T H(f)\| = \|P_T H(f_T)\| \leq \|P_T\| \cdot \|H\| \cdot \|P_T f\| = \|H\| \cdot \|P_T f\|$$

for all $f \in \mathcal{L}_e$ and all $T \in \mathcal{T}$. It can be shown that $\|H\|$ is the smallest such bound, see [31].

It is clear that a bounded causal operator on $\mathcal{L}_e$ is also a well defined bounded causal operator on $\mathcal{L}$. This follows since if $f \in \mathcal{L}$ then $\|P_T H(f)\| \leq \|H\| \cdot \|f\|$ for all $T \in \mathcal{T}$. We also have the reverse implication: A bounded causal operator on $\mathcal{L}$ is also a well defined bounded causal operator on $\mathcal{L}_e$, because $P_T H(u) = P_T H(u_T)$, and $u_T \in \mathcal{L}$. We have thus shown that

$$H \text{ is causal and bounded on } \mathcal{L}_e \Leftrightarrow H \text{ is causal and bounded on } \mathcal{L}$$

Examples

We will first introduce notation that will be used extensively in the lecture notes.

$\mathbf{RL}_\infty^{m\times m}$ — The space consisting of proper real rational matrix functions with no poles on the imaginary axis.

$\mathbf{RH}_\infty^{m\times m}$ — The subspace of $\mathbf{RL}_\infty^{m\times m}$ consisting of functions with no poles in the closed right half plane.

[5]The definition implies that $H(0) = 0$, which means that operator (system) is assumed to have a zero transient response. This is often a reasonable assumption since the initial condition often can be represented as a input or output disturbance of the system.

Example 8. Each operator $G \in \mathbf{RH}_\infty^{m\times m}$ has a state space realization $G(s) = C(sI - A)^{-1}B + D$ and corresponding weighting function $g(t) = Ce^{At}B\theta(t) + D\delta(t)$. The operation on $u \in \mathbf{L}_p^m[0,\infty)$ is defined in terms of the convolution

$$y(t) = (Gu)(t) = (g * u)(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t),$$

which shows that G is causal. Proposition 1 below shows that the operator is bounded on all $\mathbf{L}_{pe}^m[0,\infty)$.

Example 9. An operator $G \in \mathbf{RL}_\infty^{m\times m}$ is generally noncausal. It can be split into a causal term G_c and an anticausal term G_{ac}, such that $G = G_c + G_{ac}$. This is done using partial fractions expansion in such a way that $G_c \in \mathbf{RH}_\infty^{m\times m}$ and $G_{ac}(-s) \in \mathbf{RH}_\infty^{m\times m}$, i.e., G_c contains the stable poles and G_{ac} contains the unstable poles. As an example, we have

$$G(s) = \frac{2}{(s+1)(s-1)} = \underbrace{\frac{-1}{s+1}}_{G_c} + \underbrace{\frac{1}{s-1}}_{G_{ac}}$$

We have already seen in Remark 3 that $1/(s-1)$ cannot be bounded on $\mathbf{L}_{pe}[0,\infty)$. However, it turns out that it is a bounded anticausal operator on $\mathbf{L}_p(-\infty,\infty)$. In fact, any $G(s) = C(sI - A)^{-1}B + D$, with A unstable (all eigenvalues in the right half plane) defines an anticausal operator on $\mathbf{L}_p(-\infty,\infty)$ by the convolution

$$(Gu)(t) = \int_t^\infty Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

In the general case an operator $G \in \mathbf{RL}_\infty^{m\times m}$ is defined by convolution with its weighting function $g(t) = g_c(t) + g_{ac}(t)$, where[6] we have $g_c(t) = \mathcal{L}^{-1}\{G_c(s)\}$ and $g_{ac}(t) = \mathcal{L}^{-1}\{G_{ac}(s)\}$. We get (the direct term can be included in either of g_c and g_{ac} as a dirac distribution)

$$(Gu)(t) = \int_{-\infty}^\infty g(t-\tau)u(\tau)d\tau = \int_{-\infty}^t g_c(t-\tau)u(\tau)d\tau + \int_t^\infty g_{ac}(t-\tau)u(\tau)d\tau.$$

The next proposition can be used to show boundedness of the linear operators in the previous two examples.

Proposition 1. *The operator defined by the convolution*

$$(Hu)(t) = \int_{-\infty}^\infty h(t-\tau)u(\tau)d\tau$$

where $h \in \boldsymbol{L}_1(-\infty,\infty)$ is bounded on $\boldsymbol{L}_p(-\infty,\infty)$, $p \geq 1$ with gain $\|H\| \leq \|h\|_1$ (and equal to[7] $\|h\|_1$ for $\boldsymbol{L}_\infty(-\infty,\infty)$). Furthermore, if $h(t) = 0$, for $t \leq 0$, then H is also causal.

Remark 4. Note that the proposition is also valid when the operator is considered as a mapping $H : \mathbf{L}_p[0,\infty) \to \mathbf{L}_p(-\infty,\infty)$ (Note $H : \mathbf{L}_p[0,\infty) \to \mathbf{L}_p[0,\infty)$ if H is causal.)

[6] We are here considering one sided Laplace transforms. For $G_c = C_c(sI - A_c)^{-1}B_c$, where A_c is stable, we have $g_c(t) = C_c e^{A_c t}B_c$ for $t \geq 0$ and zero otherwise. Then $\mathcal{L}(g_c(t)) = \int_0^\infty e^{-st}g_c(t)dt$ with absolute convergence for Re $s \geq 0$ (in fact, for Re $s > \lambda_{\max}(A_c)$. For g_{ac} we use a one sided Laplace transform defined over negative times

[7] This is what you prove in Homework set 1

Proof. We follow the proof in [4]. Let $u \in \mathbf{L}_p(-\infty,\infty)$, and let $\frac{1}{p}+\frac{1}{q}=1$. Then

$$\left|\int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau\right| \leq \int_{-\infty}^{\infty} |h(t-\tau)|^{1/p}|u(\tau)| \cdot |h(t-\tau)|^{1/q} d\tau.$$

We can now use Hölders inequality $\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q$ with $f(\tau) = |h(t-\tau)|^{1/p}|u(\tau)| \in \mathbf{L}_p$ and $g(\tau) = |h(t-\tau)|^{1/q} \in \mathbf{L}_q$. This gives

$$\left|\int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau\right| \leq \left(\int_{-\infty}^{\infty} |h(t-\tau)| \cdot |u(\tau)|^p d\tau\right)^{1/p} \left(\int_{-\infty}^{\infty} |h(t-\tau)| d\tau\right)^{1/q},$$

where we note that the last term is $\|h\|_1^{1/q}$. If we take $\mathbf{L}_p$-norms on both sides of this inequality then we get

$$\begin{aligned}\|h*u\|_p &\leq \|h\|_1^{1/q}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |h(t-\tau)||u(\tau)|^p d\tau\right) dt\right)^{1/p} \\ &\leq \|h\|_1^{1/q} \cdot \|h\|_1^{1/p} \cdot \|u\|_p = \|h\|_1 \cdot \|u\|_p\end{aligned}$$

where we used that $\|h*f\|_1 \leq \|h\|_1 \cdot \|f\|_1$ for any $h, f \in \mathbf{L}_1$. □

It now follows from Proposition 1 and the two examples above that

- Each $G \in \mathbf{RH}_\infty$ is a bounded causal operator on $\mathbf{L}_p[0,\infty)$.
- Each $G \in \mathbf{RL}_\infty$ is a bounded operator on $\mathbf{L}_p(-\infty,\infty)$.

Remark 5. We will only consider systems with casual operators. However, noncausal operators will be used as "analysis filters" or "multipliers" in the discussion on IQCs. They will only be used for analysis of norm bounded signals i.e., signals in $\mathcal{H}$ for function spaces where the time axis is bi-infinite, e.g. $\mathcal{T} = \mathbf{R}$.

The gain bound in Proposition 1 can be improved for $\mathbf{L}_2$-spaces.

Proposition 2. *We have*

1. *Let $G \in \boldsymbol{RL}_\infty$ be an operator on $\boldsymbol{L}_2(-\infty,\infty)$ then*

$$\|G\| = \max_{\omega\in[0,\infty]} \sigma_{\max}(G(j\omega))$$

which often is denoted $\|G\|_{\boldsymbol{H}_\infty}$.

2. *Let $G \in \boldsymbol{RH}_\infty$ be an operator on $\boldsymbol{L}_2[0,\infty)$ then*

$$\|G\| = \max_{\omega\in[0,\infty]} \sigma_{\max}(G(j\omega)).$$

Proof. See [44]. The idea is to consider the frequency domain representation of the operator

$$\widehat{y}(j\omega) = G(j\omega)\widehat{u}(j\omega)$$

If the input $\widehat{u}$ has a Dirac at the frequency where the optimization problem below takes in maximum then it is possible to achieve the gain bound. □

Example 10. An operator defined by a nonlinearity $\varphi : \mathbf{R} \to \mathbf{R}$ as in Example 1 is both causal and anti-causal. Such operators are called memoryless.

Example 11. A nonlinear operator defined by a state space representation as in Example 2 is causal since the integration is assumed to be done forward in time.

Well-posedness and Stability

In the system (4) we assume that H_1 and H_2 are causal operators on $\mathcal{L}_e$. Well-posedness is defined as follows:

Definition 2 (Well-posedness). The system in (4) is well-posed if for any $u_1, u_2 \in \mathcal{L}_e$ there exist a solution $e_1, e_2 \in \mathcal{L}_e$. Furthermore, the loop signals e_1, e_2 depends causally on u_1 and u_2.

Definition 3 (Stability). The system (4) is stable if it is well-posed and if there are positive constants c_1, c_2, c_3, c_4 such that

$$\begin{aligned} \|e_{1T}\| &\le c_1\|u_{1T}\| + c_2\|u_{2T}\| \\ \|e_{2T}\| &\le c_3\|u_{1T}\| + c_4\|u_{2T}\| \end{aligned}$$

for all $T \in \mathcal{T}$.

Remark 6. If the system is stable and if u_1 and u_2 are norm bounded, i.e., $u_1, u_2 \in \mathcal{L}$, then $e_1, e_2 \in \mathcal{L}$.

Remark 7. A well posed system is not the same as a stable system. In a system that is well-posed but not stable, there may not be a (time) uniform gain as above definite. For example, if we can have $\|e_T\| = \mathcal{O}(e^{\gamma T}\|u_T\|)$ then the system is not stable.

Remark 8. Well-posedness is a generic property for any good model of a physical system. Conditions for well-posedness are discussed in detail in [31].

Let us truncate all terms on both sides of both equations in (4). We use the notation $P_T e_1 = e_{1T}$ and the fact that causality implies that $P_T H_1(e_1) = P_T H_1(e_{1T})$. We get

$$\begin{aligned} e_{1T} &= u_{1T} - P_T H_2(e_{2T}) \\ e_{2T} &= u_{2T} + P_T H_1(e_{1T}) \end{aligned} \tag{6}$$

If the system (4) is well-posed then its truncated version is a well defined equation system in the normed space $\mathcal{L}$ for all $T \in \mathcal{T}$. This means that we can take norms on both sides of the equations in (6). This will be used in the derivation of the small gain theorem.

4 The Small Gain Theorem

The small gain theorem is a fundamental result in stability theory. It generally gives conservative results but this can sometimes be alleviated by the use of loop transformations and multipliers, as is discussed in Section 6.

Theorem 1. *Assume that*

(i) the system in (4) is well-posed,

(ii) $\|H_1\| \cdot \|H_2\| < 1$.

Then the system is stable.

Proof. Consider the truncated system equations in (6). Using $e_{2T} = u_{2T} + P_T H_1(e_{1T})$ in the first equation gives

$$e_{1T} = u_{1T} - P_T H_2(u_{2T} + P_T H_1(e_{1T})) \tag{7}$$

$$e_{2T} = u_{2T} - P_T H_1(e_{1T}) \tag{8}$$

If we take norms in (7) then we get

$$\|e_{1T}\| \le \|u_{1T}\| + \|H_2\| \cdot \|u_{2T}\| + \|H_2\| \cdot \|H_1\| \cdot \|e_{1T}\|$$

Hence,

$$\|e_{1T}\| \le \frac{1}{1 - \|H_1\| \cdot \|H_2\|}\|u_{1T}\| + \frac{\|H_2\|}{1 - \|H_1\| \cdot \|H_2\|}\|u_{2T}\| \tag{9}$$

Finally, take norms of (8) and use (9). We get

$$\|e_{2T}\| \le \frac{\|H_1\|}{1 - \|H_1\| \cdot \|H_2\|}\|u_{1T}\| + \frac{1}{1 - \|H_1\| \cdot \|H_2\|}\|u_{2T}\| \tag{10}$$

$\square$

Example 12. Consider the system in (4) when H_1 is an LTI operator with transfer function $G(s) = C(sI - A)^{-1}B + D$ and when $\|H_2\| \le 1$ (for both signal spaces considered below). The small gain theorem ensures that the closed loop system is stable if $\|G\| < 1$. If we let the signal space $\mathcal{L}_e$ be $\mathbf{L}_{2e}[0,\infty)$ then the stability condition becomes

$$\|G\|_{\mathbf{H}_\infty} = \sup_{\omega \in [0,\infty]} |G(j\omega)| < 1$$

If the signal space is $\mathbf{L}_{\infty e}[0,\infty)$ then the stability condition becomes

$$\|G\|_1 = \int_0^\infty |Ce^{At}B|\, dt + |D| < 1$$

We can now argue that the $\mathbf{L}_1$-norm condition gives a more conservative condition for stability than the $\mathbf{H}_\infty$-norm. This follows since (the weighting function $g(t) = Ce^{At}B\theta(t) + D\delta(t)$)

$$|G(j\omega)| = |\int_0^\infty g(t)e^{-i\omega t}\, dt| \le \int_0^\infty |Ce^{At}B|\, dt + |D|$$

Hence, if $\|G\|_1 < 1$, then $\|G\|_{\mathbf{H}_\infty} < 1$. So is there any point in using the function space $\mathbf{L}_\infty[0,\infty)$? There is an important point. The stability bounds (9) and (10) gives bounds on the magnitudes of e_{1T}, e_{2T} that hold at any time instant when we use $\mathbf{L}_{\infty e}[0,\infty)$ whereas we get energy bounds when we use $\mathbf{L}_{2e}[0,\infty)$. The choice of signal space must reflect our requirements on the real system.

Example 13. Let $H_1 = G \in \mathbf{RH}_\infty$ and a let H_2 be a sector bounded nonlinearity $H_2 = \varphi(x) \in \text{sector}[-k,k)$. If the signal space is $\mathbf{L}_2[0,\infty)$ then the system is stable if $\|G\|_{\mathbf{H}_\infty} < 1/k$.

5 The Passivity Theorem

The passivity theorem is another fundamental result in stability theory. It has gained widespread application in analysis of electric circuits, see [4], and mechanical systems, see [3].

The passivity theorem exploits the additional structure of the inner product in a Hilbert space. We will assume that the inner product satisfies the following properties

$$\langle y, u\rangle_T := \langle y_T, u_T\rangle = \langle y, u_T\rangle = \langle y_T, u\rangle$$

and, as before, $\|u_T\|$ is a nondecreasing function of T and if $u \in \mathcal{H}$ then $\lim_{T\to\infty} \|u_T\| = \|u\|$, where $\|u\| = \sqrt{\langle u, u\rangle}$. These properties are satisfied in our standard spaces $l_{2e}(Z_+)$ and $\mathbf{L}_{2e}[0,\infty)$.

Definition 4. A causal operator $H : \mathcal{H}_e \to \mathcal{H}_e$ is

- *passive* if $\langle Hu, u \rangle_T \geq 0$ for all $u \in \mathcal{H}_e$, $\forall T \geq 0$
- *strictly output passive (SOP)* if there exists an $\varepsilon > 0$ such that

$$\langle Hu, u \rangle_T \geq \varepsilon \|P_T H(u)\|^2, \quad \forall u \in \mathcal{H}_e,\ \forall T \geq 0$$

Remark 9. Note that we do not require the operator to be bounded in the definition of passivity, see Example 16 for a passive operator with infinite gain. However, a strictly output passive operator is always bounded since

$$\varepsilon \|P_T H(u)\|^2 \leq \langle Hu, u \rangle_T \leq \|P_T H(u)\| \cdot \|u_T\|,$$

which implies that $\|H\| \leq 1/\varepsilon$.

Example 14. An LTI system $G(s) \in \mathbf{RH}_\infty^{m \times m}$ is

- *passive* if $G(j\omega) + G(j\omega)^* \geq 0$ for all ω,
- *SOP* if there exists $\varepsilon > 0$ such that $\frac{1}{2}(G(j\omega) + G(j\omega)^*) \geq \varepsilon G(j\omega)^* G(j\omega),\ \forall \omega,$

We prove this in Example 20 in Section 7.

Example 15. In this example we consider the operator H defined by the input-output map of the nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \quad x(0) = 0 \\ y &= h(x) \end{aligned}$$

where $f(0) = 0$ and $h(0) = 0$. Then H is SOP if there exists a continuously differentiable positive semidefinite function V with $V(0) = 0$ such that

$$\begin{aligned} \frac{\partial V}{\partial x} f(x) &= -k h(x)^T h(x), \\ \frac{\partial V}{\partial x} g(x) &= h^T(x) \end{aligned}$$

where $k > 0$. The system is passive if the above holds with $k = 0$. The proof follows since $\dot{V}(x) = \frac{\partial V}{\partial x}(f(x) + g(x)u) = -k|y|^2 + y^T u$. Integration gives

$$V(x(T)) - V(x(0)) = \int_0^T y^T u \, dt - k \int_0^T |y|^2 \, dt$$

Since, $V(x(0)) = 0$ and $V(x(T)) \geq 0$, we get $\langle y, u \rangle_T \geq k \|y_T\|^2$.

Example 16. Consider the following simplified version of the LuGre-friction model [21, 2]

$$\begin{aligned} \frac{dz}{dt} &= v - \frac{|v|}{g(v)} z,\ z(0) = 0 \\ g(v) &= \frac{1}{\sigma_0}(F_C + (F_S - F_C) e^{-(v/v_s)^2}) \\ F &= \sigma_0 z \end{aligned} \tag{11}$$

where F denotes the friction force, v is the relative velocity of the surfaces, σ_0 is a stiffness coefficient, F_S is the Stribeck friction, and F_C is the Columb friction. It is assumed that $F_S \geq F_C > 0$ This friction model is passive as an operator $H : v \mapsto F$ on $\mathbf{L}_{2e}[0, \infty)$ since

$$Fv = \sigma_0 (z \frac{dz}{dt} + \frac{|v|}{g(v)} z^2) \geq \sigma_0 z \frac{dz}{dt}.$$

Integration gives

$$\langle F, v \rangle_T \geq \frac{1}{2} \sigma_0 z(T)^2 \geq 0,$$

which proves passivity. It is easy to see that the friction operator is unbounded since a small input pulse can make z stay at a nonzero value when the input has turned to zero. This means that the $\mathbf{L}_2$-norm of the output is infinity.

We will next prove one of the simpler formulations of the passivity theorem.

Theorem 2 (The Passivity Theorem). *Assume that*

(i) *the system in (4) is well-posed,* $u_2 = 0$

(ii) $H_1 : \mathcal{H}_e \to \mathcal{H}_e$ *is strictly output passive*

(iii) $H_2 : \mathcal{H}_e \to \mathcal{H}_e$ *is passive*

Then the system is stable in the sense $\|e_{2T}\| \leq \frac{1}{\varepsilon} \|u_{1T}\|$, *for all* $T \geq 0$, *where* ε *is from the definition of strict output passivity.*

Remark 10. The theorem shows that e_2 is bounded but note that e_1 may not be bounded (in $\mathbf{L}_2$-norm). However, if H_2 is bounded then we also have $\|e_{1T}\| \leq c \|u_{1T}\|$ for all $T \geq 0$ for some $c > 0$.

Proof. The truncated system now becomes

$$\begin{aligned} e_{1T} &= u_{1T} - P_T H_2(e_2) \\ e_{2T} &= P_T H_1(e_1) \end{aligned}$$

We get

$$\langle u_1, H_1(e_1) \rangle_T = \langle e_1, H_1(e_1) \rangle_T + \langle H_2(e_2), e_2 \rangle_T \geq \varepsilon \|P_T H_1(e_1)\|^2$$

This gives $\|P_T H_1(e_1)\|^2 \leq \frac{1}{\varepsilon} \|u_{1T}\| \cdot \|P_T H_1(e_1)\|$, i.e., $\|P_T H_1(e_1)\| \leq \frac{1}{\varepsilon} \|u_{1T}\|$. ☐

Example 17. Consider the system in Figure 5, which models position control of a servo with friction. We assume that the friction can be modeled as the LuGre friction in Example 16 and that the PD-controller has transfer function $K(s) = k_1 + k_2 s$, where $k_1, k_2 > 0$. The system can equivalently be represented as

$$\begin{aligned} e_1 &= d - H(v) \\ v &= G e_1 \end{aligned}$$

where H denotes the LuGre friction model and $G(s) = \frac{s}{ms^2 + k_2 s + k_1}$. We know that H is passive and we have

$$\text{Re}\, \{G(j\omega)\} = k_2 |G(j\omega)|^2,$$

i.e., G is strictly output passive. Hence, it follows from the Passivity theorem that $\|v_T\| \leq c \|d_T\|$ for some $c > 0$.

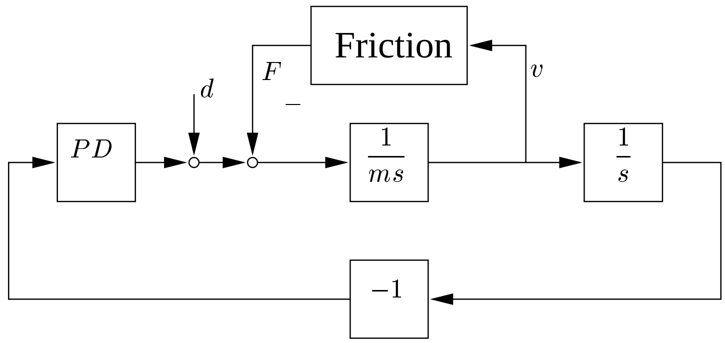

Figure 5: Servo control system with friction.

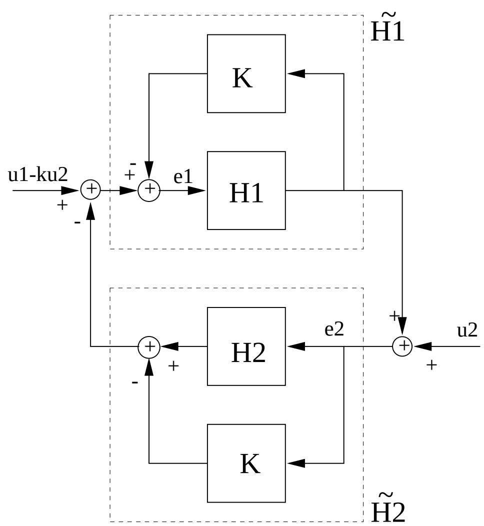

Figure 6: A Loop Transformation

6 Loop Transformations and Multipliers

The small gain theorem and the passivity theorem generally give conservative stability conditions. Loop transformations and the introduction of multipliers in the feedback loop are means to reduce conservatism.

Loop Transformations

Figure 6 shows a loop transformation of the system in (4), which we assume to be well-posed. Here $K : \mathcal{L}_e \to \mathcal{L}_e$ is a suitably chosen linear bounded and causal operator. The loop transformation is well-posed if $\widetilde{H_1} = (I + H_1K)^{-1}H_1$ is a well defined operator on $\mathcal{L}_e$. Then the transformed system is well-posed and stability of the system (4) is equivalent to stability of its transformed version.

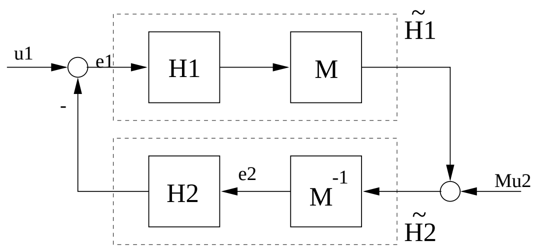

Figure 7: Introduction of multipliers

Multipliers

Figure 7 shows how a multiplier and its inverse have been introduced in the feedback loop. If both M and its inverse M^{-1} are bounded causal operators on $\mathcal{L}_e$ then stability of the system in Figure 7 implies stability of the system in (4). It is also possible to consider noncausal filters M but then several technical conditions need to be introduced.

The main point with the loop transformations and the multipliers is that it may be easier to prove stability for the transformed system than the original system.

We will in the next section discuss the IQC framework for stability analysis in Hilbert spaces. The introduction of multipliers and loop transformations is done implicitly and with great simplicity in the IQC framework. This is very convenient in advanced systems analysis. We will in a later section discuss the connection between the IQC technique and the classical loop transformation and multiplier ideas discussed above.

Equivalence between Possitivity and Unity Gain

We will end this section with a peculiar little result which exemplifies that basic mathematical ideas often extends to much more general situations.

Proposition 3. *Let $H : \mathcal{H} \to \mathcal{H}$ and assume that $H + I$ is invertible on $\mathcal{H}$. Define $S : \mathcal{H} \to \mathcal{H}$ as $S = (H - I)(H + I)^{-1}$. Then we have the following property*

$$\langle f, Hf \rangle \geq 0, \ \forall f \in \mathcal{H} \quad \Leftrightarrow \quad \|S\| \leq 1.$$

Remark 11. The proposition is a generalization of the conformal mapping $S(z) = \frac{z-1}{z+1}$ between the right half complex plane and the unic circle to nonlinear operators on a Hilbert space.

Proof. Let $g \in \mathcal{H}$. Then $f = (H + I)^{-1}(g)$ satisfies

(i) $S(g) = (H - I)(f)$

(ii) $g = (H + I)(f)$

If we use (i) and (ii) respectively then we get

$$\begin{aligned} \|S(g)\|^2 &= \|H(f)\|^2 + \|f\|^2 - 2\langle H(f), f \rangle \\ \|g\|^2 &= \|H(f)\|^2 + \|f\|^2 + 2\langle H(f), f \rangle \end{aligned}$$

After subtraction we get

$$\|g\|^2 - \|S(g)\|^2 = 4\langle H(f), f \rangle$$

which proves the claim. $\square$

7 Adjoint operators and Quadratic Forms

The integral quadratic constraints, which we discuss in the next section, are defined in terms of time-invariant quadratic forms. In order to introduce the time invariant quadratic forms we need to discuss the Hilbert adjoint operator, self-adjoint operators, and positive-definiteness of self-adjoint operators.

Definition 5. Let $H : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. Then the Hilbert adjoint H^* of H is the operator $H^* : \mathcal{H} \to \mathcal{H}$ such that

$$\langle Hf, g\rangle = \langle f, H^* g\rangle \quad \forall f, g \in \mathcal{H}$$

Example 18. A matrix $M \in \mathbf{R}^{n\times n}$ defines a bounded linear operator on the Hilbert space $\mathbf{R}^n$ equipped with the standard inner product $\langle x, y\rangle = x^T y$. The Hilbert adjoint M^* is the transpose of the matrix, i.e., $M^* = M^T$ (if the matrix is complex-valued then $M^* = \overline{M}^T$). This follows since

$$\langle Mx, y\rangle = x^T M^T y = \left\langle x, M^T y\right\rangle .$$

Example 19. Let $H \in \mathbf{RH}_\infty^{m\times m}$ be an operator on $\mathbf{L}_2^m(-\infty,\infty)$ with state space realization $H(s) = C(sI - A)^{-1}B + D$, where A is a stable matrix. Then H has Hilbert adjoint $H^*(s) = H(-s)^T = -B^T(sI + A^T)^{-1}C^T + D^T$. We will derive this in the time-domain. Let $h_0(t) = Ce^{At}B\theta(t)$, then

$$\begin{aligned}\langle Hf, g\rangle &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} h_0(t-\tau)f(\tau)d\tau + Df(t)\right)^T g(t)dt\\ &= \int_{-\infty}^{\infty} f(\tau)^T\left(\int_{\tau}^{\infty} h_0(t-\tau)^T g(\tau)dt + D^T g(t)\right)d\tau = \langle f, H^* g\rangle\end{aligned}$$

which shows that the adjoint is an anti-causal operator with state space realization $H^*(s) = H(-s)^T = -B^T(sI + A^T)^{-1}C^T + D^T$.

More generally, the adjoint of an operator $H \in \mathbf{RL}_\infty^{m\times m}$ is $H^*(s) = H(-s)^T$. This can be shown by spliting H into its causal and anticausal term and then compute the adjoint of these two terms and finally add them to get the result. However, a more direct way is to consider the frequency domain representation of the inner product

$$\begin{aligned}\langle Hf, g\rangle &= \int_{-\infty}^{\infty} (H(j\omega)\widehat{f}(j\omega))^*\widehat{g}(j\omega)d\omega\\ &= \int_{-\infty}^{\infty} \widehat{f}(j\omega)^*(H(j\omega)^*\widehat{g}(j\omega))d\omega = \langle f, H^* g\rangle\end{aligned}$$

and use the fact $H(j\omega)^* = H(-j\omega)^T$.

We have now seen two examples where it was possible to construct the adjoint. Next we state the reassuring fact that there always exists an Hilbert adjoint. Several useful properties are also stated.

Theorem 3. *The Hilbert adjoint H^* in Definition 5 exists uniquely and it is a linear operator with $\|H^*\| = \|H\|$. Furthermore, for bounded linear operators $H, H_1, H_2 : \mathcal{H} \to \mathcal{H}$ we have the following properties*

a) $(\alpha H)^* = \alpha H^*$	*b)* $(H_1 + H_2)^* = H_1^* + H_2^*$	*c)* $(H^*)^* = H$
d) $(H_1 H_2)^* = H_2^* H_1^*$	*e)* $\|T^*T\| = \|TT^*\| = \|T\|^2$	*f)* $(H^*)^{-1} = (H^{-1})^*$

where in the last statement we assume that H is invertible.

Proof. See [12] for a full proof. The existence and uniqueness is a consequence of the Riesz representation theorem. The properties $a) - f)$ are rather straightforward to derive. In fact, the proof is completely analogous to the matrix case. $\square$

We will next introduce the concept of self-adjoint operator and positive definiteness of a self-adjoint operator.

Definition 6. A bounded linear operator $H : \mathcal{H} \to \mathcal{H}$ is self-adjoint if $H^* = H$. A self-adjoint operator is

Positive semi-definite, denoted $H \geq 0$ if and only if $\langle Hf, f \rangle \geq 0$ for all $f \in \mathcal{H}$.

Positive definite, denoted $H > 0$, if and only if there exists $\varepsilon > 0$ such that

$$\langle Hf, f \rangle \geq \varepsilon \|f\|^2, \quad \forall f \in \mathcal{H}.$$

H is said to be negative semi-definite if $-H$ is positive semi-definite and H is negative definite if $-H$ is positive definite.

The integral quadratic constraints in the next section are defined in terms of time-invariant quadratic forms on a Hilbert space. A bounded self-adjoint operator $\Phi = \Phi^* : \mathcal{H} \to \mathcal{H}$ defines a (bounded) quadratic form $\sigma : \mathcal{H} \to \mathbf{R}$ as $\sigma(f) = \langle \Phi f, f \rangle$. The quadratic form is positive semi-definite if $\sigma(f) \geq 0$ for all $f \in \mathcal{H}$ and strictly positive definite if there exists $\varepsilon > 0$ such that $\sigma(f) \geq \varepsilon \|f\|^2$, for all $f \in \mathcal{H}$. Negative semi-definiteness and negative definiteness are defined analogously. It follows from Definition 6 that σ is positive semi-definite (positive definite) if and only if $\Phi \geq 0$ ($\Phi > 0$).

For a subspace $\widetilde{\mathcal{H}} \subset \mathcal{H}$ we also have that $\Phi = \Phi^* : \mathcal{H} \to \mathcal{H}$ defines a quadratic form $\sigma : \widetilde{\mathcal{H}} \to \mathbf{R}$ by the relation $\sigma(f) = \langle \Phi f, f \rangle$, $f \in \widetilde{\mathcal{H}}$. It is obvious that $\Phi \geq 0$ in this case also implies that $\sigma \geq 0$. The reverse implication is not at all clear. However, it turns out that the reverse implication holds when $\mathcal{H} = \mathbf{L}_2(-\infty, \infty)$ and $\widetilde{\mathcal{H}} = \mathbf{L}_2[0, \infty)$. Here we use that $\mathbf{L}_2[0, \infty) \subset \mathbf{L}_2(-\infty, \infty)$ if for each $f \in \mathbf{L}_2[0, \infty)$ we define $f(t) = 0$ for $t \leq 0$. We use this assumption from now on.

Proposition 4. *Let $\Phi = \Phi^* \in \boldsymbol{RL}_\infty^{m \times m}$ and define the quadratic form $\sigma(f) = \langle \Phi f, f \rangle$ on $\boldsymbol{L}_2[0, \infty)$. Then the following are equivalent*

(i) $\sigma(f) \geq 0$ *for all* $f \in \boldsymbol{L}_2[0, \infty)$

(ii) $\Phi(j\omega) \geq 0$ *for all* $\omega \geq 0$.

Proof. The proof is taken from [20]. The implication $(ii) \Rightarrow (i)$ is more or less obvious since $\mathbf{L}_2[0, \infty) \subset \mathbf{L}_2(-\infty, \infty)$ and $\Phi \geq 0$ implies that $\sigma \geq 0$ on $\mathbf{L}_2(-\infty, \infty)$. For the other direction we use that the quadratic form is time-invariant on $\mathbf{L}_2(-\infty, \infty)$. Indeed, if $S_\tau : \mathbf{L}_2(-\infty, \infty) \to \mathbf{L}_2(-\infty, \infty)$ is the shift operator defined by $(S_\tau f)(t) = f(t - \tau)$, then we have

$$\begin{aligned} \sigma(S_\tau f) = \langle \Phi S_\tau f, S_\tau f \rangle &= \int_{-\infty}^{\infty} (\widehat{f}(j\omega) e^{-j\omega\tau})^* \Phi(j\omega) \widehat{f}(j\omega) e^{-j\omega\tau} d\omega \\ &= \int_{-\infty}^{\infty} \widehat{f}(j\omega)^* \Phi(j\omega) \widehat{f}(j\omega) d\omega = \langle \Phi f, f \rangle = \sigma(f). \end{aligned}$$

Hence, if $\sigma \geq 0$ on $\mathbf{L}_2[0, \infty)$ then $\sigma \geq 0$ on $\mathbf{L}_2[\tau, \infty)$ for any $\tau > -\infty$. Next, we use that $\cup_{\tau > -\infty} \mathbf{L}_2[\tau, \infty)$ is dense in $\mathbf{L}_2(-\infty, \infty)$ and that σ is continuous on $\mathbf{L}_2(-\infty, \infty)$ to infer that $\sigma \geq 0$ on $\mathbf{L}_2[0, \infty)$ implies $\sigma \geq 0$ on $\mathbf{L}_2(-\infty, \infty)$. The later is equivalent to $\Phi(j\omega) \geq 0$ for all $\omega \geq 0$. $\square$

Example 20. We will here prove that $G \in \mathbf{RH}_\infty^{m\times m}$ is strictly output passive if $\frac{1}{2}(G(j\omega) + G(j\omega)^*) \geq \varepsilon G(j\omega)^* G(j\omega)$ for some $\varepsilon > 0$. This follows since

$$\begin{aligned}\langle Gu, u\rangle_T - \varepsilon\|P_T G u_T\|^2 &= \langle Gu_T, u_T\rangle - \varepsilon\|P_T G u_T\|^2 \\ &\geq \frac{1}{2}\left\langle (G + G^*)u_T, u_T\right\rangle - \varepsilon\|Gu_T\|^2 \\ &= \left\langle (\frac{1}{2}(G + G^*) - \varepsilon G^* G)u_T, u_T\right\rangle \geq 0,\end{aligned}$$

where we used the above proposition in the last inequality.

8 Integral Quadratic Constraints

Integral Quadratic Constraints (IQCs) give useful characterizations of the structure of a given operator on an Hilbert space. The IQCs are defined in terms of quadratic forms which are defined in terms of self-adjoint operators. The resulting stability theory unifies and extends the classical passivity based multiplier theory. The stability conditions are computationally attractive and we will discuss a method for computing the multipliers that appear in the stability criterion later.

We consider systems on the form (4) for the special case when H_1 is defined in terms of a causal and bounded LTI transfer function G, and when $H_2 = -\Delta$, where Δ is a bounded and causal operator on $\mathcal{H}$. The system equations become[8]

$$\begin{aligned} v &= Gw + e \\ w &= \Delta(v) \end{aligned} \tag{12}$$

We will be particularly interested in the case when the operators are defined on either of the extended spaces $\mathcal{H}_e = \mathbf{L}_{2e}^m[0,\infty)$ or $\mathcal{H}_e = l_{2e}^m[0,\infty)$.

Next we define the IQC for operators on $\mathcal{H}_e$. It is important to notice that the IQC is defined on the Hilbert space $\mathcal{H}$ and does not involve truncations of the signals. This is makes it much easier to obtain general and flexible results compared to when multipliers and loop transformations are used in the framework of the small gain theorem or the passivity theorem. We will discuss this in the next section.

Definition 7 (IQC). Let Π be a bounded and self-adjoint operator. Then Δ satisfies the IQC defined by Π if

$$\sigma_\Pi(v, \Delta(v)) = \left\langle \begin{bmatrix} v \\ \Delta(v) \end{bmatrix}, \Pi \begin{bmatrix} v \\ \Delta(v) \end{bmatrix} \right\rangle \geq 0, \quad \forall v \in \mathcal{H} \tag{13}$$

We often call Π the *multiplier* that defines the IQC. We will sometimes use the shorthand notation $\Delta \in \mathrm{IQC}(\Pi)$ to mean that Δ satisfies the IQC defined by Π.

Remark 12. If $\mathcal{H} = \mathbf{L}_2^m[0,\infty)$, then Π can be taken as a transfer function satisfying $\Pi(j\omega) = \Pi(j\omega)^*$. The condition in (13) reduces to

$$\sigma_\Pi(v, \Delta(v)) = \int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{bmatrix} \geq 0, \quad \forall v \in \mathbf{L}_2^m[0,\infty) \tag{14}$$

[8] A disturbance in the second equation can be included in e since G is linear and bounded.

If $\mathcal{H} = l_2^m[0,\infty)$ then Π can be taken as a transfer function satisfying $\Pi(e^{j\omega}) = \Pi(e^{j\omega})^*$ for all $\omega \in [-\pi, \pi]$. The condition in (13) reduces to

$$\sigma_\Pi(v, \Delta(v)) = \int_{-\pi}^{\pi} \begin{bmatrix} \widehat{v}(e^{j\omega}) \\ \widehat{\Delta(v)}(e^{j\omega}) \end{bmatrix}^* \Pi(e^{j\omega}) \begin{bmatrix} \widehat{v}(e^{j\omega}) \\ \widehat{\Delta(v)}(e^{j\omega}) \end{bmatrix} \geq 0, \quad \forall v \in l_2^m(Z_+)$$

Remark 13. The two simplest examples of multipliers are

$$\Pi_1 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \text{and} \quad \Pi_2 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

We see that Π_1 defines a valid IQC for operators that have gain less than one. The multiplier Π_2 corresponds to passivity.

Let us consider a couple of examples.

Example 21. Let φ be a nonlinearity that satisfies the sector condition $\alpha x^2 \leq \varphi(x,t)x \leq \beta x^2$, for all $(x,t) \in \mathbf{R} \times \mathbf{R}^+$. Then φ satisfies the IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} -2\alpha\beta & \beta + \alpha \\ \beta + \alpha & -2 \end{bmatrix}$$

To see this we notice that (this relation is in the time domain)

$$\begin{bmatrix} v \\ \varphi(v) \end{bmatrix}^T \Pi \begin{bmatrix} v \\ \varphi(v) \end{bmatrix} = 2(\beta v - \varphi(v))(\varphi(v) - \alpha v) \geq 0,$$

where the inequality is an immediate consequence of the sector condition. Integration gives the desired result.

Example 22. Let Δ correspond to multiplication with a real scalar $\delta \in [-1, 1]$, i.e., $(\Delta v)(t) = \delta v(t)$. Then Δ satisfies the IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} X(j\omega) & Y(j\omega) \\ Y(j\omega)^* & -X(j\omega), \end{bmatrix}$$

where $X(j\omega) = X(j\omega)^* \geq 0$ and $Y(j\omega)^* = -Y(j\omega)$. This follows since

$$\begin{aligned} &\begin{bmatrix} \widehat{v}(j\omega) \\ \delta\widehat{v}(j\omega) \end{bmatrix}^* \begin{bmatrix} X(j\omega) & Y(j\omega) \\ Y(j\omega)^* & -X(j\omega) \end{bmatrix} \begin{bmatrix} \widehat{v}(j\omega) \\ \delta\widehat{v}(j\omega) \end{bmatrix} \\ &\qquad = \widehat{v}(j\omega)^*(X(j\omega) - \delta^2 X(j\omega) + \delta(Y(j\omega) - Y(j\omega)))\widehat{v}(j\omega) \geq 0. \end{aligned}$$

Integration gives the result.

Example 23. Consider the saturation nonlinearity

$$\varphi(x) = \begin{cases} x, & |x| \leq 1 \\ \operatorname{sign}(x), & |x| > 1 \end{cases}$$

We will show that φ satisfies the IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} 0 & 1 + H(j\omega) \\ 1 + H(j\omega)^* & -2(1 + \text{Re } H(j\omega)) \end{bmatrix}$$

where H is the Fourier transform of a function $h : \mathbf{R} \to \mathbf{R}$ that satisfy the $\mathbf{L}_1$-norm constraint

$$\|h\|_1 = \int_{-\infty}^{\infty} |h(t)| dt \leq 1$$

To see this we notice that (here $*$ denotes convolution)

$$\begin{aligned}
&[v(t) - \varphi(v(t))] \cdot [\varphi(v(t)) + (h * \varphi(v))(t)] \\
&\qquad \geq [v(t) - \varphi(v(t))] \cdot [\varphi(v(t)) - \operatorname{sign}(v(t)) \sup_{v \in \mathbf{R}} |\varphi(v)| \cdot \|h\|_1] \\
&\qquad\qquad \geq [v(t) - \varphi(v(t))] \cdot [\varphi(v(t)) - \operatorname{sign}(v(t))] = 0
\end{aligned}$$

Integration and use of Parsevals theorem gives the desired result:

$$\begin{aligned}
0 &\leq \int_0^{\infty} 2[v - \varphi(v)] \cdot [\varphi(v) + h * \varphi(v)] dt \\
&= \int_{-\infty}^{\infty} 2\mathrm{Re}\, [\widehat{v}(j\omega) - \widehat{\varphi(v)}(j\omega)]^* [\widehat{\varphi(v)}(j\omega) + H(j\omega)\widehat{\varphi(v)}(j\omega)] d\omega \\
&= \int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{\varphi(v)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{\varphi(v)}(j\omega) \end{bmatrix} d\omega
\end{aligned}$$

The multipliers in this example can actually be used to describe any nonlinearity with slope restricted to the interval $[0, 1]$. This is proved in the classical paper [43]. Note that H can be viewed as a non-causal filter, i.e., the H can have poles both in the left half plane and the right half plane.

We have the following stability result.

Theorem 4. *Assume that*

(*i*) *for* $\tau \in [0, 1]$, *the interconnection* $(G, \tau\Delta)$ *is well-posed,*

(*ii*) *for* $\tau \in [0, 1]$, $\tau\Delta \in \mathrm{IQC}(\Pi)$,

(*iii*) *there exists* $\varepsilon > 0$ *such that*[9]

$$\begin{bmatrix} G \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G \\ I \end{bmatrix} \leq -\varepsilon I \tag{15}$$

Then the system in (12) is stable.

Remark 14. When $\mathcal{H} = \mathbf{L}_2^m[0, \infty)$ then (15) is equivalent to the condition

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\varepsilon I, \quad \forall \omega \in \mathbf{R}$$

and when $\mathcal{H} = l_2^m[0, \infty)$ then it is equivalent to the condition

$$\begin{bmatrix} G(e^{j\omega}) \\ I \end{bmatrix}^* \Pi(e^{j\omega}) \begin{bmatrix} G(e^{j\omega}) \\ I \end{bmatrix} \leq -\varepsilon I, \quad \forall \omega \in [-\pi, \pi]$$

[9] This means that the self-adjoint operator

$$\varepsilon I + \begin{bmatrix} G \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G \\ I \end{bmatrix}$$

is negative semi-definite.

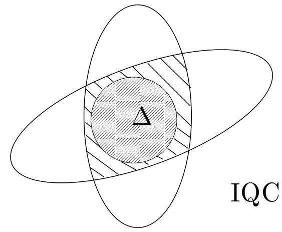

Figure 8: The more IQCs we have the better characterization we get of the uncertainty Δ. The grey area represents the set of uncertainties Δ and the shaded area represents the complete set of causal bounded operators that satisfy the IQC.

Remark 15. If

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}$$

has $\Pi_{11} \geq 0$ and $\Pi_{22} \leq 0$, then the condition $\Delta \in \mathrm{IQC}(\Pi)$ implies that $\tau\Delta \in \mathrm{IQC}(\Pi)$ for all $\tau \in [0,1]$. This is often the case in applications.

Remark 16. Assume that $\Delta \in \mathrm{IQC}(\Pi_k)$, $k = 1, \dots, N$. Then it is easy to see that $\Delta \in \mathrm{IQC}(\sum_{k=1}^N \tau_k \Pi_k)$, where $\tau_k \geq 0$. The stability test now becomes the convex feasibility test: Find $\tau_k \geq 0$ such that

$$\begin{bmatrix} G \\ I \end{bmatrix}^* \left(\sum_{k=1}^N \tau_k \Pi_k \right) \begin{bmatrix} G \\ I \end{bmatrix} \leq -\varepsilon I$$

Remark 17. In the case when $\Pi_{22} \leq 0$ the class of uncertainties $\Delta \in \mathrm{IQC}(\Pi)$ is convex and we can look at the IQC as a way to cover Δ (which may belong to a set of uncertainties) with a larger set of operators. The more IQCs we have the better characterization we have, see Figure 8.

Proof. We will prove the theorem under a somewhat stronger well-posedness assumption than necessary[10]. We will assume that there exists a unique solution $v, w \in \mathcal{H}_e$ in the system (12) for every $e \in \mathcal{H}_e$ (we did not require uniqueness in the previous well-posedness assumption). This means that $I - G\Delta$ has a causal inverse on $\mathcal{H}_e$. The proof follows if we can show that $(I - G\Delta)^{-1}$ is bounded. The idea for proving this is illustrated in Figure 9 and Figure 10. We need to show that stability of the interconnection of $(G, \tau\Delta)$ implies stability of the interconnection $(G, (\tau + \tau_\Delta)\Delta)$ for all $|\tau_\Delta| \leq \gamma$, where γ is independent of τ. We prove this in two steps below. The proof of the theorem then follows from the iterative argument that is illustrated in Figure 10.

Step 1: There exists $c_0 > 0$, which is independent of τ, such that $\|v\| \leq c_0 \|(I - \tau G\Delta)(v)\|$, $\forall v \in \mathcal{H}$.

Let us prove this. Let $w = \tau\Delta(v)$ and assume that all signals are in $\mathcal{H}$. We have

$$\begin{aligned}
0 \leq \left\langle \begin{bmatrix} v \\ w \end{bmatrix}, \Pi \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} v - Gw + Gw \\ w \end{bmatrix}, \Pi \begin{bmatrix} v - Gw + Gw \\ w \end{bmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} v - Gw \\ 0 \end{bmatrix}, \Pi \begin{bmatrix} v - Gw \\ 0 \end{bmatrix} \right\rangle + 2 \left\langle \begin{bmatrix} v - Gw \\ 0 \end{bmatrix}, \Pi \begin{bmatrix} Gw \\ w \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} Gw \\ w \end{bmatrix}, \Pi \begin{bmatrix} Gw \\ w \end{bmatrix} \right\rangle \\
&\leq \|\Pi_{11}\| \cdot \|(I - \tau G\Delta)(v)\|^2 + 2(\|\Pi_{11}\| \cdot \|G\| + \|\Pi_{12}\|) \|(I - \tau G\Delta)(v)\| \cdot \|w\| - \varepsilon \|w\|^2
\end{aligned}$$

[10] A slight variation of this proof gives the proof under the weaker well-posedness assumption.

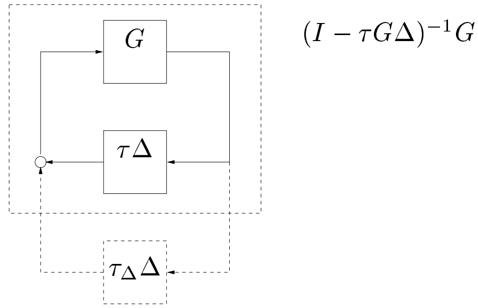

Figure 9: Stability of the feedback interconnection $(G, \tau\Delta)$ implies stability of the feedback interconnection $(G, (\tau + \tau_\Delta)\Delta)$ for all $|\tau_\Delta| \leq \gamma$, where γ is independent of τ. This means that we can insert the dashed branch in the system without loosing stability. This allows us to infer stability of (G, Δ) through an iterative argument, see Figure 10.

where the first inequality follows since $\tau\Delta \in \mathrm{IQC}(\Pi)$ and the last inequality follows from standard use of Cauchys inequality and the stability condition (15). Use of the implication (we assume $a > 0$, $c < 0$)

$$\begin{cases} ax^2 + 2bxy + cy^2 \geq 0 \\ x \geq 0 \end{cases} \qquad \Rightarrow \quad x \geq -\frac{b}{a}y + \sqrt{\frac{b^2}{a^2}y^2 - \frac{c}{a}y^2}$$

with $a = \|\Pi_{11}\|, b = \|\Pi_{11}\| \cdot \|G\| + \|\Pi_{12}\|, c = -\varepsilon, x = \|(I - \tau G\Delta)(v)\|$, and $y = \|w\|$ gives

$$\|w\| \leq \frac{1}{c_1}\|(I - \tau G\Delta)(v)\|$$

where

$$c_1 = -\frac{b}{a} + \sqrt{\frac{b^2}{a^2} + \frac{\varepsilon}{a}}.$$

On the other hand, when $\|\Pi_{11}\| = 0$ we get the same inequality with $c_1 = \varepsilon/(2(\|\Pi_{11}\| \cdot \|G\| + \|\Pi_{12}\|))$. Hence,

$$\|v\| = \|v - Gw + Gw\| \leq (1 + \|G\|/c_1)\|(I - \tau G\Delta)(v)\| = c_0\|(I - \tau G\Delta)(v)\|,$$

i.e., $c_0 = (1 + \|G\|/c_1)$. This proves the claim.

Step 2: Boundedness of $(I - \tau G\Delta)^{-1}$ for some $\tau \in [0, 1]$ implies boundedness of $(I - (\tau + \tau_\Delta)G\Delta)^{-1}$ for all $|\tau_\Delta| \leq \gamma$, where γ is independent of τ

Before we prove this we need to remark again that we only know that the system is bounded at $\tau = 0$. If we assume that $(I - \tau G\Delta)^{-1}$ is bounded, then follows from step 1 that

$$\|(I - \tau G\Delta)^{-1}\| \leq c_0$$

We will make crucial use of this inequality when we prove step 2. It is important to note that the inequality from step one by no means imply stability by itself unless we add some extra condition. The extra condition is supplied in step two, which we prove now.

Now consider the factorization

$$(I - (\tau + \tau_\Delta)G\Delta) = (I - \tau G\Delta)(I - (I - \tau G\Delta)^{-1}G\tau_\Delta\Delta)$$

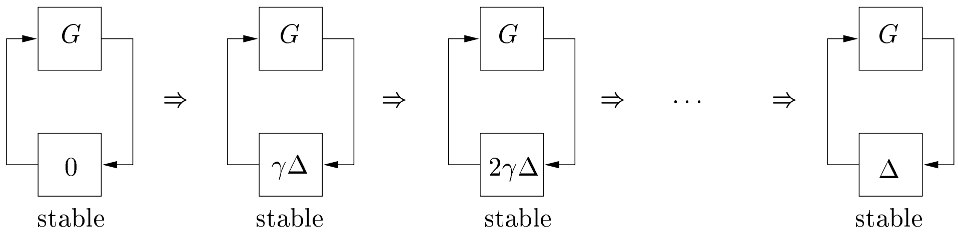

Figure 10: The left hand system is stable since G is bounded. Iterative use of the result illustrated in Figure 9 shows that all the systems in the figure are stable.

The first factor on the right hand side has a bounded inverse by assumption. To prove boundedness of the second factor we use the small gain theorem on the system in Figure 9. Due to our strong well-posedness assumption we have that $(I - (I - \tau G\Delta)^{-1} G\tau_\Delta \Delta)$ is invertible if $\|\tau_\Delta \Delta\| \cdot \|(I - \tau G\Delta)^{-1} G\| < 1$, which holds if (here we use $\|(I - \tau G\Delta)^{-1}\| \le c_0$)

$$\tau_\Delta < \gamma = \frac{1}{c_0 \|G\| \cdot \|\Delta\|} \tag{16}$$

Hence, the condition in (16) ensures boundedness of $(I - (\tau + \tau_\Delta)G\Delta)^{-1}$ and we see that γ is independent of τ. This proves the claim. $\square$

Let us consider a simple example.

Example 24. Consider the system in Figure 11. Here G is a strictly proper SISO system and φ is a nonlinearity that satisfies the sector condition $\alpha x^2 \le \varphi(x,t)x \le \beta x^2$, where we assume that $\alpha \le 0 \le \beta$. Under reasonable regularity assumptions on φ (for example continuity) we have well-posedness for all $\tau \in [0,1]$. We also have that $\tau\varphi \in \mathrm{IQC}(\Pi)$ for all $\tau \in [0,1]$ when

$$\Pi(j\omega) = \begin{bmatrix} -2\alpha\beta & \beta + \alpha \\ \beta + \alpha & -2 \end{bmatrix}$$

This follows from Example 21 since $\alpha x^2 \le \tau\varphi(x,t)x \le \beta x^2$ for all $\tau \in [0,1]$ when $\alpha < 0 < \beta$.

The system in (12) is a positive feedback interconnection and we need to include the minus sign in G. The stability condition becomes

$$\begin{bmatrix} -G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} -2\alpha\beta & \beta + \alpha \\ \beta + \alpha & -2 \end{bmatrix} \begin{bmatrix} -G(j\omega) \\ I \end{bmatrix} = -2\mathrm{Re}\ (G\beta + 1)^*(G\alpha + 1) < 0$$

multiplying this inequality with $-1/(2\beta\alpha)$ gives the stability condition

$$\mathrm{Re}\ (\overline{G(j\omega)} + 1/\beta)(G(j\omega) + 1/\alpha) < 0, \quad \forall\omega \in [0,\infty]$$

This is a version of the famous circle criterion. The stability condition is illustrated in Figure 12.

9 Relation to the Classical Methods

[11][12] The use of multipliers in stability analysis with the small gain theorem or the passivity theorem can generally reduce conservatism of the analysis extensively. We will here discuss

[11] This section is optional reading.
[12] The material is taken from [8].

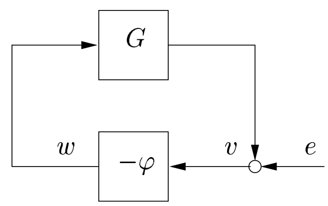

Figure 11: The feedback system for Example 24.

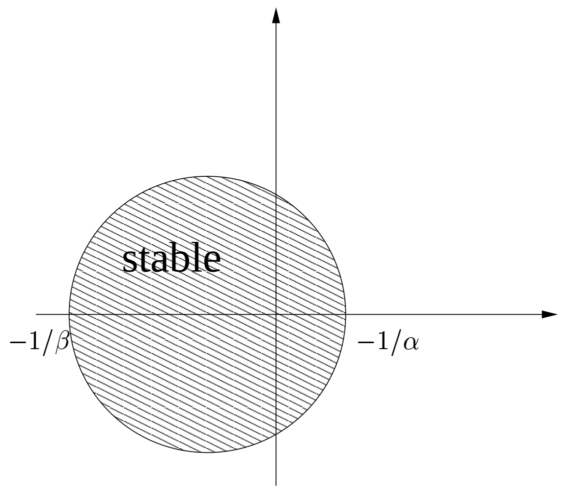

Figure 12: Graphical illustration of the circle criterion. The system is stable if the Nyquist curve of G is within the shaded area.

the classical multiplier theory and relate it to the IQC approach for stability analysis. We limit our discussion to the methodology that was introduced in [43], see also [31] and [4]. The theory is restricted to square systems for reasons that will become apparent. The main tool in the derivation of the results is the passivity theorem.

Theorem 5 (Passivity Theorem). *Assume that the feedback interconnection of G and Δ in (12) is well-posed and that the following conditions hold*

$$\begin{aligned} &\langle u_T, G u_T \rangle \leq -\varepsilon \|u_T\|^2, \\ &\langle u_T, \Delta u_T \rangle \geq 0, \end{aligned}$$

for all $u \in \boldsymbol{L}_{2e}^m[0,\infty)$. The system is then stable.

Proof. The proof is similar to the proof of Theorem 2. See, for example, [4] for a full proof. $\square$

We will next follow the arguments in [43] and [4] that lead to the multiplier theorem. The idea is the following. Assume that we want to study stability of system S_1 in Figure 13. We introduce an invertible multiplier M into the system. This results in the system S_2 in Figure 13. The multiplier is assumed to be a bounded linear operator.

The multiplier M and its inverse are assumed to be bounded but not necessarily causal. The passivity theorem requires causal operators in the feedback interconnection and it can

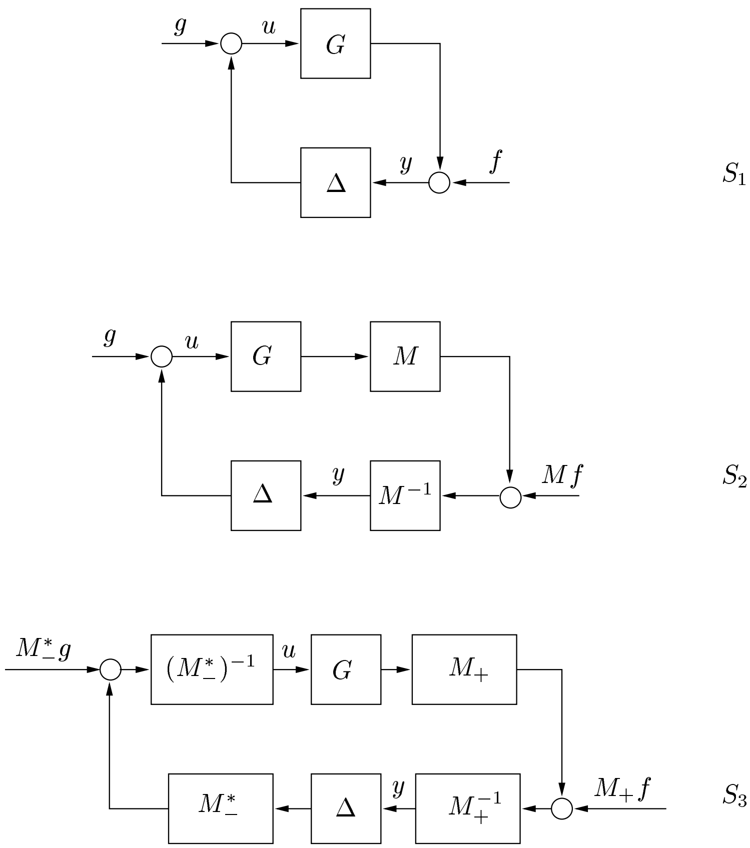

Figure 13: In the classical input–output theory a multiplier M is inserted in the loop resulting in system S_2. The passivity theorem cannot be applied if M or M^{-1} is noncausal. In this case it is required that M can be factored into $M = M_- M_+$, where M_-^*, M_+ and their inverses are causal and bounded. If such a factorization exists, stability of S_1 is equivalent to stability of S_3. The stability conditions can be stated in terms of IQCs involving the multiplier M.

therefore not be applied to system S_2 if M or M^{-1} is noncausal. In this case it is required that there exists a factorization $M = M_- M_+$, where $M_+, M_+^{-1}, M_-^*, (M_-^*)^{-1}$ are bounded and causal. If such a factorization exists we use the following lemma from [43].

Lemma 1. *The following are equivalent:*

(i) *For some $\varepsilon > 0$,*

$$\begin{aligned} &\langle v, MGv\rangle \le -\varepsilon\|v\|^2, \\ &\langle v, M^*\Delta(v)\rangle \ge 0, \end{aligned} \tag{17}$$

for all $v \in \boldsymbol{L}_2^m[0,\infty)$.

(ii) *For some $\varepsilon > 0$,*

$$\begin{aligned} &\left\langle u_T, M_+G(M_-^*)^{-1}u_T\right\rangle \le -\varepsilon\|u_T\|^2, \\ &\left\langle u_T, M_-^*\Delta(M_+^{-1}u_T)\right\rangle \ge 0, \end{aligned} \tag{18}$$

for all $u \in \boldsymbol{L}_{2e}^m[0,\infty)$ and for all $T \ge 0$.

Proof. Let $u \in \mathbf{L}_{2e}^m[0,\infty)$. Then,

$$\begin{aligned} \left\langle u_T, M_+G(M_-^*)^{-1}u_T\right\rangle &= \left\langle M_-^* v, M_+Gv\right\rangle \\ &= \langle v, MGv\rangle \le -\varepsilon\|(M_-^*)^{-1}\|^2\|u_T\|^2. \end{aligned}$$

This follows since $v = (M_-^*)^{-1}u_T \in \mathbf{L}_2^m[0,\infty)$ and from the first condition in (17). In the same way we get

$$\left\langle u_T, M_-^*\Delta(M_+^{-1}u_T)\right\rangle = \left\langle M_+v, M_-^*\Delta(v)\right\rangle = \langle v, M^*\Delta(v)\rangle \ge 0,$$

where $v = M_+^{-1}u_T \in \mathbf{L}_2^m[0,\infty)$. □

Consider now system S_3 in Figure 13. Stability and well-posedness of system S_1 and S_3 are equivalent conditions. This follows since all the multipliers in S_3 are bounded and causal. We arrive at the multiplier theorem below by applying the passivity theorem to system S_3. The conditions in the passivity theorem follow from the assumptions in the theorem statement and from Lemma 1.

Theorem 6 (Multiplier Theorem). *Assume that*

(i) *the feedback interconnection of G and Δ is well-posed,*

(ii) *Δ satisfies the IQC defined by*

$$\Pi(j\omega) = \begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix}, \tag{19}$$

(iii) *M can be factored into $M = M_- M_+$, where M_+, M_-^* and their inverses are causal and bounded,*

(iv) *there exists $\varepsilon > 0$ such that*

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \le -\varepsilon I, \quad \forall \omega \in \boldsymbol{R}.$$

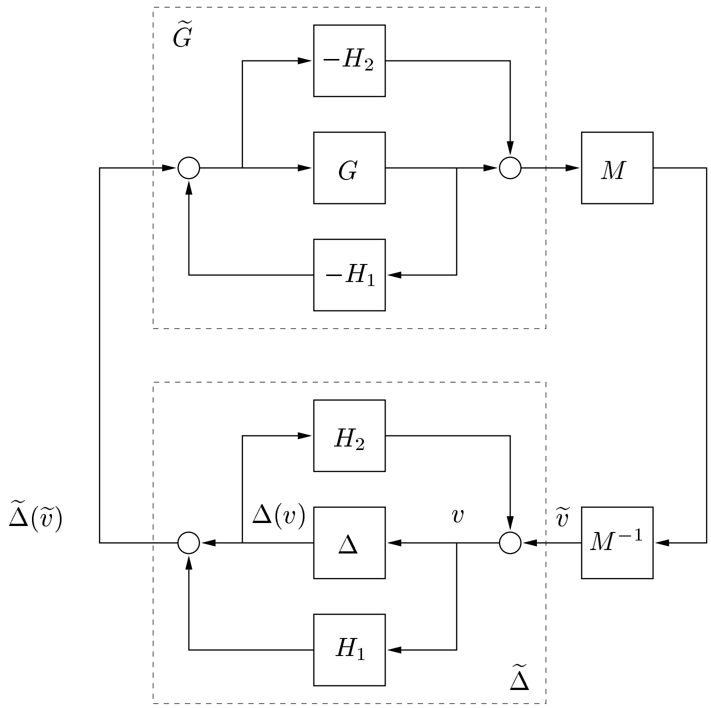

Figure 14: Loop transformations can be used to transform Δ into a new perturbation $\widetilde{\Delta}$ that is suitable for application of the multiplier theorem.

Then the interconnection of G and Δ is stable.

Remark 18. If we compare this result with the corresponding result obtained with Theorem 4 we see that the factorization condition is not needed in the IQC framework. The price paid for this is that well-posedness is required for every feedback interconnection of G and $\tau\Delta$, when $\tau \in [0,1]$. This condition is in most applications weak. In fact, we have seen in Remark 15 that if it holds at $\tau = 1$ then it often holds for all $\tau \in [0,1]$. Note that $\tau\Delta$ satisfies the IQC defined by (19) for every $\tau \in [0,1]$.

It is often necessary to transform the feedback loop in order to obtain a system that is suitable for application of the multiplier theorem. Figure 14 shows such a loop transformation. Here H_1 and H_2 are bounded causal linear operators. We assume that the loop transformation is well-posed in the sense that the operators

$$\widetilde{G} = (G - H_2)(I + H_1 G)^{-1} \quad \text{and} \quad \widetilde{\Delta} = (\Delta + H_1)(I - H_2\Delta)^{-1}$$

are well-defined on $\mathbf{L}_{2e}^m[0,\infty)$. We can formulate the following loop transformation result.

Proposition 5 (Loop Transformation). *Assume that*

(i) *the feedback interconnection of G and Δ is well-posed,*

(ii) *Δ satisfies the IQC defined by*

$$\Pi = \begin{bmatrix} I & -H_2 \\ H_1 & I \end{bmatrix}^* \begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix} \begin{bmatrix} I & -H_2 \\ H_1 & I \end{bmatrix}, \tag{20}$$

where the transformation operator

$$\begin{bmatrix} I & -H_2 \\ H_1 & I \end{bmatrix}$$

and $(I - H_2\Delta)$ *are invertible on* $\boldsymbol{L}_2^m[0,\infty)$,

(iii) M *can be factored into* $M = M_- M_+$, *where* M_+, M_-^* *and their inverses are causal and bounded,*

(iv) *there exists* $\varepsilon > 0$ *such that*

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\varepsilon I, \quad \forall \omega \in \boldsymbol{R}.$$

Then the feedback interconnection of G *and* Δ *is stable.*

Proof. We need to show that $\widetilde{\Delta}$ and $\widetilde{G}$ satisfy condition (ii) and (iv) in Theorem 6. Let us verify condition (ii). We notice that

$$\begin{bmatrix} \widetilde{v} \\ \widetilde{\Delta}(\widetilde{v}) \end{bmatrix} = \begin{bmatrix} I & -H_2 \\ H_1 & I \end{bmatrix} \begin{bmatrix} v \\ \Delta(v) \end{bmatrix},$$

where the notation refers to Figure 14. The assumptions on the transformation operator implies that $\widetilde{\Delta}$ is well-defined. It remains to show that assumption (ii) in the proposition implies (ii) in Theorem 6. This follows since

$$2\left\langle \widetilde{v}, M^* \widetilde{\Delta}(\widetilde{v}) \right\rangle = \left\langle \begin{bmatrix} v \\ \Delta(v) \end{bmatrix}, \Pi \begin{bmatrix} v \\ \Delta(v) \end{bmatrix} \right\rangle \geq 0,$$

for all v and hence for all $\widetilde{v}$ in $\mathbf{L}_2^m[0,\infty)$. Condition (iv) is verified in a similar way. □

The invertibility condition on the transformation operator and the factorization condition on M is not needed for the corresponding result derived in the IQC framework. The proposition also indicates a very fruitful approach to obtain multipliers for the IQC framework. Loop transformations and multipliers from the classical theory can be used to obtain the IQC multiplier in (20). Hence, it is possible to include loop transformations in the IQC multipliers.

10 The S-Procedure Lossless Theorem

The S-procedure is frequently used in system theory to derive stability and performance results for nonlinear and uncertain systems. In fact, the idea has been used in the former Soviet Union since the work of Lure and Postnikov [13]. The idea has since then been developed by many researchers. The most notable early results are due to Yakubovich, who pioneered the use of the S-procedure in systems analysis and optimal control, see, for example, [37, 39] and the references therein. The S-procedure became popular in the robust control community during the 1990s, largely due to a new development by Megretski and Treil [20]. We prove a version of Megretski and Treils result in this section and show how it can be used to prove necessary conditions for stability.

The basic idea behind the S-procedure is simple. Define the quadratic forms $\sigma_k : \mathcal{H} \to \mathbf{R}$ as

$$\sigma_k(f) = \langle \Phi_k f, f \rangle, \quad k = 0, 1, \ldots, N \tag{21}$$

where Φ_k are linear bounded self-adjoint operators on $\mathcal{H}$. Now consider the following two problems

S_1 : $\sigma_0(f) \le 0$ for all $f \in \mathcal{H}$ such that $\sigma_k(f) \ge 0$, $k = 1, \dots, N$.

S_2 : There exists $\tau_k \ge 0$, $k = 1, \dots, N$ such that

$$\sigma_0(f) + \sum_{k=1}^{N} \tau_k \sigma_k(f) \le 0, \quad \forall f \in \mathcal{H}.$$

It is a obvious fact that S_2 implies S_1. The two conditions S_1 and S_2 are in general not equivalent. However, there are some special cases when $S_1 \Leftrightarrow S_2$ and the S-procedure is then called lossless. Yakubovich proved losslessness of the S-procedure in [37] for the following two cases

1. $\mathcal{H} = \mathbf{R}^n$ and $N = 1$.
2. $\mathcal{H} = \mathbf{C}^n$ and $N = 2$.

Megretski and Treils losslessness result holds for the case of any finite number of time-invariant quadratic forms on $\mathbf{L}_2$.

Before stating a number of important lossless results for the S-procedure we supply some remarks and give an application of the S-procedure in the finite dimensional case.

- Note that there generally is a massive computational advantage in using the S-procedure. To understand this we notice that the constraint in S_1 generally is nonconvex. For example, in the case when $\mathcal{H} = \mathbf{R}^n$ we have

  $$\sigma_k(f) = f^T \Phi_k f,$$

  where $\Phi_k = \Phi_k^T \in \mathbf{R}^{n \times n}$ in general may be indefinite. The problem in S_2 is then equivalent to the linear matrix inequality

  $$\Phi_0 + \sum_{k=1}^{N} \tau_k \Phi_k \le 0,$$

  which can be solved efficiently. The situation is similar for the robust control applications we consider.

- We often use the S-procedure in applications where it can be lossy. This will in applications for control system stability mean that we obtain sufficient but not necessary conditions for stability. However, the computational advantage discussed in the previous remark justifies the potential conservatism.

Example 25. We will here derive a necessary and sufficient condition for quadratic stability of the system

$$\begin{aligned} \dot{x} &= Ax + Bw, \quad x(0) = x_0 \\ v &= Cx \end{aligned}$$

where the input and output satisfies the sector constraint

$$\sigma_1(v, w) = (\beta v - w)(w - \alpha v) = \frac{1}{2} \begin{bmatrix} v \\ w \end{bmatrix}^T \begin{bmatrix} -2\beta\alpha & \beta + \alpha \\ \beta + \alpha & -2 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \ge 0,$$

where $\alpha < \beta$ are real numbers. In order to have quadratic stability it is necessary and sufficient that there exists $P = P^T > 0$ such that the Lyapunov function $V(x) = x^T P x$ satisfies

$$x^T P(Ax + Bw) < 0, \ \forall (x, w) \ne 0 \text{ such that } \sigma_1(Cx, w) \ge 0.$$

This is equivalently stated as

$$\sigma_0(x,w) := \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} A^TP+PA & PB \\ B^TP & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0, \ \forall (x,w) \neq 0 \text{ s.t. } \sigma_1(Cx,w) \geq 0.$$

It follows from [37] that the S-procedure is lossless for this case of two quadratic forms (and strict/nonstrict inequality). Hence, the above criterion is equivalent to the existence of $\tau \geq 0$ such that $\sigma_0(x,w) + \tau\sigma_1(Cx,w) < 0$ for all $(x,w) \neq 0$. It is easily seen that we need $\tau > 0$ for this to hold. We can then normalize such that $\tau = 1$ (and $P/\tau \to P$). We have thus shown that quadratic stability of a linear system with sector uncertainty is equivalent to feasibility of the linear matrix inequality: $\exists P = P^T > 0$ such that

$$\begin{bmatrix} A^TP+PA-2\beta\alpha C^TC & PB+(\beta+\alpha)C^T \\ B^TP+C(\beta+\alpha) & -2 \end{bmatrix} < 0.$$

We will next formulate the S-procedure lossless result for the case of time-invariant quadratic forms on a Hilbert space. We state a somewhat more general result than in [20]. To do this we will use the following properties given in [39], where [20] was extended to a more general case.

Assumption 1. Let the quadratic forms $\sigma_k : \mathcal{H} \to \mathcal{H}$ be defined as in (21) and let $S_\tau : \mathcal{H} \to \mathcal{H}$ be the shift operator defined by $(S_\tau f)(t) = f(t-\tau)$. We assume that the Hilbert space, its inner product, and the self-adjoint operators Φ_k are such that the following properties hold

(i) if $f \in \mathcal{H}$ then $S_\tau f \in \mathcal{H}$ for all $\tau \geq 0$

(iia) $\langle \Phi_k S_\tau f_1, f_2 \rangle \to 0$ as $\tau \to \infty$

(iib) $\langle \Phi_k f_1, S_\tau f_2 \rangle \to 0$ as $\tau \to \infty$

(iii) $\sigma_k(S_\tau f) = \sigma_k(f)$ for all $\tau \geq 0$ and all $f \in \mathcal{H}$

Example 26. If $\Phi = \Phi^* \in \mathbf{RL}_\infty^{m\times m}$ and $\mathcal{H} = \mathbf{L}_2[0,\infty)$, and $\sigma(f) = \langle \Phi f, f\rangle$ then all the above properties hold due to the time-invariance of Φ and the standard properties of the $\mathbf{L}_2$ integrals.

Theorem 7 (S-Procedure Lossless Theorem). *Assume the quadratic form satisfies the properties in Assumption 1 and that there exists $f^* \in \mathcal{H}$ such that $\sigma_k(f^*) > 0$ for $k = 1,\dots,N$. Then the S-procedure is lossless, i.e., the following are equivalent*

$S_1:$ *$\sigma_0(f) \leq 0$ for all $f \in \mathcal{H}$ such that $\sigma_k(f) \geq 0$, $k = 1,\dots,N$.*

$S_2:$ *There exists $\tau_k \geq 0$, $k = 1,\dots,N$ such that*

$$\sigma_0(f) + \sum_{k=1}^{N} \tau_k\sigma_k(f) \leq 0, \quad \forall f \in \mathcal{H}.$$

Proof. The direction $S_2 \to S_1$ is obvious so it remains to prove $(S_1 \Rightarrow S_2)$. Define

$$\begin{aligned} \mathcal{K} &= \{(\sigma_0(f), \sigma_1(f), \dots, \sigma_N(f)) : f \in \mathcal{H}\}, \\ \mathcal{N} &= \{(n_0, n_1, \dots, n_N) : n_k > 0, \ k = 0, 1, \dots, N\}. \end{aligned}$$

We will first prove that the closure of $\mathcal{K}$ is convex. Then S_1 implies that $\overline{\mathcal{K}} \cap \mathcal{N} = \emptyset$ and we can use the separating hyperplane theorem to prove that S_2 holds.

Convexity of $\overline{\mathcal{K}}$: Let $f_1, f_2 \in \mathcal{H}$ and define

$$k_1 = ((\sigma_0(f_1), \sigma_1(f_1), \dots, \sigma_N(f_1)) \in \mathcal{K}$$
$$k_2 = ((\sigma_0(f_2), \sigma_1(f_2), \dots, \sigma_N(f_2)) \in \mathcal{K}$$

We have

$$\begin{aligned}\sigma_k(\sqrt{\lambda} f_1 + \sqrt{1-\lambda} S_\tau f_2) = \lambda \sigma_k(f_1) + (1-\lambda)\sigma_k(f_2) + \\ \sqrt{\lambda(1-\lambda)}(\langle \Phi f_1, S_\tau f_2\rangle + \langle \Phi S_\tau f_2, f_1 \rangle) \\ \to \lambda\sigma_k(f_1) + (1-\lambda)\sigma_k(f_2),\end{aligned}$$

as $\tau \to \infty$. Hence

$$(\sigma_0(\sqrt{\lambda} f_1 + \sqrt{1-\lambda} S_\tau f_2), \dots, \sigma_N(\sqrt{\lambda} f_1 + \sqrt{1-\lambda} S_\tau f_2)) \to \lambda k_1 + (1-\lambda) k_2,$$

as $\tau \to \infty$ and it follows that $\lambda k_1 + (1-\lambda)k_2 \in \overline{\mathcal{K}}$. This proves the claim.

The separation argument: The statement in S_1 implies that $\overline{\mathcal{K}} \cap \mathcal{N} = \emptyset$. Hence, since $\overline{\mathcal{K}}$ and $\mathcal{N}$ are convex and $\mathcal{N}$ is open there exists a separating hyperplane. In other words, there exists a nonzero $N+1$-tuple $(c_0, c_1, \dots, c_N)$ such that

$$c_0 n_0 + c_1 n_1 + \ldots + c_N n_N > 0, \quad \forall (n_0, n_1, \dots, n_N) \in \mathcal{N} \tag{22}$$
$$c_0 \kappa_0 + c_1 \kappa_1 + \ldots + c_N \kappa_N \le 0, \quad \forall (\kappa_0, \kappa_1, \dots, \kappa_N) \in \overline{\mathcal{K}} \tag{23}$$

Consider (22). For any given $\varepsilon > 0$, we have $(n_0, \varepsilon, \dots, \varepsilon) \in \mathcal{N}$, for all $n_0 > 0$. This implies that $c_0 \ge 0$. We can in the same way show that $c_k \ge 0$, $k = 1, \dots, N$. Let $\kappa_k = \sigma_k(f^*)$, then by assumption $\kappa_1, \dots, \kappa_N > 0$. Using this in (23) shows that $c_0 > 0$. This shows that S_2 holds with $\tau_k = c_k / c_0$, for $k = 1, \dots, N$. □

The next proposition shows that the condition in the IQC-theorem sometimes also can be necessary and not only sufficient for stability.

Proposition 6. *Consider the system*

$$v = Gw + e$$
$$w = \Delta(v)$$

where $G \in \boldsymbol{RH}_\infty^{m\times m}$, $e \in \boldsymbol{L}_2^m[0,\infty)$, and Δ is any bounded causal operator on $\boldsymbol{L}_2^m[0,\infty)$ such that $\Delta \in \mathrm{IQC}(\Pi_k)$, for $k = 1, \dots, N$. Here the IQCs are defined as usual

$$\sigma_{\Pi_k}(v, w) = \int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{w}(j\omega) \end{bmatrix}^* \Pi_k(j\omega) \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{w}(j\omega) \end{bmatrix} d\omega \ge 0, \ \forall w = \Delta(v), \ v \in \boldsymbol{L}_2[0,\infty).$$

Assume condition (i) and (ii) of Theorem 4 holds and that there exists a pair $(v^*, w^*) \in \boldsymbol{L}_2^{2m}[0,\infty)$ such that $\sigma_{\Pi_k}(v^*, w^*) > 0$, for $k = 1, \dots, N$. Under these conditions a necessary and sufficient condition for stability is that there exist $\tau_k \ge 0$ such that

$$\sum_{k=1}^{N} \tau_k \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi_k(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} d\omega < 0, \qquad \forall \omega \in [0, \infty]. \tag{24}$$

Proof. Sufficiency follows from Theorem 4 and Remark 16. To prove necessity we introduce

$$
\begin{aligned}
\mathcal{H} &= \{(v,w,e) \in \mathbf{L}_2^{3m}[0,\infty) : v = Gw + e\} \\
\sigma_0(v,w,e) &= \|Gw\|^2 + \|w\|^2 - \gamma\|e\|^2. \\
\sigma_k(v,w,e) &= \sigma_{\Pi_k}(v,w)
\end{aligned}
$$

Stability of the system means that

$$\sigma_0(v,w,e) \le 0, \quad \text{for all } (v,w,e) \in \mathcal{H} \text{ such that } \quad \sigma_k(v,w,e) \ge 0$$

This is by the S-procedure lossless theorem equivalent to the existence of $\tau_k \ge 0$ such that

$$\sigma_0(v,w,e) + \sum_{k=1}^{N} \tau_k \sigma_k(v,w,e) \le 0, \;\; (v,w,e) \in \mathcal{H}.$$

On the subspace $(v,w,0) \in \mathbf{L}_2^{3m}[0,\infty) : v = Gw\} \subset \mathcal{H}$ this is equivalent to

$$
\begin{aligned}
&\|Gw\|^2 + \|w\|^2 + \sum_{k=1}^{N} \tau_k \sigma_k(Gw,w,0) \\
&\qquad = \left\langle w, (\sum_{k=1}^{N} \tau_k \begin{bmatrix} G \\ I \end{bmatrix}^* \Pi_k \begin{bmatrix} G \\ I \end{bmatrix} + G^*G + 1)w \right\rangle \le 0, \quad \forall w \in \mathbf{L}_2^m[0,\infty)
\end{aligned}
$$

This is by Proposition 4 equivalent to

$$\sum_{k=1}^{N} \tau_k \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi_k(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \le -(G(j\omega)^* G(j\omega) + I), \;\; \forall \omega \in \mathbf{R}.$$

This proves that (24) is necessary for stability. $\square$

11 Uncertain Systems

We will here discuss how to treat various forms of system uncertainty with IQCs. Both uncertainty in the system model and various disturbance and noise signals will be considered.

System uncertainty System uncertainty can be due to approximations in the modeling of the system, errors during identification, change of parameters and nonlinearities due to wear, change of operating conditions (for example in gain scheduled systems), etc. Next follows a list of uncertainties with a short discussion of their scope of application. A list of IQCs for these uncertainties can be found in, for example, [19, 17] and the toolbox [18].

LTI Dynamic Uncertainty: This type of uncertainty is used to represent unmodeled dynamics or model error from identification. It is represented as a stable transfer function with bounded $\mathbf{H}_\infty$-norm. It is common to normalize such that $\|\Delta\|_{\mathbf{H}_\infty} = \sup_{\omega \in \mathbf{R}} \overline{\sigma}(\Delta(j\omega)) \le 1$ and insert weights $W(s)$ that are used to determine the frequency distribution of the uncertainty, i.e., where it is large and small. One can consider either additive or multiplicative uncertainty, see Figure 15.

Parametric Uncertainty Parametric uncertainty can be used to model uncertain gains or uncertainty in the location of real poles or zeros of the system.

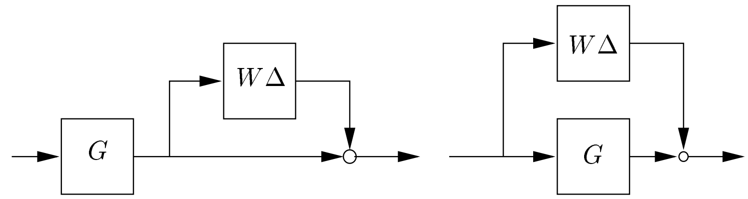

Figure 15: The left block diagram illustrates multiplicative output uncertainty and the right block diagram illustrates additive uncertainty.

General $\mathbf{L}_2$-bounded uncertainty In situations when we do not have much knowledge of the uncertainty then we use the least informative IQC possible

$$\sigma(v,w) = \int_0^\infty (|v(t)|^2 - \gamma|w(t)|^2)dt \geq 0.$$

Hence, the only thing we assume about the uncertainty is causality and a norm bound. This can be used to characterize fast time-varying parameters or time-varying and/or nonlinear operators.

Slowly Time-varying Parameters Slowly time-varying parameters can be used to represent a change in the operating conditions of the system. This can, for example, be used for analysis of some gain-scheduled system.

Memoryless Nonlinearities The IQCs for memoryless nonlinearities in previous sections are valid for a large class of sector bounded nonlinearities. This allows for uncertainty in our knowledge of the true nonlinearity.

Disturbance Signals We can use IQCs to characterize the spectral contents of load disturbances and measurement noise in the system. Early contributions along this line can be found in [17, 23].

Definition 8. A signal set $\mathcal{E} \subset \mathbf{L}_2^q[0,\infty)$ satisfies the IQC defined by $\Psi = \Psi^* \in \mathbf{RL}_\infty^{q\times q}$ ($\mathcal{E} \in \mathrm{IQC}(\Psi)$) if

$$\sigma_\Psi(e) = \int_{-\infty}^{\infty} \widehat{e}(j\omega)^* \Psi(j\omega)\widehat{e}(j\omega)d\omega \geq 0 \tag{25}$$

for all $e \in \mathcal{E}$.

We give two examples.

Dominant Harmonics: Let $e \in \mathbf{L}_2^q[0,\infty)$ be a bandpass signal with $\mathrm{supp}\ \widehat{\mathrm{e}} \in [-\mathrm{b},-\mathrm{a}] \cup [\mathrm{a},\mathrm{b}]$, where $\mathrm{supp}\ \widehat{\mathrm{e}}$ denotes the support of the Fourier transform of e. Then we can use

$$\Psi(j\omega) = \begin{cases} 0, & |\omega| \in [a,b], \\ -\infty I, & \text{otherwise.} \end{cases}$$

in (25). Rational approximations of Ψ can easily be obtained.

Signals with Given Spectral Characteristic: Consider a signal with *spectrum*

$$|\widehat{e}(j\omega)|^2 = \frac{\|e\|^2}{\|H\|_2^2}|H(j\omega)|^2 \tag{26}$$

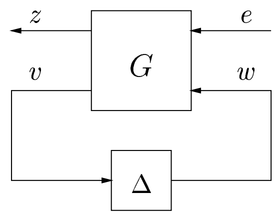

Figure 16: The LFT in (27).

where H is a given transfer function. Such signals can be used to model *filtered deterministic white noise* or the initial conditions response of a linear system. If Ψ satisfies

$$\int_{-\infty}^{\infty} \Psi(j\omega)|H(j\omega)|^2\, d\omega \geq 0$$

then the IQC (25) holds for all signals with spectrum (26). This follows since

$$\int_{-\infty}^{\infty} \Psi(j\omega)|\hat{e}(j\omega)\,|^2 d\omega = \frac{\|e\|^2}{\|H\|_2^2} \int_{-\infty}^{\infty} \Psi(j\omega)|H(j\omega)|^2\, d\omega \geq 0,$$

Linear Fractional Transformations

It is common in robust control to represent an uncertain system with disturbance signals as a Linear Fractional Transformation (LFT). We will see later that this is not crucial for the treatment of robust control systems. However, it is a convenient mathematical notation and it has a crucial role in many robust control papers and toolboxes, see, for example [1]. If the transfer function $G \in \mathbf{RH}_\infty^{(q+m)\times(q+m)}$ has block structure

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

then the (lower) LFT with respect to Δ is defined as

$$\mathcal{F}_l(G, \Delta) = G_{11} + G_{12}\Delta(I - G_{22}\Delta)^{-1}G_{21}. \tag{27}$$

This LFT corresponds to the block diagram in Figure 16. As an example consider the feedback system in Figure 17. The system on LFT form is given in Figure 18 where φ is the saturation nonlinearity and

$$G = \begin{bmatrix} P & P & 1 \\ -KP & -KP & -K \\ P & P & 0 \end{bmatrix}.$$

An IQC for the diagonal operator

$$\begin{bmatrix} \varphi & 0 \\ 0 & \Delta \end{bmatrix}$$

can easily be obtained from IQCs of the two diagonal elements. Indeed, if φ satisfies the IQC defined by Π_1 and Δ satisfies the IQC defined by Π_2, where the matrices has block structure

$$\Pi_i = \begin{bmatrix} \Pi_{i(11)} & \Pi_{i(12)} \\ \Pi_{i(12)}^* & \Pi_{i(22)} \end{bmatrix},$$

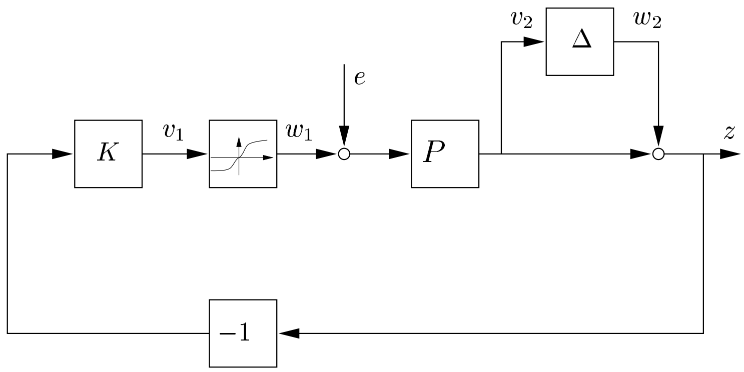

Figure 17: Control system with saturation and uncertainty.

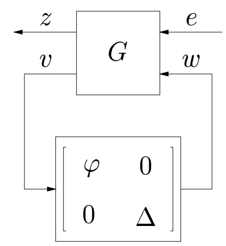

Figure 18: The system in Figure 17 on LFT form.

then the diagonal operator satisfies the IQC defined by

$$\Pi = \left[\begin{array}{cc|cc} \Pi_{1(11)} & & \Pi_{1(12)} & \\ & \Pi_{2(11)} & & \Pi_{2(12)} \\ \hline \Pi_{1(12)}^* & & \Pi_{1(22)} & \\ & \Pi_{2(12)}^* & & \Pi_{2(22)} \end{array}\right].$$

This is easily seen by writing out the expression for the IQC.

Diagonal uncertainty structures are normally called *structured uncertainty* in the robust control literature.

Robust Performance Analysis

Consider now the system

$$\begin{aligned} \begin{bmatrix} z \\ v \end{bmatrix} &= G \begin{bmatrix} e \\ w \end{bmatrix} \\ w &= \Delta(v) \end{aligned} \tag{28}$$

see also Figure 16. Assume $G \in \mathbf{RH}_\infty^{(m+q)\times(m+q)}$. We want to investigate if the closed loop system satisfies various performance objectives. The most common performance measure is the $\mathbf{L}_2$-gain of the system. This corresponds to the IQC

$$\sigma_P(z, e) = \int_0^\infty (|z(t)|^2 - \gamma^2 |e(t)|^2) dt \leq 0.$$

Other examples are the $\mathbf{L}_2 \to \mathbf{L}_\infty$ gain and various weighted sensitivity measures. Robust performance is formally defined as follows.

Definition 9. Assume $e \in \mathcal{E} \subset \mathbf{L}_2^q[0,\infty)$. Then the system in (28) has robust performance with respect to the performance IQC σ_P if

(i) the system is stable

(ii) $\sigma_P(z,e) \le 0$ for all $z = \mathcal{F}_l(G,\Delta)e$, $e \in \mathcal{E}$.

To derive a condition for robust performance assume that we have the noise IQC

$$\sigma_\Psi(e) = \int_{-\infty}^{\infty} \widehat{e}(j\omega)^* \Psi(j\omega)\widehat{e}(j\omega)d\omega \ge 0, \quad e \in \mathcal{E} \tag{29}$$

and the IQC

$$\sigma_\Pi(v,\Delta(v)) = \int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{bmatrix} d\omega \ge 0, \quad \forall v \in \mathbf{L}_2^m[0,\infty), \tag{30}$$

for the uncertainty. We assume that Π has the block structure

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}.$$

We can now prove the following robust $\mathbf{L}_2$-performance result.

Proposition 7. *Assume that $\mathcal{E}$ satisfies (29) and Δ satisfies (30). Then the system (28) has robust $\mathbf{L}_2$-gain γ if*

(i) *it is stable*

(ii) *the frequency domain inequality*

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & \Pi_{11}(j\omega) & 0 & \Pi_{12}(j\omega) \\ \hline 0 & 0 & -\gamma^2 I + \Psi(j\omega) & 0 \\ 0 & \Pi_{12}^*(j\omega) & 0 & \Pi_{22}(j\omega) \end{array}\right] \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \le 0,$$

holds for all $\omega \in [0,\infty]$.

Furthermore, if condition (i) and (ii) in Theorem 4 hold and the frequency domain inequality above holds strictly then the system is also stable.

Proof. The result follows from the trivial direction of the S-procedure. Let

$$\mathcal{H} = \left\{ (z,v,e,w) \in \mathbf{L}_2^{2m+2q}[0,\infty) : \begin{bmatrix} z \\ v \end{bmatrix} = G \begin{bmatrix} e \\ w \end{bmatrix} \right\}.$$

We need

$$\sigma_P(z,e) \le 0, \text{ for all } (z,v,w,e) \in \mathcal{H} \text{ such that } \sigma_\Psi(e) \ge 0,\ \sigma_\Pi(v,w) \ge 0.$$

This is clearly the case if $\sigma(z,v,e,w) := \sigma_P(z,e) + \sigma_\Psi(e) + \sigma_\Pi(v,w) \le 0$ for all $(z,v,w,e) \in \mathcal{H}$. Using that $(z,v) = G(e,w)$ gives the equivalent statement

$$\sigma(z,v,e,w) = \int_{-\infty}^{\infty} \begin{bmatrix} \widehat{e} \\ \widehat{w} \end{bmatrix}^* \begin{bmatrix} G \\ I \end{bmatrix}^* \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & \Pi_{11} & 0 & \Pi_{12} \\ \hline 0 & 0 & -\gamma^2 I + \Psi & 0 \\ 0 & \Pi_{12}^* & 0 & \Pi_{22} \end{array}\right] \begin{bmatrix} G \\ I \end{bmatrix} \begin{bmatrix} \widehat{e} \\ \widehat{w} \end{bmatrix} d\omega \le 0 \tag{31}$$

for all $(e,w) \in \mathbf{L}_2^{m+q}[0,\infty)$. Application of Proposition 4 shows that the frequency domain inequality in (ii) is equivalent to (31). The last claim is easy to verify. $\square$

12 The Kalman Yakubovich Popov Lemma

We will next show that the frequency domain criterion

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0, \infty] \tag{32}$$

is equivalent to a number of conditions on the system matrices in the realization of the transfer functions G and Π. The discrete time case can be treated similarly.

We will first derive an LQ optimal control formulation of (32). Let Π have the realization

$$\Pi = \begin{bmatrix} (j\omega I - A_\pi)^{-1} B_\pi \\ I \end{bmatrix}^* M_\pi \begin{bmatrix} (j\omega I - A_\pi)^{-1} B_\pi \\ I \end{bmatrix}, \tag{33}$$

where $B_\pi = \begin{bmatrix} B_{\pi,v} & B_{\pi,w} \end{bmatrix}$ and A_π is Hurwitz. Using (33) and $G(s) = C_G(sI - A_G)^{-1} B_G + D_G$ (where A_G is Hurwitz) shows that (32) can be formulated as[13]

$$\begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix} > 0 \tag{34}$$

where

$$A = \begin{bmatrix} A_\pi & B_{\pi,v} C_G \\ 0 & A_G \end{bmatrix}, \quad B = \begin{bmatrix} B_{\pi,v} D_G + B_{\pi,w} \\ B_G \end{bmatrix},$$

and

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = - \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & C_G & D_G \\ \hline 0 & 0 & I \end{array}\right]^T M_\pi \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & C_G & D_G \\ \hline 0 & 0 & I \end{array}\right].$$

From Proposition 4 it follows that (34) is equivalent to existence of $\varepsilon > 0$ such that

$$\begin{aligned} \varepsilon \|w\|^2 &\le \int_{-\infty}^{\infty} \begin{bmatrix} (j\omega I - A)^{-1} B \widehat{w}(j\omega) \\ \widehat{w}(j\omega) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1} B \widehat{w}(j\omega) \\ \widehat{w}(j\omega) \end{bmatrix} d\omega \\ &= \int_0^\infty (x^T Q x + 2x^T S w + w^T R w) dt, \end{aligned} \tag{35}$$

for all pairs $(x, w) \in \mathbf{L}_2[0, \infty)$ such that $\dot{x} = Ax + Bw,\ x(0) = 0,\ w \in \mathbf{L}_2^m[0, \infty)$. This is an *LQ optimal control problem.* The Kalman Yakubovich Popov Lemma shows that (34) and the LQ optimal control problem above are equivalent to an LMI condition, a Riccati equation condition, and an eigenvalue condition on the Hamiltonian matrix corresponding to the LQ problem.

[13] Here we used the following rule for system composition: If

$$G_i(s) = C_i(sI - A_i)^{-1} B_i + D_i = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right]$$

for $i = 1, 2$, then

$$G_1 G_2 = \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array}\right].$$

Theorem 8 ("KYP-Lemma"). *Assume the pair of matrices (A, B) is stabilizable and A has no eigenvalues on the imaginary axis*[14]*. Then the following statements are equivalent:*

(i) *there exists $\epsilon > 0$ such that*[15]

$$\int_0^\infty (x^T Q x + 2x^T S w + w^T R w)dt \geq \epsilon \int_0^\infty (|x|^2 + |w|^2)dt,$$

for all pairs $(x, w) \in \boldsymbol{L}_2[0, \infty)$ such that $\dot{x} = Ax + Bw,\ x(0) = 0$.

(ii) *we have*

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} > 0, \quad \forall \omega \in [0, \infty]$$

(iii) *there exists $P = P^T$ such that*

$$\begin{bmatrix} PA + A^T P & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0.$$

(iv) *$R > 0$, and the Riccati equation*

$$Q + PA + A^T P = (PB + S)R^{-1}(B^T P + S^T) \tag{36}$$

has a stabilizing solution $P = P^T$, i.e., $\widehat{A} = A - BR^{-1}(PB + S)^T$ is Hurwitz.

(v) *$R > 0$, and the Hamiltonian matrix*

$$H = \begin{bmatrix} A - BR^{-1}S^T & BR^{-1}B^T \\ Q - SR^{-1}S^T & -A^T + SR^{-1}B^T \end{bmatrix}$$

has no eigenvalues on the imaginary axis.

Proof. See, for example, [30]. □

Optimization of IQCs

Let us consider the feasibility problem: Find $\tau_k \geq 0$ such that

$$\sum_{k=1}^N \tau_k \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi_k(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0, \infty]. \tag{37}$$

It is no loss of generality to assume that

$$\sum_{k=1}^N \tau_k \Pi_k(j\omega) = \begin{bmatrix} (j\omega I - A_\pi)^{-1}B_\pi \\ I \end{bmatrix}^* M_\pi(\tau) \begin{bmatrix} (j\omega I - A_\pi)^{-1}B_\pi \\ I \end{bmatrix},$$

where again $B_\pi = \begin{bmatrix} B_{\pi,v} & B_{\pi,w} \end{bmatrix}$, A_π is Hurwitz, and M_π is linear in the τ_k, i.e.,

$$M_\pi(\tau) = \sum_{k=1}^N \tau_k M_k,$$

[14]The condition that A has no eigenvalues on the imaginary axis can be removed, but then condition (ii) needs to be slightly changed.

[15]This corresponds to (35) since there A was Hurwitz and then we have $\|(sI - A)^{-1}Bw\| \leq c\|w\|$ for some $c > 0$. Hence, we could use $\varepsilon = (c + 1)\epsilon$ in (35)

where each M_k is a real valued symmetric matrix. We can again use the state space realization $G(s) = C_G(sI - A_G)^{-1}B_G + D_G$ to formulate (37) as: Find $\tau_k \geq 0$ such that

$$\sum_{K=1}^{N} \tau_k \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} Q_k & S_k \\ S_k^T & R_k \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} > 0 \tag{38}$$

where the matrices are defined in the same way as before. By the KYP lemma (38) is equivalent to the following feasibility problem for linear matrix inequalities: Find $P = P^T$ and $\tau_k \geq 0$ such that

$$\begin{bmatrix} PA + A^T P & PB \\ B^T P & 0 \end{bmatrix} + \sum_{k=1}^{N} \tau_k \begin{bmatrix} Q_k & S_k \\ S_k^T & R_k \end{bmatrix} > 0.$$

Such problems can be solved using, for example, LMIlab [7].

The Bounded Real Lemma

As a special case of the equivalence $(ii) \Leftrightarrow (iii)$ in Theorem 8 we consider the important bounded real lemma.

Let $G(s) = C(sI - A)^{-1}B + D$, where A is Hurwitz. Then the following are equivalent statements

(i) $\|G\|_{\mathbf{H}_\infty} < 1$,

(ii) $G(j\omega)^* G(j\omega) < I, \ \forall \omega \in [0, \infty]$,

(iii) there exists $P = P^T > 0$ such that

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} C^T C & C^T D \\ D^T C & -(I - D^T D) \end{bmatrix} < 0.$$

To see this we first note that the equivalence between (i) and (ii) follows since $\|G\|_\infty = \sup_{\omega \in [0,\infty]} \sigma_{\max}(G(j\omega))$ and since the condition $\sigma_{\max}(G(j\omega)) < 1$ is equivalent with the condition $G(j\omega)^* G(j\omega) < I$. The equivalence between (ii) and (iii) follows from the KYP Lemma, since

$$\begin{aligned} & G(j\omega)^* G(j\omega) < I \\ \Leftrightarrow \quad & \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0 \\ \Leftrightarrow \quad & \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \underbrace{\begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}}_{\begin{bmatrix} C^T C & C^T D \\ D^T C & -(I - D^T D) \end{bmatrix}} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} < 0. \end{aligned}$$

We finally note that $P > 0$ since A is Hurwitz and since $C^T C \geq 0$. Another important special case, the positive real lemma, will be proven as a homework problem.

13 IQC analysis of Complex Systems

In this section we consider IQC analysis of complex systems, i.e., system of high complexity. The section contains an alternative view of the development of the material in the previous sections. In fact, we show how the ideas in the previous sections can be used as a theoretical foundation for a Matlab toolbox for systems analysis. One such Matlab toolbox is the IQCbeta toolbox, which was developed at LIDS-MIT in 1997. The most current version of the toolbox can be found at http://web.mit.edu/cykao/www/index.html.

The system under consideration can in general be written as, see also the block diagram in Figure 19,

$$\begin{aligned} z &= \sum_{j=1}^{N} G_{0j} w_j + e_0 \\ v_i &= \sum_{j=1}^{N} G_{ij} w_j + e_i \\ w_i &= \Delta_i(v_i) \end{aligned} \tag{39}$$

where the G_{ij} are stable LTI transfer functions, Δ_i are bounded causal operators, and the disturbance signals e_i belong to subsets $\mathcal{E}_i \subset \mathbf{L}_2[0,\infty)$. We assume that we want to find an upper bound on the $\mathbf{L}_2$-gain of the closed loop system, i.e., an as small as possible $\gamma > 0$ such that

$$\int_0^\infty (|z|^2 - \gamma^2 |e|^2) dt \le 0,$$

for all input output pairs of (39). We will show how this can be done in a way that can be implemented in a software package as Matlab.

We next use IQCs to characterize the operators Δ_k and the signals e_k, $k = 0, 1, \ldots, N$. Assume that $\Delta_k \in \mathrm{IQC}(\Pi_k(\lambda_{\pi_k}))$, where $\lambda_{\pi_k} \in \Lambda_{\pi_k}$ is a parameterization of the IQCs. It is assumed that Π_k is linear in λ_{π_k} and that Λ_{π_k} is a convex cone. We further assume that Π_k has the realization

$$\Pi_k(j\omega, \lambda_{\pi_k}) = \begin{bmatrix} (j\omega I - A_{\pi_k})^{-1} B_{\pi_k} \\ I \end{bmatrix}^* M_{\pi_k}(\lambda_{\pi_k}) \begin{bmatrix} (j\omega I - A_{\pi_k})^{-1} B_{\pi_k} \\ I \end{bmatrix}, \tag{40}$$

where A_{π_k} is Hurwitz, $B_{\pi_k} = \begin{bmatrix} B_{\pi_k,v} & B_{\pi_k,w} \end{bmatrix}$, and M_{π_k} is linear in λ_{π_k}. The IQC $\Delta_k \in \mathrm{IQC}(\Pi_k(\lambda_{\pi_k}))$ can now be formulated in state space as

$$\begin{aligned} &\int_0^\infty \mathcal{Q}_{\pi_k}(x_{\pi_k}, v_k, w_k, \lambda_{\pi_k}) dt \ge 0, \ \forall (x_{\pi_k}, v_k, w_k) \in \mathbf{L}_2[0,\infty) \text{ such that} \\ &\begin{cases} \dot{x}_{\pi_k} = A_{\pi_k} x_{\pi_k} + B_{\pi_k,v} v_k + B_{\pi_k,w} w_k, \ x_{\pi_k}(0) = 0, \\ w_k = \Delta_k(v_k) \end{cases} \\ &\text{where} \quad \mathcal{Q}_{\pi_k}(x_{\pi_k}, v_k, w_k, \lambda_{\pi_k}) := \begin{bmatrix} x_{\pi_k} \\ v_k \\ w_k \end{bmatrix}^T M_{\pi_k}(\lambda_{\pi_k}) \begin{bmatrix} x_{\pi_k} \\ v_k \\ w_k \end{bmatrix}. \end{aligned} \tag{41}$$

Similarly, we assume that the disturbance signals satisfies the IQCs $\mathcal{E}_k \in \mathrm{IQC}(\Psi_k(\lambda_{\psi_k}))$ (see Definition 8), where λ_{ψ_k} is a linear parameterization of the IQCs. Again we assume that λ_{ψ_k} belongs to a convex cone Λ_{ψ_k} and that the Ψ_k have state space realizations

$$\Psi_k(j\omega, \lambda_{\psi_k}) = \begin{bmatrix} (j\omega I - A_{\psi_k})^{-1} B_{\psi_k} \\ I \end{bmatrix}^* M_{\psi_k}(\lambda_{\psi_k}) \begin{bmatrix} (j\omega I - A_{\psi_k})^{-1} B_{\psi_k} \\ I \end{bmatrix},$$

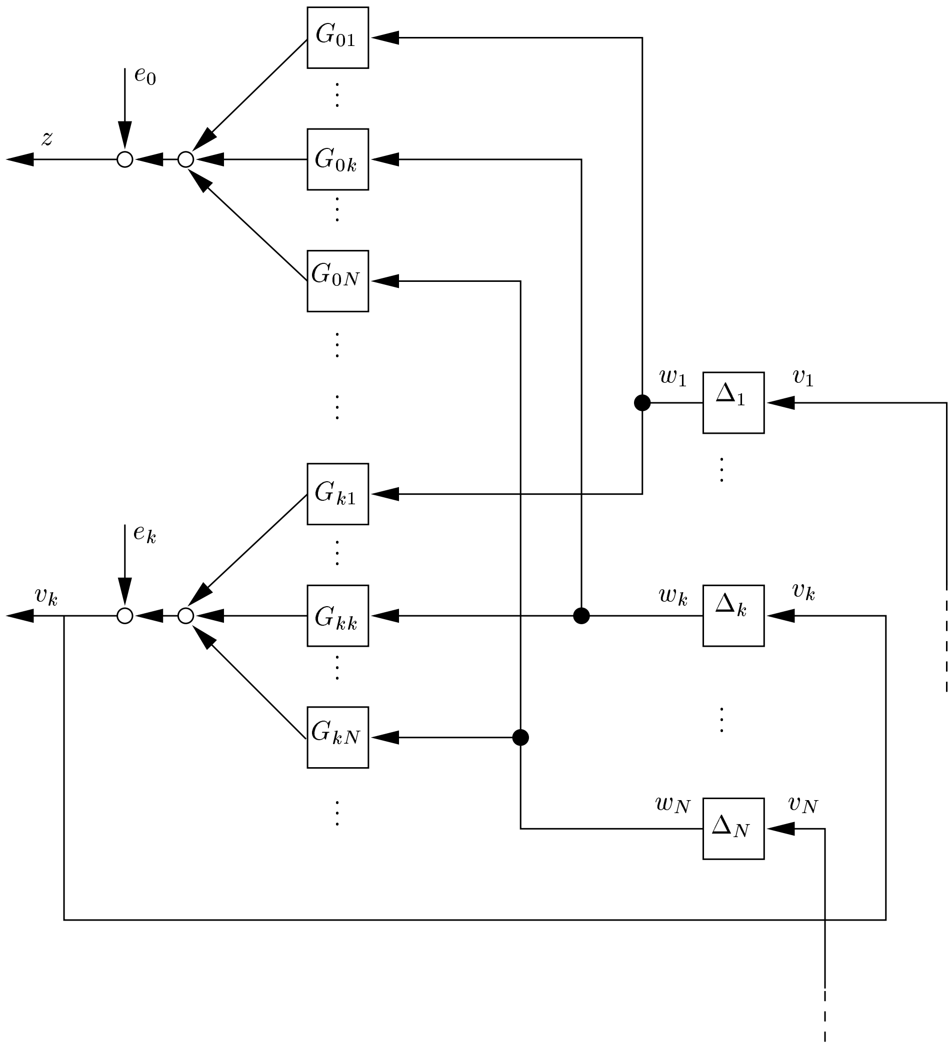

Figure 19: A block diagram of the system in (39).

where A_{ψ_k} is Hurwitz and M_{ψ_k} is affine in λ_{ψ_k}. Then the IQCs $\mathcal{E}_k \in \text{IQC}(\Psi_k(\lambda))$ can equivalently can be formulated as

$$\begin{aligned}
&\int_0^\infty \mathcal{Q}_{\psi_k}(x_{\psi_k}, e_k, \lambda_{\psi_k})dt \geq 0, \text{ for all } (x_{\psi_k}, e_k) \in \mathbf{L}_2[0,\infty) \text{ such that} \\
&\dot{x}_{\psi_k} = A_{\psi_k} x_{\psi_k} + B_{\psi_k} e_k, \; x_{\psi_k}(0) = 0, \; e_k \in \mathcal{E}_k \\
&\text{where} \quad \mathcal{Q}_{\psi_k}(x_{\psi_k}, e_k, \lambda_{\psi_k}) := \begin{bmatrix} x_{\psi_k} \\ e_k \end{bmatrix}^T M_{\psi_k}(\lambda_{\psi_k}) \begin{bmatrix} x_{\psi_k} \\ e_k \end{bmatrix}
\end{aligned} \tag{42}$$

Examples of affine parameterization of IQCs can, for example, be found in the manual for IQCbeta [18].

Let us define the set valued functions[16] $\mathfrak{D}_k : \mathbf{L}_2^{m_k}[0,\infty) \times \Lambda_{\pi_k} \to \mathcal{P}(\mathbf{L}_2^{m_k}[0,\infty))$ defined as $w_k \in \mathfrak{D}_k(v_k, \lambda_{\pi_k})$, where

$$\begin{aligned}
\mathfrak{D}_k(v_k, \lambda_{\pi_k}) = \{w_k \in \mathbf{L}_2^{m_k}[0,\infty) : \int_0^\infty \mathcal{Q}_{\pi_k}(x_{\pi_k}, v_k, w_k, \lambda_{\pi_k})dt \geq 0;& \\
\dot{x}_{\pi_k} = A_{\pi_k} x_{\pi_k} + B_{\pi_k,v} v_k + B_{\pi_k,w} w_k; \; x_{\pi_k}(0) = 0\}.&
\end{aligned}$$

Let us also introduce the sets

$$\begin{aligned}
\mathfrak{E}_k(\lambda_{\psi_k}) = \{e_k \in \mathbf{L}_2^{m_k}[0,\infty) : \int_0^\infty \mathcal{Q}_{\psi_k}(x_{\psi_k}, e_k, \lambda_{\psi_k})dt \geq 0;& \\
\dot{x}_{\psi_k} = A_{\psi_k} x_{\psi_k} + B_{\psi_k} e_k; \; x_{\psi_k}(0) = 0\}.&
\end{aligned}$$

We will initially assume that the closed loop system is stable, which means that all signals in the loop belongs to $\mathbf{L}_2$. The operators Δ_k in (39) can then be replaced by $\mathfrak{D}_k$ and the noise signals e_k can be replaced by arbitrary signals $e_k \in \mathfrak{E}_k$. This follows since

- every $w_k = \Delta_k(v_k)$ also belongs to $\mathfrak{D}_k$ due to the IQC constraint (41)
- every $e_k \in \mathcal{E}_k$ also belongs to $\mathfrak{E}_k$ due to the IQC constraint (42)

This implies that all possible solutions of the original system also are valid solutions of the new system, which is illustrated in Figure 20.

Next we use state space realizations of the G_{ij} to obtain a realization of the linear part of the system on the form

$$\begin{aligned}
\dot{x}_G &= A_G x_G + \sum_{k=1}^N B_{G,k} w_k, \quad x_G(0) = 0 \\
z &= C_0 x_G + \sum_{k=1}^N D_{0,k} w_k + e_0 \\
v_i &= C_i x_G + \sum_{k=1}^N D_{i,k} w_k + e_i, \quad i = 1, \ldots, N
\end{aligned} \tag{43}$$

An upper bound to our robust performance condition can now be obtained as (here $w^T = \left[w_1^T, \ldots, w_N^T\right]^T$, $v^T = \left[v_1^T, \ldots, v_N^T\right]^T$, and finally $e^T = \left[e_1^T, \ldots, e_N^T\right]^T$)

$$\inf \gamma \quad \text{subj to} \quad \begin{cases} \int_0^\infty (|z|^2 - \gamma^2 |e|^2)dt \leq 0, \; \forall (z, v, w, e) \in \mathbf{L}_2 \text{ s.t.} \\ (43), \; w_k \in \mathfrak{D}_k(v_k, \lambda_{\pi_k}), \text{ and } e_k \in \mathfrak{E}_k(\lambda_{\psi_k}) \\ \gamma \geq 0, \; \lambda_{\pi_k} \in \Lambda_{\pi_k}, \; \lambda_{\psi_k} \in \Lambda_{\Psi_k}, \quad \forall k. \end{cases} \tag{44}$$

[16] $\mathcal{P}(\mathbf{L}_2^{m_k}[0,\infty))$ denotes the set of all subsets of $\mathbf{L}_2^{m_k}[0,\infty)$

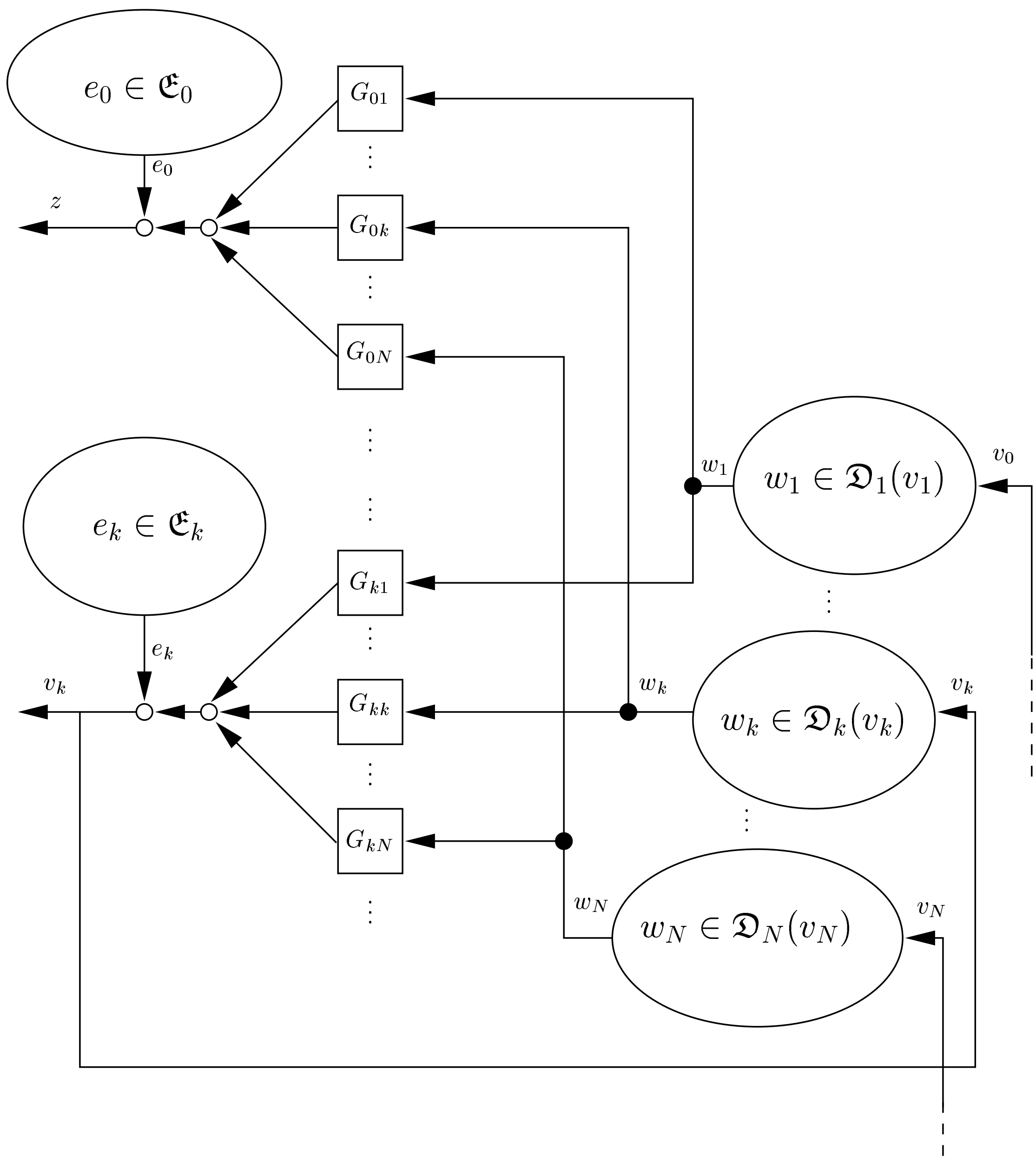

Figure 20: IQC relaxation of the system in (39).

The above optimization problem is generally not convex since the IQC constraints $w_k \in \mathfrak{D}_k(v_k, \lambda_{\pi_k})$, and $e_k \in \mathfrak{E}_k(\lambda_{\psi_k})$ are not convex in general. However, it is possible to use the S-procedure to obtain a convex optimization problem. The following steps will do the job

- Combine the dynamics in (43) with the dynamics in $\mathfrak{D}_k$ and $\mathfrak{E}_k$. The total state space equation for the optimization problem (44) can now be written $\dot{x} = Ax + B_1 w + B_2 e$, $x(0) = 0$, where $x^T = \left[x_G^T, x_{\pi_1}^T, \ldots x_{\pi_N}^T, x_{\psi_0}^T, \ldots, x_{\psi_N}^T\right]^T$, $w^T = \left[w_1^T, \ldots, w_N^T\right]^T$, and finally $e^T = \left[e_0^T, e_1^T, \ldots, e_N^T\right]^T$. The matrix A will be Hurwitz.

- In order to define the IQCs in terms of the complete state space vector we introduce the quadratic forms

$$\widetilde{\mathcal{Q}}_{\pi_k}(x, w, e, \lambda_{\pi_k}) := \mathcal{Q}_{\pi_k}(x_{\pi_k}, v_k, w_k, \lambda_{\pi_k})$$
$$\widetilde{\mathcal{Q}}_{\psi_k}(x, w, e, \lambda_{\psi_k}) := \mathcal{Q}_{\psi_k}(x_{\psi_k}, e_k, \lambda_{\psi_k})$$

  where v_k is defined as a function of x, w, e from the state space equation in (43).

- Define[17] $\mathcal{Q}_p(x, w, e, \gamma) = |z|^2 - \gamma^2 |e|^2$. Then the performance constraint in (44) can equivalently be written

$$\int_0^\infty \mathcal{Q}_p(x, w, e, \gamma) dt \le 0, \ \forall (x, w, e) \in \mathcal{H} \text{ s.t. } \begin{cases} \int_0^\infty \widetilde{\mathcal{Q}}_{\pi_k}(x, w, e, \lambda_{\pi_k}) dt \ge 0 \\ \int_0^\infty \widetilde{\mathcal{Q}}_{\psi_k}(x, w, e, \lambda_{\psi_k}) dt \ge 0 \end{cases}$$

  where $\mathcal{H} = \{(x, w, e) \in \mathbf{L}_2[0, \infty) : \dot{x} = Ax + B_1 w + B_2 e\}$. This is by the S-procedure implied by[18] the condition: There exists $\tau_{\pi_k}, \tau_{\psi_k} \ge 0$ such that

$$\int_0^\infty (\mathcal{Q}_p(x, w, e, \gamma) + \sum_k [\tau_{\pi_k} \widetilde{\mathcal{Q}}_{\pi_k}(x, w, e, \lambda_{\pi_k}) + \tau_{\psi_k} \widetilde{\mathcal{Q}}_{\psi_k}(x, w, e, \lambda_{\psi_k}]) dt \le 0, \quad \forall (x, w, e) \in \mathcal{H}. \quad (45)$$

- Linearity of the quadratic form gives $\tau_{\pi_k} \widetilde{\mathcal{Q}}_{\pi_k}(x, w, e, \lambda_{\pi_k}) = \widetilde{\mathcal{Q}}_{\pi_k}(x, w, e, \tau_{\pi_k} \lambda_{\pi_k})$, but $\tau_{\pi_k} \lambda_{\pi_k} \in \Lambda_{\pi_k}$, since Λ_{π_k} is a convex cone. The same holds for the other quadratic forms. This means that we can remove all the τ from the problem.

- If we replace (45) by its strict counter part then we also have robust stability (this follows as in Proposition 7) given that the two technical conditions (i) and (ii) in Theorem 4 hold.

- Define $\lambda = (\lambda_{\pi_1}, \ldots, \lambda_{\psi_N})$, $\Lambda = \{(\lambda_{\pi_1}, \ldots, \lambda_{\psi_N}) : \lambda_{\pi_k} \in \Lambda_{\pi_k}, \lambda_{\psi_k} \in \Lambda_{\psi_k}\}$ and

$$\mathcal{Q}(x, w, e, \lambda, \gamma) = -\mathcal{Q}_P(x, w, e, \gamma) - \sum_{k=1}^N \widetilde{\mathcal{Q}}_{\pi_k}(x, w, e, \lambda_{\pi_k}) - \sum_{k=0}^N \widetilde{\mathcal{Q}}_{\psi_k}(x, w, e, \lambda_{\psi_k}).$$

[17] We just use that $z = C_0 x_G + e_0 + \sum_{k=1}^N D_{0,k} w_k$ and that x has x_G as its first component

[18] even equivalent if there exists $(x^*, w^*, e^*) \in \mathcal{H}$ such that $\int_0^\infty \widetilde{\mathcal{Q}}_{\pi_k}(x^*, w^*, e^*, \lambda_{\pi_k}) dt \ge \varepsilon(\|x^*\|^2 + \|w^*\|^2 + \|e^*\|^2)$ and $\int_0^\infty \widetilde{\mathcal{Q}}_{\psi_k}(x^*, w^*, e^*, \lambda_{\psi_k}) dt \ge \varepsilon(\|x^*\|^2 + \|w^*\|^2 + \|e^*\|^2)$ for $k = 1, \ldots, N$

Then it follows from the above that the optimization problem

$$
\begin{aligned}
&\inf \gamma \quad \text{subject to} \\
&\begin{cases}
\int_0^\infty \mathcal{Q}(x, w, e, \lambda, \gamma) dt \geq \varepsilon(\|x\|^2 + \|w\|^2 + \|e\|^2) \\
\dot{x} = Ax + B_1 w + B_2 e, \ x(0) = 0 \\
\gamma \geq 0, \ \varepsilon > 0, \ \lambda \in \Lambda
\end{cases}
\end{aligned}
\tag{46}
$$

gives an upper bound on the induced $\mathbf{L}_2$-gain of the system in (39).

- We will have

$$
\mathcal{Q}(x, w, e, \lambda, \gamma) = \begin{bmatrix} x \\ \hline w \\ e \end{bmatrix}^T \begin{bmatrix} Q(\lambda, \gamma) & S(\lambda, \gamma) \\ S(\lambda, \gamma)^T & R(\lambda, \gamma) \end{bmatrix} \begin{bmatrix} x \\ \hline w \\ e \end{bmatrix}
$$

where all matrices Q, S, R are affine in (λ, γ). It is now possible to use Theorem 8 (KYP lemma) to obtain an LMI optimization problem, which is equivalent to (46). It can be formulated as

$$
\begin{aligned}
&\inf \gamma \quad \text{subject to} \\
&\begin{cases}
\exists P = P^T, \gamma \geq 0, \lambda \in \Lambda \text{ such that} \\
\begin{bmatrix} PA + A^T P & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} Q(\lambda, \gamma) & S(\lambda, \gamma) \\ S(\lambda, \gamma)^T & R(\lambda, \gamma) \end{bmatrix} > 0.
\end{cases}
\end{aligned}
\tag{47}
$$

We have now presented the theoretical background behind IQCbeta. More details are given in the manual [18], which can be obtained from:
http://web.mit.edu/cykao/www/index.html. See also the transparencies for next lecture.

14 Applications

Applications of IQC analysis have been reported in the following publication

- Analysis of an antiwindup scheme was considered in [10].
- An selector system was analysed in [9]
- Robust stability analysis of the longitudinal control system of a tail-less aircraft was discussed in [11]

During the course we discussed [9] in detail.

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