Using the Circle Criterion for Dynamic Output Feedback Control

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Reference

The work in this presentation is based on

"Sufficient Conditions for Dynamical Output Feedback Stabilization via the Circle Criterion",
A. Shiriaev[1,3], R. Johansson[2], and A. Robertsson[2].
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Outline

- Introduction and Preliminaries
- Stabilization of Nonlinear Systems via the Circle Criterion:
  - Examples
  - Bilinear Matrix Inequalities (BMI)
  - Solution to BMI via Separation Principle
- Reduced *vs.* "full-order observers"
- Observer Design for Systems Stabilized via the Circle Criterion
- Concluding remarks

Introduction

Output feedback control of nonlinear systems

- Static feedback
- Dynamic feedback
  - Feedback from observers
    - Controller-based observer design
    - Observer-based control design
    - Certainty equivalence designs
  - No direct observer interpretation

The finite escape time phenomenon is a major obstacle for a general separation principle for nonlinear systems.

Recently Praly, Arcak *et. al.* have reported on controller-observer separation for some classes of nonlinear systems where *reduced order observers* but not so-called *"full-order observers"* qualify.

Arcak, M. (2002): "A Global Separation Theorem for a New Class of Nonlinear Observers" In *Proceedings of 41st IEEE Conf on Decision and Control (CDC'03)*. Las Vegas, USA.

Arcak, M. and P. Kokotović (2001): "Nonlinear Observers: A circle criterion design and robustness analysis" In *Automatica*. 37(12)

Example from Arcak (2002)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\left(u - x_2^5\right), \qquad y = x_1$$

Surge subsystem of the Moore-Greitzer model

$$\begin{aligned} \frac{d}{dt}\phi &= -\psi + \frac{3}{2}\phi + \frac{1}{2} - \frac{1}{2}(1+\phi)^3 \\ \frac{d}{dt}\psi &= \frac{1}{\beta^2}\Big(\phi + 1 - u\Big), \qquad y = \psi \end{aligned}$$

A more general model class

$$\begin{aligned} \frac{d}{dt}x &= Ax + B_1 u + B_2 \triangle(d,t) \\ d &= N_2 x, \qquad y = N_1 x \end{aligned}$$

(+ assumptions on $\triangle(d,t)$ ...)

Consider the nonlinear system

$$\frac{d}{dt}x = Ax + Bw, \quad v = Cx, \tag{1}$$
$$w = \phi(v,t) \tag{2}$$

where $x \in R^n$, $w \in R^1$, $v \in R^1$ and ϕ is a scalar real continuous function.

The equation (1) is said to describe a linear part of the system (or simply is a *linear block*), the equation (2) describes a *nonlinear block*.

It is assumed that the nonlinear block satisfies the relations

$$\mu_1 \cdot v \leq \phi(v,t) \leq \mu_2 \cdot v, \quad \forall v, t.$$

THEOREM 1—CIRCLE CRITERION

Consider the system

$$\begin{aligned} \tfrac{d}{dt}x &= Ax + Bw, \quad v = Cx, \\ w &= \phi(v,t) \end{aligned}$$

Suppose that

1) $\mu_1 \cdot v \leq \phi(v,t) \leq \mu_2 \cdot v, \forall\, v, t$;
2) $\det(i\omega I_n - A) \neq 0 \;\forall\, \omega \in R^1$ and $\exists \mu_0 \in [\mu_1, \mu_2]$: $A + \mu_0 \cdot BC$ is stable;
3) The ‘frequency condition’

$$\mathbf{Re}\left[(\mu_2\, C(i\omega I_n - A)^{-1}B - 1)^*(1 - \mu_1\, C(i\omega I_n - A)^{-1}B)\right] < 0,$$

is valid $\forall\, \omega \in R^1$. Then $\exists c > 0$, $\delta > 0$ such that for any solution of the system the following relation holds

$$|x(t)| \leq c \cdot e^{-\delta \cdot t}|x(0)|, \quad \forall\, t \geq 0$$

The main steps

1. To divide the original system into *linear* and *nonlinear* parts

$$\frac{d}{dt}x = Ax + Bw, \quad v = Cx,$$
$$w = \phi(v,t)$$

2. Find quadratic form (may be several) which describes *nonlinear block*, like for a sector nonlinearity

$$\mu_1 \cdot v \leq \phi(v,t) \leq \mu_2 \cdot v \quad \Leftrightarrow \quad \Big(\mu_2 v - w\Big)\Big(w - \mu_1 v\Big) \geq 0.$$

3. Check *'frequency condition'* with such quadratic form to prove that some Lyapunov function candidate has *negative definite time derivative* along the system solutions

4. Check that this Lyapunov function candidate is *positive definite (bounded from below)*

As an example consider the following dynamical system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}(u - x_2^5) \qquad (3)$$
$$y = x_1$$

The problem is to design a controller by output feedback so that the origin of the system (3) becomes globally asymptotically stable.

Hint: If the both variables x_1, x_2 were measured, the static controller can be derived, for example, by exact canceling the nonlinearity x_2^5, i. e.

$$u = k_1 x_1 + k_2 x_2 + x_2^5.$$

But in this case the variable x_2 is not available!

Let us consider a dynamical controller of the form

$$u = \lambda_1 x_1 + \lambda_2 z + (c_1 x_1 + c_3 z)^5, \qquad \frac{d}{dt} z = \lambda_3 x_1 + \lambda_4 z$$

where λ_i, c_j are some constants – parameters of the controller to be defined.

The closed loop system is then

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \lambda_1 & 1 & \lambda_2 \\ \lambda_3 & 0 & \lambda_4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\left((c_1 x_1 + c_3 z)^5 - x_2^5\right)$$

When does the closed loop system satisfy the conditions of the Circle criterion (i.e. asymptotic stability)?

To analyze the closed loop system stability, one needs a key observation:

The nonlinearity

$$w = (c_1 x_1 + c_3 z)^5 - x_2^5$$

and the linear output

$$v = c_1 x_1 - x_2 + c_3 z$$

satisfies the *passivity relation* (the infinite sector constraint)

$$\begin{aligned} v \cdot w &= \Big(c_1 x_1 - x_2 + c_3 z\Big)\left[\Big(c_1 x_1 + c_3 z\Big)^5 - x_2^5\right] \\ &= \Big(c_1 x_1 - x_2 + c_3 z\Big)^2 p(x_1, x_2, z) \geq 0 \quad \forall\, x_1,\, x_2,\, z \end{aligned}$$

As stated in the *Circle Criterion* the closed loop system is exponentially stable provided that

1. the *frequency condition*

$$\text{Re}\left\{ [\, c_1, -1, c_3 \,] \left(j\omega I_3 - \begin{bmatrix} 0 & 1 & 0 \\ \lambda_1 & 1 & \lambda_2 \\ \lambda_3 & 0 & \lambda_4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} < 0$$

   holds $\forall\, \omega \in R_+$;

2. the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ \lambda_1 & 1 & \lambda_2 \\ \lambda_3 & 0 & \lambda_4 \end{bmatrix}$$

   is strictly Hurwitz.

These conditions are equivalent to the fact that:
There exists the 3×3 matrix $P = P^T > 0$ so that

$$\begin{bmatrix} 0 & 1 & 0 \\ \lambda_1 & 1 & \lambda_2 \\ \lambda_3 & 0 & \lambda_4 \end{bmatrix}^T P \;+\; P \begin{bmatrix} 0 & 1 & 0 \\ \lambda_1 & 1 & \lambda_2 \\ \lambda_3 & 0 & \lambda_4 \end{bmatrix} < 0$$

$$P \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \;=\; -\left[c_1, -1, c_3 \right]^T$$

These relations are *Bilinear Matrix Inequalities*.

Frequency condition: Introduce the quantities

$$\begin{aligned} \alpha &= 1 - c_1 \\ \beta &= c_3\lambda_3 - \lambda_2\lambda_3 + c_3\lambda_3\lambda_4 - c_1\lambda_4^2 - c_1\lambda_1 + \lambda_4^2 \\ \gamma &= (\lambda_1\lambda_4 - \lambda_2\lambda_3)(c_3\lambda_3 - c_1\lambda_4) \end{aligned}$$

The two cases when **FC** holds:

1. if the parameters are so that

$$\beta < 0 \quad \Rightarrow \quad \gamma < 0, \quad \alpha < 0$$

2. if the parameters are so that

$$\beta \geq 0 \quad \Rightarrow \quad 4\alpha\gamma - \beta^2 > 0, \quad \alpha < 0$$

The matrix A is Hurwitz provided:

$$\lambda_4 < -1, \; \lambda_1 < \lambda_4, \; \lambda_1\lambda_4 > \lambda_2\lambda_3, \; (-1-\lambda_4)(\lambda_4-\lambda_1) > (\lambda_4\lambda_1-\lambda_3\lambda_2)$$

Important Remark: The last inequalities have at least one solution! For instance:

$$(\lambda_1, \; \lambda_2, \; \lambda_3, \; \lambda_4, \; c_1, \; c_3) = (-5, \; -2, \; -7, \; -3, \; 2, \; 1)$$

Furthermore, it can be shown that the set of solutions is unbounded in the space of parameters R^6!

In other words, if one is interested in optimization of some performance (not just stabilization), it is possible over a rich set of exponentially stabilizing controllers.

Consider the case when this performance is *robustness* of the closed loop system with respect to *parametric uncertainty* in the system.

Suppose there is an uncertain constant ε in front of the nonlinearity

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\left(u - \varepsilon \cdot x_2^5\right) \\ y &= x_1 \end{aligned}$$

Suppose that the nominal value for ε is chosen as $\varepsilon_0 = 1$

and the dynamical stabilizing controller has been designed as discussed above for this nominal value.

How to find the interval of uncertainty $[\varepsilon_{min},\ \varepsilon_{max}]$ so that the closed loop system with nominal controller remains stable? How to enlarge this interval?

LEMMA 1

Consider the nonlinear system

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\left(u - \varepsilon \cdot x_2^5\right) \\ y &= x_1 \end{aligned}$$

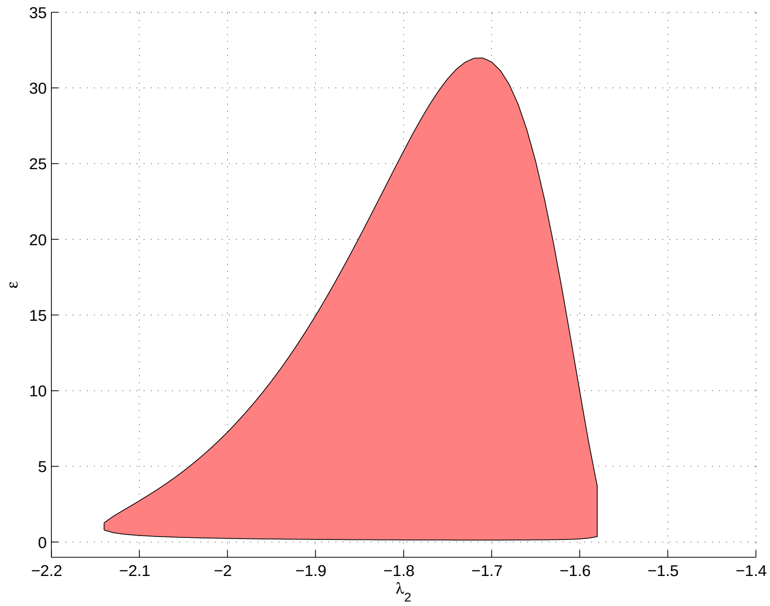
with the dynamical controller

$$u = -5x_1 - 2z + (2x_1 + z)^5, \qquad \frac{d}{dt}z = -7x_1 - 3z$$

that was designed to stabilize the system with nominal value $\varepsilon = \varepsilon_0 = 1$. The closed loop system remains asymptotically stable if the true value of the constant parameter ε belongs to the range

$$\varepsilon \in \left[0.243,\ 7.26\right]$$

□



The light red area corresponds to values of ε as a function of the parameter λ_2 for which the asymptotic stability of the closed system preserved. The other parameters have the nominal values . The largest uncertainty interval for ε is attained for $\lambda_2 = -1.71$, and is $\left[0.132, 31.97\right]$.

The derivation of the stabilizing controller is based on the *Separation Principle* and the validity of additional *structural assumptions*!

Step 1 Derive static feedback controller, that is, we allow to use x_1 and x_2 in feedback, that renders the closed loop system to satisfy the *Circle criterion*.

Step 2 Interpret some linear combination of z and x_1 as estimate for unmeasured variable x_2, and derive the error dynamics to satisfy the *Circle criterion*.

Step 3 Check structural assumption, that allows to verify that the combined system satisfies the *Circle criterion*.

Consider a nonlinear control system of the form

$$\begin{aligned} \frac{d}{dt}x &= Ax + B_1 u + B_2 \triangle(d,t) \\ y &= N_1 x \\ d &= N_2 x \end{aligned}$$

where

- y is a vector of measurements;
- d is a vector that is not available;
- $\triangle(d,t)$ is a scalar nonlinearity;
- u is a control variable.

ASSUMPTION 1

A feedback controller

$$u = K_1 y + K_2 d + K_\triangle \triangle(d,t)$$

with some matrices K_1, K_2, $K_\triangle$ renders the closed loop system to satisfy conditions of the *Circle Criterion* with Quadratic Constraint

$$\int_0^{t_n} \begin{bmatrix} Md(\tau) \\ \triangle(d(\tau),\tau) \end{bmatrix}^T \begin{bmatrix} \Pi_{11} & \Pi_{12}^* \\ \Pi_{12} & 0 \end{bmatrix} \begin{bmatrix} Md(\tau) \\ \triangle(d(\tau),\tau) \end{bmatrix} d\tau \geq 0, \quad \forall\, n$$

Here M, Π_{11}, Π_{12} are constant matrices of appropriate dimensions.

Namely, the matrices K_1, K_2, $K_\triangle$ are such that

1. The *'frequency condition'*

$$\begin{bmatrix} MN_2A_{j\omega}^{-1}(B_1K_\triangle + B_2) \\ 1 \end{bmatrix}^* \begin{bmatrix} \Pi_{11} & \Pi_{12}^* \\ \Pi_{12} & 0 \end{bmatrix} \begin{bmatrix} MN_2A_{j\omega}^{-1}(B_1K_\triangle + B_2) \\ 1 \end{bmatrix} < 0$$

   holds $\forall\, \omega \in R_+$. Here

$$A_{j\omega}^{-1} = (j\omega I_n - \{A + B_1(K_1N_1 + K_2N_2)\})^{-1};$$

2. the matrix $(A + B_1(K_1N_1 + K_2N_2))$ is strictly Hurwitz. ■

Introduce new coordinates as

$$\begin{bmatrix} r \\ y \\ d \end{bmatrix} = Nx = \begin{bmatrix} N_0 \\ N_1 \\ N_2 \end{bmatrix} x$$

In new coordinates the system is

$$\begin{aligned} \dot r &= A_{11}r + A_{12}y + A_{13}d + B_{11}u + B_{12}\triangle(d,t) \\ \dot y &= A_{21}r + A_{22}y + A_{23}d + B_{21}u + B_{22}\triangle(d,t) \\ \dot d &= A_{31}r + A_{32}y + A_{33}d + B_{31}u + B_{32}\triangle(d,t) \end{aligned}$$

ASSUMPTION 2
B_{12}, B_{22} are zeros matrices of appropriate dimensions. ■

ASSUMPTION 3

There exists a matrix Φ such that

$$A_{31} = \Phi A_{21}$$

and such that the system

$$\frac{d}{dt} e = (A_{33} - \Phi A_{23})\, e + B_{32}\left(\triangle(d,t) - \triangle(d-e,t)\right)$$

satisfies conditions of the *Circle Criterion* applied to the constraint

$$\int_0^{t_n} \Big(d_1(\tau) - d_2(\tau)\Big)^T \Pi_\triangle \Big(\triangle(d_1(\tau),\tau) - \triangle(d_2(\tau),\tau)\Big)\, d\tau \geq 0$$

Here $\Pi_\triangle$ is a constant matrix of appropriate dimension.

□

Namely

1. The *frequency condition*

$$\text{Re}\left\{\Pi_\triangle \left(j\omega I_k - (A_{33} - \Phi A_{23})\right)^{-1} B_{32}\right\} < 0$$

   holds $\forall\, \omega \in R_+$;
2. the matrix $(A_{33} - \Phi A_{23})$ is strictly Hurwitz. ■

THEOREM 2

Suppose Assumptions [1–3] hold, then there exists a controller of the form

$$\begin{aligned} u &= R_z z + R_y y + R_\triangle \triangle(C_y y + C_z z, t) \\ \frac{d}{dt} z &= \Lambda_z z + \Lambda_y y + \Lambda_u u + \Lambda_\triangle \triangle(C_y y + C_z z, t) \end{aligned}$$

so that the closed loop system satisfies the conditions of the Circle criterion. One of the solutions looks as

$$\begin{aligned} \Lambda_z &= A_{33} - \Phi A_{23} & \Lambda_y &= (A_{32} - \Phi A_{22}) + (A_{33} - \Phi A_{23})\Phi \\ \Lambda_u &= B_{31} - \Phi B_{21} & \Lambda_\triangle &= B_{32} \\ R_z &= K_2 & R_y &= K_1 + K_2\Phi \\ R_\triangle &= K_\triangle & & \\ C_z &= I_k & C_y &= \Phi \end{aligned}$$

□

Conclusions (Part I)

- There is a gap between the stability theory and the methods used in stabilization of nonlinear systems. Even in the example of the well accepted and well known stability test as the *Circle Criterion* the way to write stabilizing controller might not be clear;
- The difficulty comes from the necessity to solve the *Bilinear Matrix Inequalities* that computationally might be intractable;
- The solution then could come (it is proven) from the decomposition of the system, and checking the validity of the *Separation principle*;
- This enables us to prove the solvability the *Bilinear Matrix Inequalities* and find the rich set of stabilizing controllers.

Consider a nonlinear systems of the form

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu \\ y &= Cx \\ z &= Lx \\ u &= \psi(z,t) \end{aligned}$$

with the nonlinearity satisfying the sector condition

$$\left(\kappa_2 z - u\right)^T \left(u - \kappa_1 z\right) \geq 0,$$

where κ_1 and κ_2 are constant matrices.

Suppose that the system satisfies *Circle Criterion* based on this quadratic constraint.

How to redesign the controller $u = \psi(z,t)$ if the signal z is not available?

In case when y is only measured, one can try to reconstruct the variable z by the standard Luenberger observer

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + K\left(y - C\hat{x}\right), \quad \hat{z} = L\hat{x}$$

and design the feedback control based on the estimate

$$u = \psi\left(\hat{x}, t\right),$$

Questions:

- **For which K is the closed loop system stable?**
- **If it is stable for some K, how does the Lyapunov function look like for the closed loop system? Is it quadratic?**

THEOREM 3

The closed loop system

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu \\ y &= Cx \\ \frac{d}{dt}\hat{x} &= A\hat{x} + Bu + K\left(y - C\hat{x}\right) \\ \hat{z} &= L\hat{x} \\ u &= \psi(\hat{z}, t) \end{aligned}$$

is asymptotically stable for any observer gain K that makes the matrix $(A - KC)$ strictly Hurwitz. Furthermore, for any such K the Lyapunov function of the closed loop system is quadratic. ■

Proof follows from the fact that the closed loop system augmented by the observer satisfies to the *Circle Criterion* itself.

Step 1 The closed loop system satisfies to the new quadratic constraint

$$(\kappa_2\hat{z} - u)^T(u - \kappa_1\hat{z}) \geq 0$$

Furthermore the *'frequency condition'* with such quadratic constraint holds due to the property of any Luenberger observer:

the transfer function from the input u to $\hat{x}$ is the same as to x!

Step 2 The minimal stability property follows from the fact that

the separation principle holds for linear systems!

Important Remark The Lyapunov functions for original system and for the system augmented by the observer

$$V(x) = x^T P x, \quad W(x, \hat{x}) = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}^T Q \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

have no direct (straightforward) connections!

The previous approaches were based on the description of those observer gains K so that the matrix Q was constructed from P!

The simplest example is the back-stepping with a SPR linear subsystem!

Consider the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix} u, \qquad y = Cx = x_1$$

with $\alpha > 0$. Consider the controller as

$$u = \gamma\, x_2 \cdot \sin(t)$$

and denote

$$w = \sin(t) \cdot u = \sin^2(t) \cdot \gamma\, x_2,$$

then the 'nonlinearity' w satisfies the constraint: for any solution, except $x_1 = x_2 = 0$, of the closed loop system

$$\exists \{t_k\} : \int_0^{t_k} \left\{ w(\tau) \cdot x_2(\tau) - w^2(\tau) \right\} d\tau > 0$$

with $\gamma \in (0, 1)$.

If for example $\alpha = 3$, this constraint leads to the validity of conditions of the *Circle Criterion*.

The Lyapunov function is of simple form (quadratic), described by the diagonal matrix

$$P = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Consider the observer

$$\frac{d}{dt}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix} u + K\,(x_1 - \hat{x}_1)$$

and the feedback controller

$$u = \gamma\,\hat{x}_2 \cdot \sin(t),$$

which depends now only on the observer state.

Then the augmented system is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & K_1 & 0 & 1-K_1 \\ 0 & K_2 & -\alpha & -K_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \sin(t) \times u$$

Let again $\alpha = 3$ and the observer gains be

$$K_1 = 5, \quad K_2 = 3$$

This leads to the matrix

$$A - KC = \begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix}$$

which is strictly Hurwitz.

The Lyapunov function is then

$$Q = \begin{bmatrix} 1.6500 & -0.1250 & -0.1500 & 0.1250 \\ -0.1250 & 0.6292 & 0.1250 & -0.1292 \\ -0.1500 & 0.1250 & 0.1500 & -0.1250 \\ 0.1250 & -0.1292 & -0.1250 & 0.1292 \end{bmatrix}$$

To observe that it is positive definite, we have typed its eigenvalues

$$\lambda(Q) = \left\{0.0139,\ 0.1789,\ 0.6677,\ 1.6979\right\}.$$

Conclusions (Part II)

- The separation principle for this particular class of the non-linear systems is proven;
- The important part is that the Lyapunov function of the augmented system is found quadratic;
- The proof is based on the well-known arguments, but the final result may be new

Consider again the following dynamical system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\left(u - x_2^5\right) \quad (4)$$
$$y = x_1$$

Introduce a dynamical controller of the form

$$u = \lambda_1 x_1 + \Lambda_2 z + \Big(c_1 x_1 + C_3 z\Big)^5, \qquad \frac{dz}{dt} = \gamma x_1 + A_z z, \quad z \in R^m \quad (5)$$

The closed loop system is then

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0_{1\times m} \\ \lambda_1 & 1 & \Lambda_2 \\ \gamma & 0_{m\times 1} & A_z \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0_{m\times 1} \end{bmatrix} \times$$
$$\times \left((c_1 x_1 + C_3 z)^5 - x_2^5\right)$$

Question: When does the closed loop system satisfy the conditions of the *Circle criterion* with the nonlinearity

$$w = (c_1 x_1 + C_3 z)^5 - x_2^5$$

with the *fictitious* linear output

$$v = c_1 x_1 - x_2 + C_3 z$$

and the infinite sector constraint

$$\begin{aligned} v \cdot w &= \left(c_1 x_1 - x_2 + C_3 z\right) \left[\left(c_1 x_1 + C_3 z\right)^5 - x_2^5\right] \\ &= \left(c_1 x_1 - x_2 + C_3 z\right)^2 p(x_1, x_2, z) \geq 0 \quad \forall\, x_1,\, x_2,\, z \end{aligned}$$

Let us check the *frequency condition* for the values of ω that approach infinity, that is if

$$\omega^2 \cdot \text{Re} \left\{ \begin{bmatrix} c_1 & -1 & C_3 \end{bmatrix} \left(j\omega I_{m+2} - \begin{bmatrix} 0 & 1 & 0_{1\times m} \\ \lambda_1 & 1 & \Lambda_2 \\ \gamma & 0_{m\times 1} & A_z \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \\ 0_{m\times 1} \end{bmatrix} \right\} < 0$$

holds for $\omega \to +\infty$.

$$\omega^2 \cdot \mathrm{Re}\left\{ \begin{bmatrix} c_1 & -1 & C_3 \end{bmatrix} \begin{bmatrix} j\omega & -1 & 0_{1\times m} \\ -\lambda_1 & j\omega - 1 & -\Lambda_2 \\ -\gamma & 0_{m\times 1} & j\omega I_m - A_z \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0_{m\times 1} \end{bmatrix} \right\}$$

$$= \omega^2 \cdot \mathrm{Re}\left\{ \begin{bmatrix} c_1, -1, C_3 \end{bmatrix} \begin{bmatrix} * & S_{12} & * \\ * & S_{22} & * \\ * & S_{32} & * \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0_{m\times 1} \end{bmatrix} \right\}$$

$$\approx \omega^2 \cdot \mathrm{Re}\left\{ \frac{-c_1 \det \begin{bmatrix} -1 & 0 \\ 0 & (j\omega I_m - A_z) \end{bmatrix} - j\omega \det(j\omega I_m - A_z)}{\det(j\omega I_m - A_z) \det \begin{bmatrix} j\omega & -1 \\ -\lambda_1 & (j\omega - 1) \end{bmatrix}} \right\}$$

$$= \quad \omega^2 \cdot \mathrm{Re}\left\{ \frac{c_1 - j\omega}{-\omega^2 - \lambda_1 - j\omega} \right\} = \omega^2 \cdot \frac{\omega^2(1 - c_1) - c_1\lambda_1}{(\omega^2 + \lambda_1)^2 + \omega^2}$$

Therefore, irrespective of the controller parameters(!), to satisfy the conditions of *the Circle criterion* the parameter c_1 by necessity should satisfy the inequality

$$1 - c_1 < 0$$

and in particular $c_1 = 0$ will violate the condition.

Remark: No contraint on λ_1

$$u = \lambda_1 x_1 + \Lambda_2 z + \left(c_1 x_1 + C_3 z\right)^5$$

Consider the surge subsystem of the Moore-Greitzer model

$$
\begin{aligned}
\frac{d}{dt}\phi &= -\psi + \frac{3}{2}\phi + \frac{1}{2} - \frac{1}{2}(1+\phi)^3 \\
\frac{d}{dt}\psi &= \frac{1}{\beta^2}\Big(\phi + 1 - u\Big) \\
y &= \psi
\end{aligned}
$$

How to determine a stabilizing controller using only available measurements ψ?

Following the way we have considered in the previous Example, we can try to analyze stability of the closed loop system with the controller

$$
\begin{aligned}
u &= \lambda_1\psi + \lambda_2 z + \alpha_u\left(1 - (1 + c_\psi\psi + c_z z)^3\right) \\
\frac{d}{dt}z &= \lambda_3\psi + \lambda_4 z + \alpha_z\left(1 - (1 + c_\psi\psi + c_z z)^3\right)
\end{aligned}
$$

The closed loop system looks as

$$\frac{d}{dt}\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix} = \underbrace{\begin{bmatrix}\frac{3}{2} & -1 & 0\\ \frac{1}{\beta^2} & -\frac{\lambda_1}{\beta^2} & -\frac{\lambda_2}{\beta^2}\\ 0 & \lambda_3 & \lambda_4\end{bmatrix}}_{A}\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ -\frac{\alpha_u}{\beta^2}\\ \alpha_z\end{bmatrix}}_{B_1} w_1 + \underbrace{\begin{bmatrix}\frac{1}{2}\\ 0\\ 0\end{bmatrix}}_{B_2} w_2$$

where

$$\begin{aligned} w_1 &= 1 - \left(1 + c_\psi \psi + c_z z\right)^3 \\ w_2 &= 1 - \left(1 + \phi\right)^3 \end{aligned}$$

There are at least two quadratic constraints:

$$\begin{aligned} (-c_\psi \psi - c_z z) w_1 &= (-c_\psi \psi - c_z z)\left(1 - \left(1 + c_\psi \psi + c_z z\right)^3\right) \\ &= (-c_\psi \psi - c_z z)^2\, p_1(\psi, z) \ \geq\ 0 \end{aligned}$$

$$\begin{aligned} &(c_\psi \psi + c_z z - \phi)\,(w_2 - w_1) = \\ &\qquad = (c_\psi \psi + c_z z - \phi)\left(\left(1 + c_\psi \psi + c_z z\right)^3 - \left(1 + \phi\right)^3\right) \\ &\qquad = (c_\psi \psi + c_z z - \phi)^2\, p_2(\phi, \psi, z) \ \geq\ 0 \end{aligned}$$

The stability conditions of the *Circle Criterion* with such constraints are:

I) there exist $\tau_1 \geq 0$, $\tau_2 \geq 0$, $\tau_1 + \tau_2 > 0$ such that the **frequency condition**

$$\tau_1 \,\mathrm{Re}\,\left\{\xi_1^* C_1 A_{j\omega}^{-1}\left(B_1\xi_1 + B_2\xi_2\right)\right\} + \\ + \tau_2 \,\mathrm{Re}\,\left\{\xi_2^* C_2 A_{j\omega}^{-1}\left(B_1\xi_1 + B_2\xi_2\right)\right\} < 0$$

holds $\forall\, \omega \in R_+$ and $\forall\, \xi_1$, $\forall\, \xi_2 \in C^1$. Here

$$A_{j\omega}^{-1} = \left(j\omega I_3 - A\right)^{-1}, \quad C_1 = [\,0, \quad -c_\psi \quad -c_z\,], C_2 = [\,-1, \quad c_\psi, \quad c_z\,]$$

II) the matrix A is strictly Hurwitz.

LEMMA 2

The frequency inequality is valid if and only if the following three conditions hold

1. The polynomial

$$p_1(s) = a_2 s^2 + a_1 s + a_0$$

is negative for all $s \geq 0$, where the coefficients a_2–a_0 are given in the handouts.

2. The polynomial

$$p_2(s) = b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0$$

is positive for all $s \geq 0$, where the coefficients b_4–b_0 are given in the handouts.

3. The equality

$$8c_z\alpha_z c_\psi \alpha_u \beta - 4c_\psi \alpha_u \beta - 4{c_z}^2{\alpha_z}^2\beta^2 + 4c_z\alpha_z\beta^2 - \beta^2 - 4{c_\psi}^2{\alpha_u}^2 = 0$$

LEMMA 3

The matrix A is strictly Hurwitz if and only if the following inequalities

$$\begin{aligned}
\frac{\lambda_1}{\beta^2} &> \lambda_4 + \frac{3}{2} \\
\frac{\lambda_2\lambda_3}{\beta^2} + \frac{3}{2}\lambda_4 + \frac{1}{\beta^2} &> \frac{3\lambda_1}{2\beta^2} \\
0 &> \lambda_4 + \frac{3}{2}\lambda_2\lambda_3 \\
\left(\frac{\lambda_1}{\beta^2} - \lambda_4 - \frac{3}{2}\right)\left(\frac{\lambda_2\lambda_3}{\beta^2} + \frac{3}{2}\lambda_4 + \frac{1}{\beta^2} - \frac{3\lambda_1}{2\beta^2}\right) &> -\frac{1}{\beta^2}\left(\lambda_4 + \frac{3}{2}\lambda_2\lambda_3\right)
\end{aligned}$$

are valid. ■

□

THEOREM 4

Consider any constants γ_1–γ_4 such that

$$\gamma_2 > \frac{3}{2}\beta^2, \quad 1 > \gamma_1 + \frac{3}{2}\gamma_2, \quad -\frac{3}{4}\beta^2 > \gamma_3, \quad \gamma_2 > -2\gamma_3, \quad \gamma_4 > \frac{3}{2}\beta^2$$

The set of such constants γ_1–γ_4 is not empty.

Then the dynamical output controller

$$\begin{aligned}
u &= \lambda_1\psi + \lambda_2 z + \alpha_u\left(1 - (1 + c_\psi\psi + c_z z)^3\right) \\
\frac{d}{dt}z &= \lambda_3\psi + \lambda_4 z + \alpha_z\left(1 - (1 + c_\psi\psi + c_z z)^3\right)
\end{aligned}$$

□

with the parameters

$$\lambda_1 = \gamma_2 + \gamma_1\gamma_4 \qquad \lambda_2 = \gamma_1$$

$$\lambda_3 = \frac{\gamma_4^2}{\beta^2}(\gamma_1 - 1) + \gamma_4\left(\frac{3}{2} + \frac{\gamma_2}{\beta^2}\right) - 1 \qquad \lambda_4 = \frac{3}{2} + \frac{\gamma_4}{\beta^2}(\gamma_1 - 1)$$

$$\alpha_u = \gamma_3 \qquad \alpha_z = \frac{1}{2} + \frac{\gamma_3\gamma_4}{\beta^2}$$

$$c_\psi = \gamma_4 \qquad c_z = 1$$

renders the closed loop system to satisfy all the conditions of the Circle criterion. In other words, the closed loop system is globally exponentially stable. ■

The derivation of the stabilizing controller is based on the *Separation Principle* and the validity of additional *structural assumptions*!

Step 1 Derive static feedback controller, that is, we allow to use ψ and ϕ in feedback, that renders the closed loop system to satisfy the *Circle criterion*.

Step 2 Interpret some linear combination of z and ψ as estimate for unmeasured variable ϕ, and derive the error dynamics to satisfy the *Circle criterion*.

Step 3 Check structural assumption, that allows to verify that the combined system satisfies the *Circle criterion*.