1991 Bode Prize Lecture

The Joy of Feedback: Nonlinear and Adaptive

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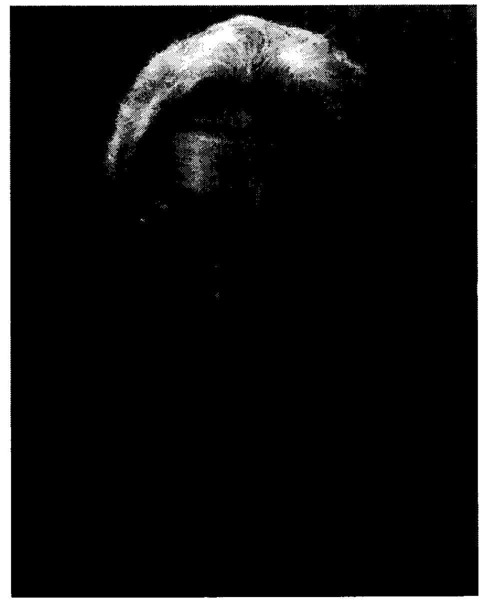

Feedback Everywhere

It is a joy to feel this feedback from so many of you here today. You just heard from Alan Laub, our Society's president, about my quarter of a century in Urbana, Illinois, the birthplace of Hendrik Bode. Indeed, much of what I know about systems, control and feedback I learned from my colleagues and students at the University of Illinois, the conferrer of one of Bode's honorary doctorates. Alan hinted that there may be a parallel between this lecture and the two well known "joy of" books. Well, yes, insofar as they suggest that continuous experimentation with recipes and styles leads to joys which grow with age. My own joy of feedback has been growing for thirty plus years, ever since a Bode formula led me to my first little discovery of sensitivity points. This joy is continuing to grow and I believe that after *age infinity* there will be even more joy of feedback.[1]

Publicize or Perish

In this lecture I will try to tell you why I am so optimistic about the future of feedback. However, before I do this, allow me to echo Roger Brockett's speech last night, about the need to publicize our contributions. Our profession has contributed not only to technology, but also to other scientific disciplines. We don't pay much attention to this fact, because we are all too busy discovering new system properties and design techniques. We rejoice when they lead to safer aircraft, more efficient cars, cheaper CD players, etc., but we seldom make them media successes. Other professions win front pages with press releases that sometimes are applications or modifications of our results. We don't even bother to claim credit for feedback and its many uses!

Feedback is one of the deepest and most inspiring concepts that our profession has contributed to modern civilization. It has permeated, at least in vague forms, many scientific disciplines. There are psychologists who draw feedback diagrams for their counseling sessions and publish a control theory journal. There are "biofeedbacks" and similar phenomena in biological journals. In the current issue of *American Scientist*, the body weight regulation is described as a feedback system. Economists and marketing experts employ feedback. According to a recent report in *Mosaic*, published by NSF, the quantification of *cloud feedback* is keeping in suspense meteorologists from several countries. No less than fourteen models of global warming are competing to reveal whether the cloud feedback is positive or negative.

We cannot afford to approach such "naive" uses of feedback as rigor-morticians, with prefabricated mathematical coffins. The discovery of feedback has its many layers, from "naive" to qualitative, then quantifiable, and finally, to rigorous. While it is not *a priori* clear which of the layers is the richest in content, it is certain that the qualitative style is more likely to cross disciplinary borders. How many authors in our journals dare to be qualitative, let alone "naive"?

Who Controls Chaos?

A good number of you here are trained as mathematicians and, at least in this sense, we can say that feedback control has a strong

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[1]"What happens after *age infinity*," asked Len Shaw in a banquet anecdote at the 1989 CDC. His answer, much simpler than mine, was: "See you in *eltwo*."

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standing in applied mathematics. How about physics? Until recently, physicists seemed uninterested in feedback. Now, during the decade of chaos, the situation is changing. A year or so ago, scientific and popular media informed us that "physicists can control chaos"!

Have some control engineers in this audience understood complex nonlinear dynamics, including the so-called "chaos," to the point of being able to control them? Of course! What they have not done is to inform the media that we know both, how to use chaos for feedback control, and how to use feedback control to suppress chaos. In their 1989 *Automatica* paper, Mareels and Bitmead placed an Australian gumleaf on a chaotic spot in adaptive control. Their chaotic feedback is stabilizing, remains bounded and has a deceptively simple form:

$$u_k = -\frac{1}{u_{k-1}} + \frac{1}{u_{k-2}}.$$

In a 1992 ACC paper, Abed, Wang and Lee, show how to suppress chaotic flow that occurred in an experiment reported by Singer, Wang and Bau in *Phys. Rev. Letters*, 1991. Their nonlinear model exhibiting chaos for $R = 19$ and $u = 0$ is

$$\dot{x}_1 = -px_1 + px_2$$

$$\dot{x}_2 = -x_1x_3 - x_2$$

$$\dot{x}_3 = x_1x_2 - x_3 - R + u.$$

After a bifurcation analysis, Abed and coworkers reduced chaos to a stable limit cycle. They achieved this with a feedback controller consisting only of a linear washout filter and a cubic nonlinearity:

$$\dot{x}_4 = x_3 - cx_4,\ u = -k(x_3 - cx_4)^3.$$

The simplicity of this "chaos extinguisher" and its systematic design are fascinating, but I cannot tell you more about them, because we must finally get to the main topics of this lecture.

Linear Versus Nonlinear

In the first Bode Lecture two years ago, Gunter Stein enriched us with crisp insights into the linear feedback system properties. It seems appropriate for the third Bode Lecture to make a similar attempt with nonlinear feedback, so that, of the three Bode Lectures so far, the two odd ones be about feedback.

Beyond the Worst Case

Can nonlinear feedback interest an audience conditioned to expect that most control problems can be solved by neat linear tools? A long time ago, Richard Bellman used to compare linear designs of nonlinear systems with a man, who, having lost his watch in a dark alley, is searching for it under a lamp post. Today's linear designs are more willing to confront nonlinearities. They include nonlinearities as bounded-norm operators residing in linear sectors. Effects of such nonlinearities are then reduced either with high-gain or worst-case designs. Numerous papers at this conference follow this path to achieve robust stability and performance.

In many situations such a linear design leads to success, and should be a cause for enthusiasm, but not for claims open to misinterpretations. For example, one should qualify the claim that "for unstructured bounded-norm disturbances, nonlinear controllers don't offer advantages over linear controllers." The readers must be warned that this claim is made for an undisclosed class of nonlinear controllers and refers only to their worst-case performance. The performance for less severe and more common disturbances is usually not discussed. Most bounded-norm authors agree that for highly structured or parametric uncertainties a nonlinear controller outperforms the best linear controller. But how many of them admit that this is also true for unstructured bounded-norm uncertainties?

In his very important *Systems & Control* letter of September 1989, Tamer Başar includes a nonlinear controller as a candidate for an optimal design with an unstructured bounded-norm disturbance. He then shows that the worst-case performance attained by this nonlinear controller coincides with the performance attained by the best linear design, but that in an open neighborhood of the worst-case disturbance the nonlinear controller does uniformly better than the linear controller.

An issue, more critical than the worst-case optimality, is that the norm bounds on uncertainties depend on system's operating points and/or initial conditions. George Zames, a pioneer of input-output designs, reiterated at the panel session yesterday, that these designs must be validated *ex post facto* by making sure that the designed system never leaves the linear sectors to which it was confined by the assumed norm bounds. This cautionary note is a good starting point for the technical part of my lecture.

Lecture Outline

In this lecture I will undertake three tasks. First, I will argue that, for a cautious design, a nonlinear analysis is needed to reveal when and why our linear tools fail. Second, I will illustrate a few emerging nonlinear tools with which we can overcome limitations of linear designs. Third, I will try to show that some of these tools can be made adaptive and applied to nonlinear systems with unknown parameters. I will also point to the emergence of robust designs for nonlinear "interval" plants.

Of course, you don't expect me to tackle such ambitious tasks in a systematic and rigorous way. The best I can do is to select a particular nonlinear phenomenon, illustrate it by simple examples and suggest, again through examples, some methods to deal with the effects of the observed phenomenon. The phenomenon I have chosen for this purpose is *peaking*. Among the tools developed to counteract the effects of peaking is *nonlinear damping*. Recursive application of such tools leads to *backstepping procedures*. I will illustrate two new backstepping procedures, one for adaptive nonlinear designs and the other for observer-based nonlinear designs. I will comment on how these procedures are being modified for robust nonadaptive designs. So, the four major sections of the technical part of my lecture are:

- Fear of peaking
- Backstepping from passivity
- Adaptive and robust backstepping
- Observer-based backstepping

Although I will try to mention my sources whenever convenient, my references will remain informal and incomplete. This lecture is neither a survey nor a journal paper. It does not pretend to be representative of all the major developments in nonlinear control, but only of some of my recent joint work with:

Ioannis Kanellakopoulos,
Riccardo Marino,
Steve Morse, and
Hector Sussmann.

Most of the ideas and results are theirs, while all the misinterpretations and prejudices are mine. An incomplete list of other colleagues who have contributed to this lecture includes Eyad Abed, Tamer Başar, Bob Bitmead, Joe Chow, Randy Freeman, Jessy Grizzle, John Hauser, Petros Ioannou, Alberto Isidori, Hassan Khalil, Miroslav Krstić, Rick Middleton, Laurent Praly, Ali Saberi, Shankar Sastry, Peter Sauer, Eduardo Sontag, Mark Spong, Gang Tao, David Taylor and Andy Teel. A stimulating amount of "real life feedback" from Jim Winkelman, Doug Rhode, Davor Hrovat, Bill Powers, and other colleagues at Ford, has also influenced certain attitudes expressed in this lecture.

Fear of Peaking

With all its benefits, feedback is not free of risks and dangers. Some of them, such as the possibility to destabilize neglected high frequency modes, are common in linear systems, while others are specific to unmodeled nonlinearities. We will examine only one of the dangerous nonlinear phenomena, which, although easy to understand, is not well known.

The BB-Syndrome

To help me introduce the *peaking phenomenon*, please perform with me a series of imaginary experiments on a ball and beam (BB) system in one of your undergraduate laboratories.[2] In the notation of Fig. 1, a reasonable model of this system is

$$\text{ball: } \ddot r = -g\sin\theta + r\dot\theta^2$$

$$\text{beam: } \ddot\theta = \tau - \frac{mg\cos\theta}{mr^2+J}r - \frac{2mr\dot r}{mr^2+J}\dot\theta \triangleq u.$$

This model disregards a "jumping ball" so, if you prefer, think of BB as a bead sliding on a bar. Assuming the knowledge of J, m and g, and the exact measurements of r, $\dot r$, θ and $\dot\theta$, we will let the control u be the beam angular acceleration $\ddot\theta$, rather than the motor torque τ. With convenient numerical values the state equations of the BB-system are

$$\text{ball: } \dot x_1 = x_2$$

$$\dot x_2 = -x_3 + (x_3 - \sin x_3) + x_1x_4^2$$

$$\text{beam: } \dot x_3 = x_4$$

$$\dot x_4 = u.$$

[2]When John Hauser, supersonic jet pilot, Shankar Sastry, and I selected the BB-system for our 1989 CDC paper, John told us that he can feel nonlinear aircraft dynamics on this toy system. Perhaps even some of those we saw in Keith Glover's video?

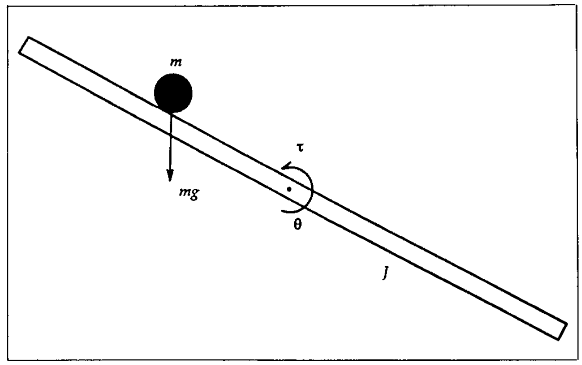

Fig. 1. The ball and beam system.

You can view this system as a chain of four integrators "perturbed" by two nonlinear terms. The perturbation term $x_3 - \sin x_3$ is confined to a linear sector and may seem tractable by bounded-norm designs. However, this certainly is not the case with the centrifugal force $x_1x_4^2$ which makes it impossible to describe the BB-system as perturbed by a bounded-norm operator. This term grows with the square of the beam angular velocity $x_4 = \dot\theta$ and is the cause for what I call *the BB-syndrome*. You will see this syndrome if you notice that the ball can be stabilized only through $\sin x_3$. However, for $|x_1x_4^2| > 1$, our "control" $\sin x_3$ is weaker than the centrifugal force $x_1x_4^2$. Worse yet: the term $x_1x_4^2$ represents a strong positive feedback which, combined with the peaking of x_4, will lead to instability and make the ball fly off the beam.

To see the peaking of x_4, suppose that the BB-system is approximated by the chain of four integrators and that a linear state feedback control is used to place the eigenvalues to the left of $-a < 0$. What is the effect of this linear control on system nonlinearities? Will the dangerous term $x_1x_4^2$ be negligible? To answer questions of this type, Hector Sussmann and I presented a *peaking analysis* in a 1991 *AC-Transactions* paper which, applied to the four-integrator plant with $Re\ \lambda < -a$, proves that, for some initial conditions on the unit sphere, the state x_4 necessarily reaches peak values of order a^3. So, if $Re\ \lambda < -10$, then the peak of x_4 is 1000. I will leave it up to you to imagine the effect of this peaking on the positive feedback term $x_1x_4^2$.

High-Gain Mirage

The phenomena in the BB-system, although easy to visualize, are hard to compute. For a simpler example let me go back to a 1986 *AC-Transactions* note in which Riccardo Marino and I analyze the peaking caused by the linear feedback control $u = -k^2x_1 - kx_2$ in a second order system

$$\dot x_1 = x_2 + w$$

$$\dot x_2 = u + \frac{1}{3}x_2^3 = -k^2x_1 - kx_2 + \frac{1}{3}x_2^3.$$

In an attempt to reduce the effect of a bounded disturbance w on the output $y = x_1$, we increased the gain k. We expected that this increase would also reduce the effect of the nonlinearity $x_2^3/3$. Our hope was that for larger k the linear term kx_2 is more likely to dominate the nonlinear term $x_2^3/3$. But our hope turned out to be a *high-gain mirage*!

In reality, the increase of k led to a decrease of the stability region because of the peaking in x_2. Exact calculations showed that all the solutions with initial conditions such that

$$kx_1^2(0) + \frac{1}{k}x_2^2(0) > 3^2$$

escape to infinity. You can easily sketch this "escape set" and see that its boundary along the x_1-axis is $\pm\ 1/\sqrt{k}$. This will tell you that *the region of local asymptotic stability vanishes as the feedback gain k increases!*

Caveat

My description of the BB-syndrome and the high-gain mirage ends with a message:

Achieving local stability
without a safeguard against
peaking is dangerous.

Theoretically, such a danger does not exist in the case of *global stability*. Since the stability properties of linear systems are always global, most linear designs ignore the danger of peaking.

Every sensible feedback design must guarantee a stability region Ω. For this purpose we sometimes use the concept of semiglobal stabilizability. We call a system *semiglobally stabilizable* to an equilibrium x^e by means of a class F of feedback controls, if for every bounded set Ω of the state space there exists a control in F that makes x^e asymptotically stable, with Ω belonging to its stability region. As my simple examples show, growth rates of nonlinear terms and linear peaking phenomena are among the key factors in determining whether a nonlinear system is semiglobally stabilizable or not.

Peaking in Cascades

Recent developments of the geometric theory of nonlinear control, summarized in Alberto Isidori's superb 1989 book *Nonlinear Control Systems*, allow us to present nonlinear systems in cascade forms like

$$\begin{aligned} \dot{x} &= f_0(x) + y^T f(x, \xi) \\ \dot{\xi} &= A\xi + Bu,\ y = C\xi. \end{aligned} \qquad \text{(CF)}$$

The nonlinear part of this cascade is unobservable from ξ, so that $\dot{x} = f_0(x)$ describes the *zero dynamics* of (CF). When $x = 0$ is a globally asymptotically stable equilibrium of $\dot{x} = f_0(x)$, it would seem that the whole cascade can be globally stabilized with ξ-feedback only, that is with $u = K\xi$. This expectation is based on the fact that the exponential decay of $\|y(t)\| \le ce^{-at}$ can be made as fast as desired by the choice of feedback gain K. It would seem that, with $y(t)$ rapidly decaying, the stability of $\dot{x} = f_0(x)$ will be preserved. To see that, in general, this idea is false, let's examine the system

$$\begin{aligned} \dot{x} &= -x + \xi x^2 \\ \dot{\xi} &= u, \end{aligned}$$

where, like in the BB-syndrome, the term ξx^2 may introduce positive feedback. With the ξ-feedback alone, say $u = -\xi$, we have $\xi(t) = \xi_0 e^{-t}$ and the x-subsystem is

$$\dot{x} = -x + x^2\,\xi_0 e^{-t}.$$

With $x(0) = x_0$ the explicit solution is

$$x(t) = \frac{2x_0}{(2 - x_0\xi_0)e^{t} + x_0\xi_0 e^{-t}}.$$

Now you can see that for $x_0\xi_0 > 2$ the state $x(t)$ tends to infinity in finite time! So, the nonlinearity x^2 is dangerous even when multiplied by $\xi_0 e^{-t}$. We can reduce this danger using $u = -a\xi$ instead of $u = -\xi$. Then the stability region is $x_0\xi_0 < a + 1$. It is semiglobal, because we can make it as large as desired by increasing the gain a, without any peaking in ξ. However, this high-gain idea fails in the following example:

$$\begin{aligned} \dot{x} &= -x + \xi_2 x^2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u. \end{aligned}$$

If in this system you use the ξ-feedback $u = -a^2\xi_1 - 2a\xi_2$ so that $\lambda_1 = \lambda_2 = -a$, you should expect that the danger of x^2 increases as the gain a increases, because $\xi_2(t)$ is peaking with a. In fact, for $x(0) = x_0$, $\xi_1(0) = \xi_0$, $\xi_2(0) = 0$, it is easy to calculate that $x(t)$ escapes whenever $x_0\xi_0 > 2/a$. So, by increasing a to speed-up the decay of $\xi(t)$, we reduce the stability region of x. As in the high-gain mirage, *the stability region vanishes* as $a \to \infty$.

Backstepping from Passivity

After so many examples of dramatic instabilities, you may wonder what happened to our joy of feedback? It will grow as we learn more about nonlinear designs which prevent disasters caused by peaking and achieve global or semiglobal stabilization. Let me start with full state feedback designs.

Passive Designs

To prevent the instabilities in a cascade (CF), we need to investigate which linear-nonlinear connecting terms propagate the effects of the peaking phenomena and how to counteract them. Sussmann and I have initiated such investigations in a 1989 *Systems & Control* letter and continued them with Saberi in a 1990 *SIAM Journal of Control* paper. Our result is that for cascade forms (CF) global stabilization with full state feedback is possible if the linear part of the cascade is weakly minimum phase, with arbitrary relative degree, and if a connection restriction is satisfied. Although only sufficient, these conditions are in a particular sense close to being necessary.

When the linear part of the cascade (CF) is of relative degree one, our design starts by finding a K to satisfy the well known *positive real condition*:

$$(A + BK)^T P + P (A + BK) = -Q \text{ and } PB = C^T$$

for some $P > 0$ and $Q \geq 0$. Then, assuming that $V(x)$ is a Lyapunov function for $\dot{x} = f_0(x)$, our feedback control for (CF) is

$$u = K\xi - \frac{1}{2}\frac{\partial V}{\partial x} f(x,\xi).$$

The global asymptotic stability property of the resulting feedback system is established using the Lyapunov function:

$$W(x) = V(x) + \xi^T P \xi.$$

Let us illustrate this design on the above third-order system in which the escape of x was caused by the peaking in ξ_2. Now a global stabilization of this system is easy. For $y = \xi_2$ we have $C = [0, 1]$ and the condition $PB = C^T$ is satisfied with $P = I$. Then $K = [-1, -1]$ yields $Q \geq 0$, and using $V(x) = x^2$, our globally stabilizing control is

$$u = -\xi_1 - \xi_2 - x^3.$$

The resulting feedback system is

$$\dot{x} = -x + \xi_2 x^2$$

$$\dot{\xi}_1 = \xi_2$$

$$\dot{\xi}_2 = -\xi_1 - \xi_2 - x^3.$$

We say that the nonlinear term $-x^3$ provides *nonlinear damping* which counteracts the peaking and prevents the escape of x.

For connoisseurs of passive charms, Parks, Landau, Anderson, Narendra and others, the idea of our design is *déjà vu*. In a 1990 *Automatica* paper, Ortega extended it to cascades in which the first subsystem is also nonlinear and can be made passive by feedback. A geometric characterization of systems that can be made passive by feedback was given in a 1991 *IEEE Transactions on Automatic Control* paper by Byrnes, Isidori and Willems.

Backstepping

As much as we enjoy the simplicity of passive designs, we must not forget that passivity restricts the system's relative degree not to be higher than one. Fortunately, several versions of a recursive procedure, called *backstepping*, are being developed to remove this relative-degree restriction.

The key idea of backstepping is to start with a system which is stabilizable with a known feedback law for a known Lyapunov function, and then to add to its input an integrator. For the augmented system a new stabilizing feedback law is explicitly designed and shown to be stabilizing for a new Lyapunov function, and so on...

This idea is so simple that most of you have probably used it without paying much attention to it. It is, therefore, surprising that this idea has become an explicit tool for systematic nonlinear design only very recently. At the risk of being unfair to many other authors, let me mention the 1988-1991 works of Tsinias, Sontag, Byrnes and Isidori, and my already quoted papers with Sussmann and Saberi which contain many other references.

The basic form of the backstepping procedure is best explained on an example of a system in "strict feedback form":

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = x_3 + f_2(x_1, x_2)$$

$$\dot{x}_3 = u + f_3(x_1, x_2, x_3).$$

Step 1. Imagine that we can use x_2 to stabilize at 0 the first equation with a feedback law $\alpha_1(x_1)$ so that $\partial V_1/\partial x_1 \, f_1(x_1, \alpha_1(x_1)) < 0$, for all $x_1 \neq 0$, where $V_1(x_1)$ is a known Lyapunov function. Note, however, that we can achieve $x_2 = \alpha_1(x_1)$ only with an error $z_2 = x_2 - \alpha_1(x_1)$. Let us also denote $z_1 = x_1$ so that x_1 and x_2 are known explicit functions of z_1 and z_2 and vice-versa. We now rewrite the first two system equations as

$$\dot{z}_1 = f_1 (z_1, \alpha_1 (z_1)) + z_2 \, \varphi_1 (z_1, z_2)$$

$$\dot{z}_2 = x_3 + f_2 (z_1, z_2 + \alpha_1 (z_1)) - \dot{\alpha}_1$$

where $\varphi_1 (z_1, z_2)$ is known, because f_1 is assumed to be differentiable and is expressed as

$$f_1 (z_1, z_2 + \alpha_1 (z_1)) \equiv f_1 (z_1, \alpha_1 (z_1)) + z_2 \, \varphi_1 (z_1, z_2)$$

Another key observation is that $\dot{\alpha}$ is also known explicitly:

$$\dot{\alpha}_1 = \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 = \frac{\partial \alpha_1}{\partial x_1} f_1(x_1, x_2)$$
$$= \frac{\partial \alpha_1}{\partial z_1} f_1(z_1, z_2 + \alpha_1(z_1)) \triangleq \beta_1(z_1, z_2).$$

Step 2. Imagine now that we can use x_3 to stabilize at 0 the above (x_1, x_2)-system with a feedback law $\alpha_2(z_1, z_2)$. To design α_2 we first construct a Lyapunov function:

$$V_2(z_1, z_2) = V_1(z_1) + 1/2 \, z_2^2.$$

With $x_3 = \alpha_2(z_1, z_2)$ we want to make $\dot{V}_2$ negative:

$$\dot{V}_2 = \frac{\partial V_1}{\partial z_1} f_1(z_1, \alpha_1(z_1))$$
$$+ z_2 \left(\frac{\partial V_1}{\partial z_1} \varphi_1(z_1, z_2) + \alpha_2 + f_2(z_1, z_2 + \alpha_1(z_1, z_2)) \right).$$

Recall that the first term was made negative in Step 1, so we choose α_2 to make the expression multiplying z_2 equal $-z_2$:

$$\alpha_2(z_1, z_2) = -z_2 - \frac{\partial V_1}{\partial z_1} \varphi_1(z_1, z_2)$$
$$- f_2(z_1, z_2 + \alpha_1(z_1)) + \beta_1(z_1, z_2).$$

However, since we cannot achieve $x_3 = \alpha_2(z_1, z_2)$, there is an error $z_3 = x_3 - \alpha_2(z_1, z_2)$ and the actual $\dot{V}_2$ is

$$\dot{V}_2 = \frac{\partial V_1}{\partial z_1} f_1(z_1, \alpha_1(z_1)) - z_2^2 + z_2 z_3.$$

We will take care of $z_2 z_3$ in Step 3. Since we know z_1, z_2 and z_3 as functions of x_1, x_2 and x_3 and vice-versa, our system can be written as

$$\dot{z}_1 = f_1(z_1, \alpha_1(z_1)) + z_2 \varphi_1(z_1, z_2)$$

$$\dot{z}_2 = z_3 + \alpha_2(z_1, z_2) + f_2(z_1, z_2 + \alpha_1(z_1)) - \beta_1(z_1, z_2)$$

$$\dot{z}_3 = u + f_3(z_1, z_2 + \alpha_1(z_1), z_3 + \alpha_2(z_1, z_2)) - \beta_2(z_1, z_2, z_3)$$

where $\beta_2(z_1, z_2, z_3)$ is the known expression for $\dot{\alpha}_2$.

Step 3. At this final step there is no need to imagine a fictitious control, because the actual control u is at our disposal. A feedback law for u is now chosen to make the derivative of $V_3 = V_2 + 1/2\ z_3^2$ negative

$$\dot{V}_3 = \dot{V}_2 + z_3 \dot{z}_3 = \frac{\partial V_1}{\partial z_1} f_1(z_1, \alpha_1(z_1)) - z_2^2 + z_2 z_3 + z_3(u + f_3 - \beta_2).$$

The main achievement of our efforts is not only that the first two terms are negative, but also that the remaining terms have z_3 as a common factor. This is crucial, because now the choice of feedback

$$u = -z_3 - f_3(z_1, z_2 + \alpha_1, z_3 + \alpha_2) + \beta_2(z_1, z_2, z_3) - z_2$$

makes the last two terms in $\dot{V}_3$ equal to $-z_3^2$ and guarantees that $\dot{V}_3 < 0$ for all nonzero z_1, z_2, z_3. So, we have achieved the desired global asymptotic stability property of the equilibrium at 0. The designed feedback law can now be expressed as a function of x_1, x_2, x_3 and is ready for implementation.

This procedure is explicit and its results are global when the system is in the special "strict feedback" form. In the case of a more general "pure feedback" form, the results may not be explicit or global, but a nonvanishing region of stability is still guaranteed.

In the above simple example, Step 1 of backstepping was used to stabilize a scalar equation. In more interesting applications, the backstepping procedure may start with a higher order subsystem for which one of the external state variables can be used as a fictitious stabilizing control. In particular, Step 1 may consist of a passive design, as discussed above. In a 1992 *Transactions on Automatic Control* note, Lozano, Brogliato and Landau give a passivity interpretation of each step of our procedure.

Someone just asked if backstepping is applicable to systems which are not feedback linearizable. Yes! For example, you can use it to globally stabilize the system

$$\dot{x}_1 = x_1 x_2^3$$

$$\dot{x}_2 = u$$

which is not controllable at zero. Start with $\alpha_1 = -x_1^k$, where $k > 1$, say $k = 4/3$ or $k = 2$, so that α_1 is differentiable. Your design will be in two steps. If you use $k = 4/3$ you can compare your solution with the one on page 319 of the 1986 Academic Press book on *Singular Perturbations*, by myself, Khalil and O'Reilly. This will show you that the colorful pedigree of backstepping includes singular perturbations.

Saturating Feedback

Let us backstep to the BB-system and examine if with backstepping we can achieve its semiglobal stabilization. Unfortunately, the critical term $x_1 x_4^2$ appears in the $\dot{x}_2$-equation and the BB-system is not in the form to which one of the existing backstepping procedures can be applied.

The BB-syndrome is one of several benchmark examples—challenges for new nonlinear and adaptive designs. At the Nonlinear Workshop last October in Santa Barbara, Andy Teel responded to the challenge with a *control saturation* design that keeps the dangerous peaking terms within prescribed bounds and thus achieves semiglobal stabilization. In a 1989 ACC paper, Esfandiari and Khalil employed a similar saturation idea to counteract the effects of peaking in high-gain observers.

A benchmark system, to which a backstepping design does not apply, is

$$\dot{x}_1 = x_2 + x_3^2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = u.$$

The main difficulty is the presence of x_3^2 in the first equation, which is not in a "pure feedback form." With $z_1 = x_1 + x_2 + x_3$, Teel brings u into the first equation

$$\dot{z}_1 = x_2 + x_3 + x_3^2 + u$$

and then designs the feedback

$$u = -x_2 - x_3 - \mathrm{SAT}\,(z_1)$$

where SAT is the usual saturation characteristic, linear in an interval centered at 0 and constant outside this interval. The resulting feedback system is

$$\dot{z}_1 = -\,\mathrm{SAT}\,(z_1) + x_3 + x_3^2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -x_2 - x_3 - \mathrm{SAT}\,(z_1).$$

It is clear that the potentially dangerous term $x_3^2$is now bounded, because the linear subsystem is asymptotically stable and its input SAT (z_1) is bounded. A further analysis proves global asymptotic stability. Using a similar approach, Teel shows that the saturating feedback

$$u = -4x_3 - 4x_4 + \mathrm{SAT}(-4x_1 - 12x_2 + 9x_3 + 2x_4)$$

achieves semiglobal stabilization of the BB-system. It is amazing that the stability region includes initial conditions with the beam in a vertical position!

Adaptive and Robust Backstepping

There are many systems with nonlinearities known from physical laws, such as kinematic nonlinearities, or energy, flow and mass balance nonlinearities. Some of these nonlinearities may appear multiplied with unknown parameters and give rise to the problem of controlling nonlinear systems with *parametric uncertainty*. For a broader class of systems, the nonlinearities themselves may be unknown. Such difficult problems may still be tractable if the uncertainties are within some known nonlinear bounds, the so-called *nonlinear interval uncertainties*.

Many exciting results have been obtained in this area in the last three years. Although output feedback results are beginning to appear, I will discuss only a couple of state feedback designs: an *adaptive* design for parametric uncertainty and a *robust* design for interval uncertainty.

Adaptive Backstepping

Adaptive state-feedback control of nonlinear plants has a short but eventful history which involves the names of Taylor, Marino, Kanellakopoulos, Sastry, Isidori, Arapostathis, Nam, Praly, Pomet, Campion, Bastin, Morse, and many others. Several breakthroughs by some of these authors were presented during the 1990 Grainger Lectures, now available as Part Two (more than 200 pages) of *Foundations of Adaptive Control*, published by Springer. One of these breakthroughs was the *adaptive backstepping* procedure, developed by Ioannis Kanellakopoulos. At different stages of this development his coauthors were Marino, Morse and myself. The procedure was simplified by Jiang and Praly, and was brought to its present *tuning function* form by Krstić and Kanellakopoulos.

I will explain adaptive backstepping on a benchmark nonlinear system:

$$\dot{x}_1 = x_2 + \theta x_1^2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = u.$$

Note that the destabilizing term θx_1^2 is now more dangerous than before, because the parameter θ is unknown. We assume, however, that θ is constant. For this system the adaptive controller design is in three steps.

Step 1. We introduce $z_1 = x_1$ and $z_2 = x_2 - \alpha_1$ and consider α_1 as a control to be used to stabilize the z_1-system with respect to the Lyapunov function $V_1 = 1/2\ z_1^2 + 1/2(\hat{\theta} - \theta)^2$. The z_1-system and the corresponding $\dot{V}_1$ are

$$\dot{z}_1 = z_2 + \alpha_1 + \hat{\theta}\lambda(z_1) - (\hat{\theta} - \theta)\lambda(z_1)$$

$$\dot{V}_1 = z_1(z_2 + \alpha_1 + \hat{\theta}\,\lambda(z_1)) + (\hat{\theta} - \theta)\,(\dot{\hat{\theta}} - z_1\lambda(z_1)).$$

The *tuning function* $\tau_1 = z_1\lambda(z_1)$ would eliminate $\hat{\theta} - \theta$ from $\dot{V}_1$ via the update law $\dot{\hat{\theta}} = \tau_1$. Then, if $z_2 = 0$, we would achieve $\dot{V}_1 = -z_1^2$ with $\alpha_1 = -z_1 - \hat{\theta}\lambda(z_1)$.

Instead of using $\dot{\hat{\theta}} = \tau_1$ as an update law, we just substitute $\tau_1(z_1)$ and $\alpha_1(z_1, \hat{\theta})$ into $\dot{V}_1$, and obtain

$$\dot{V}_1 = -z_1^2 + z_1z_2 + (\hat{\theta} - \theta)\,(\dot{\hat{\theta}} - \tau_1).$$

Only the term $-z_1^2$ is negative as desired. In the subsequent steps we must take off the other two terms.

Step 2. Introducing $z_3 = x_3 - \alpha_2$, we consider α_2 as a control to be used to stabilize the (z_1, z_2)-system

$$\dot{z}_1 = -z_1 + z_2 - (\hat{\theta} - \theta)\lambda(z_1)$$
$$\dot{z}_2 = z_3 + \alpha_2 - \frac{\partial\alpha_1}{\partial z_1}\dot{z}_1 - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}$$

For the augmented Lyapunov function $V_2 = V_1 + 1/2\ z_2^2$, let us examine $\dot{V}_2 = \dot{V}_1\ z_2\dot{z}_2$ term by term:

$$\dot{V}_2 = -z_1^2 + z_2\left[z_1 + z_3 + \alpha_2 + \frac{\partial\alpha_1}{\partial z_1}(z_1 - z_2) - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}\right]$$
$$+ (\hat{\theta} - \theta)\left[\dot{\hat{\theta}} - \tau_1 + z_2\frac{\partial\alpha_1}{\partial z_1}\lambda(z_1)\right]$$

We could eliminate $\hat{\theta} - \theta$ from $\dot{V}_2$ using the update law $\dot{\hat{\theta}} = \tau_2$, where

$$\tau_2 = \tau_1 - z_2\frac{\partial\alpha_1}{\partial z_1}\lambda(z_1).$$

Then, to make $\dot{V}_2 = -z_1^2 - z_2^2$ when $z_3 = 0$, we would design α_2 such that the bracketed term multiplying z_2 equals $-z_2$, namely

$$\alpha_2 = -z_2 - z_1 - \frac{\partial\alpha_1}{\partial z_1}(z_1 - z_2) + \frac{\partial\alpha_1}{\partial\hat{\theta}}\tau_2,$$

where τ_2 replaces $\dot{\hat{\theta}}$. However, we do not use $\dot{\hat{\theta}} = \tau_2$ as an update law, but retain $\tau_2(z_1, z_2, \hat{\theta})$ as our second *tuning function.* Substituting α_2 into the expression for $\dot{V}_2$ we obtain

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2z_3 + (\hat{\theta} - \theta)(\dot{\hat{\theta}} - \tau_2) - z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}(\dot{\hat{\theta}} - \tau_2).$$

While $-z_1^2 - z_2^2$ is negative as desired, in Step 3 we must take care of the remaining terms.

Step 3: With $z_1 = x_1$, $z_2 = x_2 - \alpha_1$, $z_3 = x_3 - \alpha_2$, the original system has been transformed into

$$\dot{z}_1 = -z_1 + z_2 - (\hat{\theta} - \theta)\lambda(z_1)$$
$$\dot{z}_2 = -z_2 + z_3 - z_1 + (\hat{\theta} - \theta)\frac{\partial\alpha_1}{\partial z_1}\lambda(z_1) - \frac{\partial\alpha_1}{\partial\hat{\theta}}(\dot{\hat{\theta}} - \tau_2)$$
$$\dot{z}_3 = u - \frac{\partial\alpha_2}{\partial z_1}\dot{z}_1 - \frac{\partial\alpha_2}{\partial z_2}\dot{z}_2 - \frac{\partial\alpha_2}{\partial\hat{\theta}}\dot{\hat{\theta}}.$$

We now design an update law $\dot{\hat{\theta}} = \tau_3$ and a feedback control u to globally stabilize this system with respect to $V_3 = V_2 + 1/2z_3^2$. To this end, we examine $\dot{V}_3 = \dot{V}_2 + z_3\dot{z}_3$ term by term:

$$\dot{V}_3 = -z_1^2 - z_2^2 - z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}(\dot{\hat{\theta}} - \tau_2)$$

$$+ z_3\Bigg[u + z_2 + \frac{\partial\alpha_2}{\partial z_1}(z_1 - z_2) + \frac{\partial\alpha_2}{\partial z_2}(z_2 - z_3 + z_1)$$

$$- \frac{\partial\alpha_2}{\partial z_2}\lambda(z_1)(\dot{\hat{\theta}} - \tau_2) - \frac{\partial\alpha_2}{\partial\hat{\theta}}\dot{\hat{\theta}}\Bigg]$$

$$+ (\hat{\theta} - \theta)(\dot{\hat{\theta}} - \tau_2 + z_3\omega\ (z_1, z_2, \hat{\theta}))$$

where

$$\omega\ (z_1, z_2, \hat{\theta}) = \frac{\partial\alpha_2}{\partial z_1}\lambda(z_1) - \frac{\partial\alpha_2}{\partial z_2}\frac{\partial\alpha_1}{\partial z_1}\lambda(z_1).$$

To eliminate $\hat{\theta} - \theta$ from $\dot{V}_3$, we choose the update law

$$\dot{\hat{\theta}} = \tau_3(z_1, z_2, z_3, \hat{\theta}) = \tau_2(z_1, z_2, \hat{\theta}) - z_3\omega(z_1, z_2, \hat{\theta}).$$

Substituting $\dot{\hat{\theta}} - \tau_2 = -z_3\omega(z_1, z_2, \hat{\theta})$ into $\dot{V}_3$, we obtain

$$\dot{V}_3 = -z_1^2 - z_2^2 + z_3\Bigg[z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}\omega\ (z_1, z_2, \hat{\theta}) + u + z_2 + \frac{\partial\alpha_2}{\partial z_1}(z_1 - z_2)$$

$$+ \frac{\partial\alpha_2}{\partial z_2}\big(z_2 - z_3 + z_1 + \lambda(z_1)z_3\omega\ (z_1, z_2, \hat{\theta})\big) - \frac{\partial\alpha_2}{\partial\hat{\theta}}\tau_3\Bigg]$$

Finally, we choose the control u such that the bracketed term multiplying z_3 becomes $-z_3$, namely

$$u = z_3 - z_2 - z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}\omega\ (z_1, z_2, \hat{\theta}) - \frac{\partial\alpha_2}{\partial z_1}(z_1 - z_2)$$

$$- \frac{\partial\alpha_2}{\partial z_2}(z_2 - z_3 + z_1 + \lambda(z_1)z_3\omega\ (z_1, z_2, \hat{\theta})) + \frac{\partial\alpha_2}{\partial\hat{\theta}}\tau_3.$$

We have thus reached our goal of global stabilization, because

$$\dot{V}_3 = -z_1^2 - z_2^2 - z_3^2,$$

which means that the equilibrium $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $\hat{\theta} = \theta$ of the original system with the update law for $\hat{\theta}$, is globally stable. It is easy to see that we have also achieved the regulation of x, namely $x_1(t) \to 0$, $x_2(t) \to 0$, $x_3(t) \to 0$ as $t \to \infty$.

Robust Backstepping Designs

By now you have seen the backstepping idea applied to two different state feedback designs. In both cases we were able to enlarge our design model step-by-step and recursively calculate stabilizing feedback controls. For nonlinear plants with interval uncertainties several new designs are being developed by Marino and Tomei, Praly and Jiang, Spong, Freeman and Kanellakopoulos, among others. For a glimpse into this new research area, I will use an example from a recent paper of my student Randy Freeman. One aspect of this example illustrates how a backstepping procedure can remove restrictive matching conditions made in the early results of Leitman, Corless, Barmish and others. In the second-order plant

$$\dot{x}_1 = x_2 + \theta x_1^2$$

$$\dot{x}_2 = u,$$

we now assume that the unknown parameter θ belongs to a known interval, say $|\theta| < \bar{\theta}$.

Our goal is to design several static feedback alternatives to adaptive controllers which always include parameter update dynamics. To apply backstepping, we first design a v-controller for

$$\dot{x}_1 = \theta x_1^2 + v$$

with $V_1 = 1/2\ x_1^2$. This first order system satisfies the matching condition and several different designs are possible. Among them are the following two designs:

$$C:\ v = \alpha_c(x_1) = -c_1x_1 - \frac{\bar{\theta}}{2}(ax_1 + \frac{1}{a}x_1^3)$$

$$S:\ v = \alpha_s(x_1) = c_1x_1 - x_1^2\bar{\theta}s_1(x_1),$$

where c_1 and a are positive design parameters and $s_1(x_1)$ is a continuous approximation of a switching function. It is easy to verify that $\dot{V}_1 \leq -c_1x_1^2$ with the first v-controller. For the second v-controller

$$\dot{V}_1 = -c_1x_1^2 - x_1^3(\bar{\theta}s_1(x_1) - \theta),$$

and you can make your own choice of $s_1(x_1)$ to achieve $\dot{V}_1 \leq -c_1/2\ x_1^2$. Thus, $\alpha_c(x_1)$ and $\alpha_s(x_1)$ are both smooth globally stabilizing control laws for the first order subsystem. We now employ backstepping to design u-controllers for the second order system with respect to $V_2 = V_1 + 1/2(x_2 - v)^2$. Note that we again have several possibilities, including C and S, to make $\dot{V}_2$ negative. The corresponding controllers CC, CS, SC and SS will all be globally stabilizing. For example, both CS and SS will be of the form:

$$u = -x_1 - c_2(x_2 - v) + \frac{\partial v}{\partial x_1}(s_2(x_1, x_2)x_1^2 + x_2),$$

and they will differ only in their expressions for v. As before, $s_2(x_1,x_2)$ is a smooth approximation of a switching function. If at either of the two steps we instead employ an adaptive update law, denoted by A, then the set of possible controllers will expand to include SA, AS, CA, etc. These controllers may exhibit different transients and possess different robustness properties.

The message from this simple example is clear: we should not and will not be dogmatic about a single approach to nonlinear robust control. On the contrary, the backstepping methodology has created the possibility of many competitive classes of nonlinear controllers. A fascinating research topic is not only to design such controllers, but also to find performance and robustness criteria to select the winners in their competition. This task is difficult. Fortunately, the difficulty is caused by abundance, rather than scarcity, of ideas for controller designs.

Observer-Based Backstepping

I can see some skeptical faces here in the front rows. Surely you don't doubt the state feedback results I just presented. They are simple enough and I hope that their correctness has been established even through my informal presentation. What you are skeptical about is the assumption that the full state is available for measurement. It hardly ever is. So, if we want some applicable results, we must address the output feedback problem, when only some of the states, constituting our output, are being measured.

For nonlinear systems, the output feedback problem has been a much bigger challenge than for the linear systems. For the purpose of this lecture I will divide this challenge in two parts. The state estimation part of the challenge, even in the noise-free setting, is that it is hard to design nonlinear observers with guaranteed convergence properties. The second part of the challenge is that there are fundamental difficulties even when a convergent nonlinear observer is available. The remainder of my lecture will be devoted not to the progress in the design of nonlinear observers, but to the advances made in observer feedback design when an observer is available.

Observer Induced Peaking

Suppose that you know a full state feedback $u(x)$ that would stabilize your plant. Suppose, moreover, that you have an observer which can give you exponentially convergent estimates $\hat{x}$ of the states x, so that

$$\| u(x(t)) - u(\hat{x}(t)) \| \le ce^{-at}.$$

Let $x = 0$ be the globally asymptotically stable equilibrium of your plant with full state feedback:

$$\dot{x} = f(x) + g(x)u(x) \triangleq f_s(x).$$

Can anything go wrong if instead of $u(x)$ you use the implementable "certainty equivalence" control $u(\hat{x})$? The same plant controlled by $u(\hat{x})$ can be written in the perturbed form as

$$\dot{x} = f_s(x) + g(x)(u(\hat{x}) - u(x)).$$

The equilibrium is still $x = 0$, but what about its stability properties? In contrast to what we know about linear systems, the global stability property of $x = 0$ in the above nonlinear system may be destroyed by the exponentially decaying estimation error term $u(\hat{x}) - u(x)$. This should come as no surprise to those of you who followed my peaking discussion earlier in the lecture.

We are facing the same peaking phenomenon, except that now it is due to the peaking in some of the components of the estimation error $\xi = x - \hat{x}$. Let me illustrate it on an example which is by now familiar to all of you

$$\dot{x}_1 = -x_1 + x_2x_1^2 + u$$

$$\dot{x}_2 = -x_2 + x_1^2,\ y = x_1.$$

Here I assume that only $y = x_1$ is available for feedback. If both x_1 and x_2 were available, then $u = -x_2x_1^2$ would make the equilibrium $x_1 = x_2 = 0$ globally asymptotically stable. Let's investigate what happens when in the same control law we replace x_2 with its estimate $\hat{x}_2$ obtained from the exponentially convergent "observer"

$$\dot{\hat{x}}_2 = -\hat{x}_2 + x_1^2.$$

With $u = -\hat{x}_2x_1^2$ the first equation of the plant becomes

$$\dot{x}_1 = -x_1 + \xi x_1^2 = -x_1 + x_1^2\xi(0)e^{-t}$$

where $\xi(t) = x_2(t) - \hat{x}_2(t) = \xi(0)e^{-t}$. You recognize in it the same equation for which we have established that $x_1(t)$ escapes to infinity in finite time whenever $\xi(0)x_1(0) > 2$!

From what you saw about the peaking phenomena before, you would be able to imagine higher order examples in which some state estimates peak with the observer gain and also multiply some dangerous nonlinearities. You would then see that, if you increase the observer gains for faster convergence, the stability region will shrink, rather than increase. In other words, there are situations in which an exponentially convergent observer causes the loss of not only global, but also semiglobal, stability of the equilibrium $x = 0$.

Nonlinear Damping

In my earlier examples, the effects of peaking were counteracted by specially designed *nonlinear damping* terms. Let's try the same idea again by designing an extra term v added to the "certainty equivalence" control:

$$u = -\hat{x}_2x_1^2 + v.$$

With this control and the same "observer" as above, the relevant equation of the plant and that of the estimation error are

$$\dot{x}_1 = -x_1 + \xi x_1^2 + v$$

$$\dot{\xi} = -\xi.$$

We now want to make the derivative of $V = 1/2\ x_1^2 + 1/2\ \xi^2$ negative:

$$\dot{V} = -x_1^2 - \xi^2 + \xi x_1^3 + x_1 v.$$

To achieve $\dot{V} < 0$, we can let v be a function of x_1, but not of ξ, because $\xi = x_2 - \hat{x}_2$ is not available for feedback. So, to enhance $-x_1^2$ we let $v = -x_1\omega$ and rewrite $\dot{V}$ as

$$\dot{V} = -\begin{bmatrix} x_1 & \frac{1}{2}\xi \end{bmatrix} \begin{bmatrix} 1+\omega & -x_1^2 \\ -x_1^2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \frac{1}{2}\xi \end{bmatrix} - \frac{3}{4}\xi^2.$$

Now, we simply make the 2×2 matrix positive definite by the choice $\omega = x_1^4$ and, hence, our nonlinear damping term is $v = -x_1^5$.

The resulting system with the observer plus nonlinear damping feedback is

$$\dot{x}_1 = -x_1 + x_2x_1^2 - \hat{x}_2x_1^2 - x_1^5$$

$$\dot{x}_2 = -x_2 + x_1^2$$

$$\dot{\hat{x}}_2 = -\hat{x}_2 + x_1^2.$$

The equilibrium $x_1 = x_2 = \hat{x}_2 = 0$ of this system is again globally asymptotically stable, as with the full state feedback.

Observer Backstepping

Can the success with the preceding simple example be extended to other nonlinear systems for which exponentially convergent observers are available? A major breakthrough in this direction is due to Marino and Tomei and their method of filtered transformations, as you can see from their two papers at this conference and the references therein.

Also presented at this conference is an alternative path via *observer backstepping* by Kanellakopoulos et al. Let's follow this path and design an observer-based feedback for the plant

$$\dot{x}_1 = x_2 + \varphi_1(y)$$

$$\dot{x}_2 = x_3 + \varphi_2(y)$$

$$\dot{x}_3 = u, \; y = x_1$$

in the case when y is required to track a given signal y_r with known $\dot{y}_r$ and $\ddot{y}_r$. In the above system the nonlinearities depend only on the output and a Krener-Isidori type observer is available

$$\dot{\hat{x}}_1 = \hat{x}_2 + k_1(\hat{x}_1 - y) + \varphi_1(y)$$

$$\dot{\hat{x}}_2 = \hat{x}_3 + k_2(\hat{x}_1 - y) + \varphi_2(y)$$

$$\dot{\hat{x}}_3 = u + k_3(\hat{x}_1 - y).$$

Note that, after the cancellation of $\varphi_1(y)$ and $\varphi_2(y)$, this results in a linear estimation error system $\dot{\xi} = A\xi$, where $\xi = x - \hat{x}$ and A is made to satisfy $PA + A^TP = -I$ by the choice of k_1, k_2 and k_3. We are now prepared for an observer-based backstepping design. Its idea is to perform backstepping in the observer and, at the same time, to account for the destabilizing effect of the estimation error by designing nonlinear damping terms.

Step 1. For the tracking error $z_1 = y - y_r$ we get

$$\dot{z}_1 = \hat{x}_2 + \xi_2 + \varphi_1(y) - \dot{y}_r.$$

In this equation we let $\hat{x}_2 = z_2 + \alpha_1$ and design $\alpha_1 = -z_1 - \varphi_1(y) + \dot{y}_r$ which is implementable and yields

$$\dot{z}_1 = -z_1 + z_2 + \xi_2$$
$$\dot{z}_2 = \dot{\hat{x}}_2 - \dot{\alpha}_1 = \hat{x}_3 + \beta_2 - \frac{\partial \alpha_1}{\partial y}\xi_2,$$

where the two important terms in the $\dot{z}_2$-equation are given explicitly, while all other terms are incorporated in β_2, which is known and implementable.

Step 2. Now we let $\hat{x}_3 = z_3 + \alpha_2$ and use α_2 to stabilize the $\dot{z}_2$-equation with $V_2 = 1/2\, z_2^2 + \xi^TP\xi$. Since now we must counteract the effect of ξ_2 which multiplies a nonlinear term, we let

$$\alpha_2 = -z_2 - \beta_2 - z_2w_2$$

where z_2w_2 will be a nonlinear damping term. To design w_2 we substitute α_2 into $\dot{V}_2$:

$$\dot{V}_2 = -\begin{bmatrix} z_2 & \frac{1}{2}\xi_2 \end{bmatrix} \begin{bmatrix} 1 + w_2 & \frac{\partial \alpha_1}{\partial y} \\ \frac{\partial \alpha_1}{\partial y} & 1 \end{bmatrix} \begin{bmatrix} z_2 \\ \frac{1}{2}\xi_2 \end{bmatrix} - \xi_1^2 - \frac{3}{4}\xi_2^2 - \xi_3^2 + z_2z_3$$

and make the 2 × 2 matrix positive definite by the choice $w_2 = (\partial\alpha_1/\partial y)^2$. This completes the design of α_2 as

$$\alpha_2 = -z_2 - \beta_2 - z_2\,(\partial\alpha_1/\partial y)^2.$$

Now $\dot{V}_2$ is made of a negative definite part plus the term z_2z_3, which will be absorbed in the final step.

Step 3. Our final task is to design a feedback law for u to globally stabilize the z-system

$$\dot{z}_1 = -z_1 + z_2 + \xi_2$$
$$\dot{z}_2 = -z_2 + z_3 - \frac{\partial \alpha_1}{\partial y}\xi_2 - \left(\frac{\partial \alpha_1}{\partial y}\right)^2 z_2$$
$$\dot{z}_3 = \dot{\hat{x}}_3 - \dot{\alpha}_2 = u + \beta_3 - \frac{\partial \alpha_2}{\partial y}\xi_2$$

where all the terms incorporated in β_3 are known and implementable. Now it should not be difficult to see how to choose u to make the derivative of $V_3 = V_2 + 1/2\, z_3^2$ negative:

$$u = -z_3 - \beta_3 - z_3\left(\frac{\partial \alpha_2}{\partial y}\right)^2.$$

This feedback control adds a nonlinear damping term to counteract the effects of peaking on z_3.

We have thus completed one more backstepping design which uses an existing observer and counteracts its destabilizing effects by nonlinear damping terms. It is fascinating that similar design procedures are being developed for systems with unknown parameters (adaptive) and with interval uncertainties (robust). I hope that this lecture will motivate you to read about these designs in the 1991 CDC papers by Marino and Tomei and Kanellakopoulos *et al.*

More Joy of Feedback

This lecture will have accomplished one of its goals if after it you share not only my fear of peaking but also my joy of being able to overcome it, at least for some classes of nonlinear systems. Of course, these are not the only classes of systems in which we need to counteract peaking, nor is the peaking phenomenon the only danger in nonlinear control. There are many more tough feedback problems ahead, and there will be more joy of inventing methods to solve them. I am using the word *inventing*, rather than *developing*, because I hope that the spirit of invention will continue to grow in our profession. Some simple feedback inventions, like backstepping and saturating controls, may have far-reaching practical consequences and stimulate the development of new theories. These theories are likely to encompass practically important classes of nonlinear systems and increase the impact of our results. We will, of course, widen their applicability by approximations, simplifications, robustifications and other marvelous arts of engineering.

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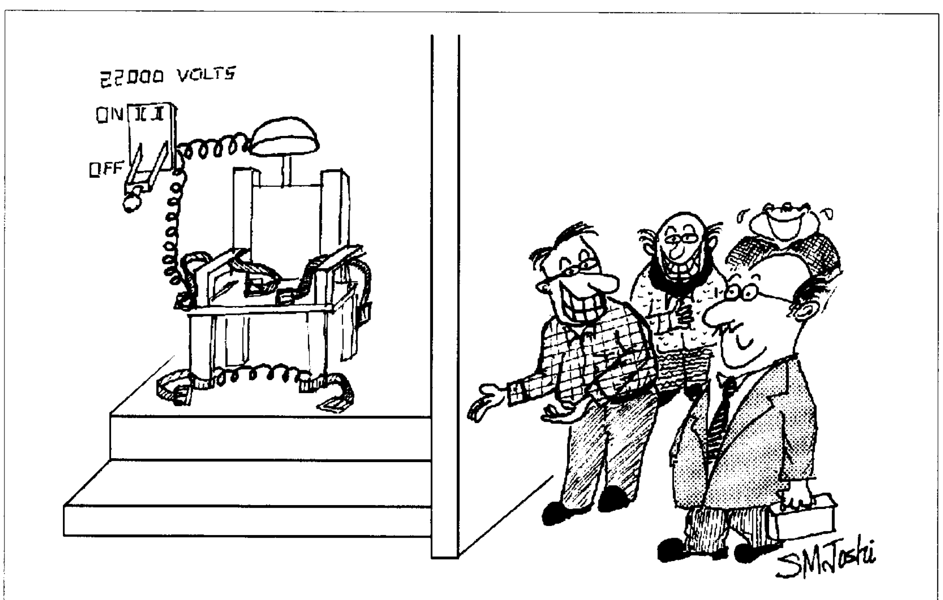
Out of Control



Mistakenly believing that he's been offered the electrical department's chair, Prof. Fenwick proceeds to take on his new job amidst the faculty members' enthusiastic cheers.