Introduction to Time-Delay Systems

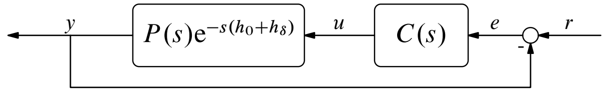
lecture no. 7

Leonid Mirkin

Faculty of Mechanical Engineering, Technion—Israel Institute of Technology

Department of Automatic Control, Lund University

Topic studied today



Assume the closed-loop system is stable,

- the smallest destabilizing deviation of loop delay h_δ

called the delay margin, μ_d.

We'll discuss the following questions:

- How to calculate μ_d?
- What μ_d can be achieved?

Topic studied today (contd)

In most cases, precise methods of computing μ_d

- ☹ apply only to single- and commensurate delay cases
- ☹ computationally involved
- ☹ unsuitable for design

This motivates developing alternative methods

- trading accuracy for simplicity

These methods

- ☹ normally conservative

yet

- ☺ numerically efficient
- ☺ design friendly (at least some of them)

Outline

Preliminaries: small gain arguments and LMIs

Delay robustness analysis: unstructured uncertainty embedding

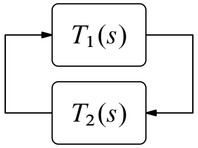
Covering options

Delay robustness analysis: Lyapunov-Krasovskiĭ approach

Time-varying extensions

Bounds on the achievable delay margin

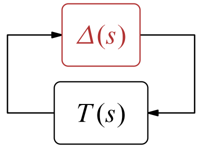
The Small Gain Theorem



Theorem (Small Gain Theorem)
The closed-loop system is stable if

$$T_1 \in H^\infty, \quad T_2 \in H^\infty, \quad \textit{and} \quad \|T_1T_2\|_\infty < 1.$$

Robust stability theorem



$\Delta(s)$ belongs to the unit ball in H^∞ ($\|\Delta\|_\infty \le 1$) but otherwise arbitrary (for scalars, $\Delta(\mathrm{j}\omega) \in \bar{\mathbb{D}}$ at each ω, where the closed unit disk $\bar{\mathbb{D}} := \{z : |z| \le 1\}$)

Theorem
Closed-loop system is stable for all Δ from the class above iff

$$T \in H^\infty \quad \textit{and} \quad \|T\|_\infty < 1.$$

Proof.

"if": follows by the SGT

"only if": if $\|T\|_\infty \ge 1$ destabilizing admissible Δ can be constructed (such that $I - \Delta(\mathrm{j}\omega)T(\mathrm{j}\omega)$ is singular for some ω) □

Linear Matrix Inequalities

Inequality

$$M_0 + \sum_{i=1}^{m} x_i M_i > 0$$

to be solved in real $x_1, \ldots, x_m$ for given $M_i = M_i'$ called LMI. E.g.,

- Lyapunov LMI

  $$AP + PA' < 0, \quad \text{in } P = P' > 0$$

  To see the connection, consider the 2×2 case. Then

  $$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

  is a basis for the 3-dimensional space of 2×2 symmetric matrices and we end up with an LMI for

  $$M_0 = 0 \quad \text{and} \quad M_i = \begin{bmatrix} P_i & 0 \\ 0 & -AP_i - P_iA' \end{bmatrix}, \quad i = 1, 2, 3.$$

Schur complement and LMIs

It's known that

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} > 0$$
$$\Updownarrow$$
$$M_{11} > 0 \quad \text{and} \quad M_{22} - M_{21}M_{11}^{-1}M_{12} > 0$$
$$\Updownarrow$$
$$M_{22} > 0 \quad \text{and} \quad M_{11} - M_{12}M_{22}^{-1}M_{21} > 0$$

For example, $\exists P > 0$ such that $P'A + A'P + C'RC + P'BR^{-1}B'P < 0$ for some given A, B, C, and $R > 0$ iff

$$\underbrace{\exists P > 0 \quad \text{such that} \begin{bmatrix} -P'A - A'P - C'RC & -P'B \\ -B'P & R \end{bmatrix} > 0}_{\text{LMI}}$$

LMIs: why ?

Because

1. they can be efficiently solved and many solvers available on the market
2. plenty of control problems can be solved in terms of LMIs

H^∞ norm via LMI

Let

$$T(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then $T \in H^\infty$ and $\|T\|_\infty < 1$ iff

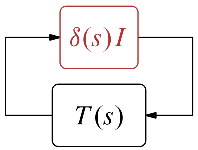
$$\exists X > 0 \quad \text{such that} \begin{bmatrix} A'X + XA & XB & C' \\ B'X & -I & D' \\ C & D & -I \end{bmatrix} < 0$$

or, equivalently,

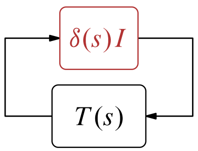
$$\exists X > 0 \quad \text{such that} \begin{bmatrix} A'X + XA + C'C & XB + C'D \\ B'X + D'C & D'D - I \end{bmatrix} < 0.$$

Structured uncertainty

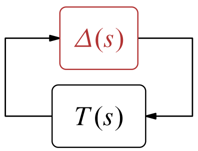
It may happen that we know more about the structure of uncertainty, e.g.:



If scalar $\delta(s)$ is only restricted to be in the unit ball in H^∞,

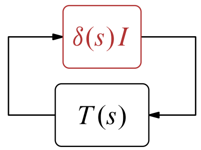


$\subset$



and SGT arguments apply. But they are conservative then.

Example



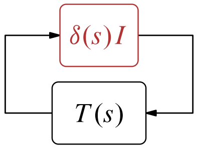
with $T(s) = \begin{bmatrix} 0.5 & \alpha \\ 0 & 0.5 \end{bmatrix}$ for some $\alpha \in \mathbb{R}$. Closed-loop system is stable iff

$$I - \delta(\mathrm{j}\omega)T(\mathrm{j}\omega) = \begin{bmatrix} 1 - 0.5\delta(\mathrm{j}\omega) & \alpha\delta(\mathrm{j}\omega) \\ 0 & 1 - 0.5\delta(\mathrm{j}\omega) \end{bmatrix}$$

is nonsingular for every admissible $\delta(\omega)$ and $\forall\omega \in \mathbb{R}$. This is indeed true as $|\delta(\mathrm{j}\omega)| \leq 1$ for all ω, so that

- system is stable for all $\|\delta\|_\infty \leq 1$ irrespective of α.

Example (contd)



To apply SGT:

$$\|T\|_\infty = \bar{\sigma}\left(\begin{bmatrix} 0.5 & \alpha \\ 0 & 0.5 \end{bmatrix}\right) = \frac{1}{2}\sqrt{2\alpha^2 + 1 + 2|\alpha|\sqrt{\alpha^2+1}},$$

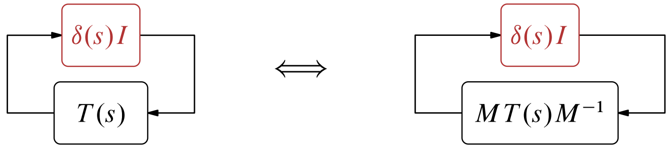
which grows with $|\alpha|$ (in fact, $\|T\|_\infty \geq 1$ iff $|\alpha| \geq \frac{3}{4}$). Thus

- by SGT stability is guaranteed only if $|\alpha| < \frac{3}{4}$

which is conservative.

Scaled robust stability theorem

Key observation:



for every nonsingular M. Thus:

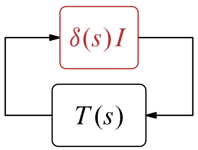
Theorem (scaled robust stability theorem)

The closed-loop system is stable for all $\|\delta\|_\infty \leq 1$ *if*

$$\exists M \quad \textit{such that } \|MTM^{-1}\|_\infty < 1.$$

This condition would become necessary as well if we allowed time-varying (in fact, arbitrarily slow) δ's. Otherwise, dynamic $M(s)$ should be used (μ).

Example (contd)



Choose $M = \left[\begin{smallmatrix} \beta & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Then

$$MT(s)M^{-1} = \begin{bmatrix} 0.5 & \alpha\beta \\ 0 & 0.5 \end{bmatrix}$$

so that $\|MTM^{-1}\|_\infty = \frac{1}{2}\sqrt{2\alpha^2\beta^2 + 1 + 2|\alpha|\beta\sqrt{\alpha^2\beta^2+1}} < 1$ iff $\beta|\alpha| < \frac{3}{4}$. In other words,

- SRST guarantees stability for every α if β is sufficiently small

(in fact, $\|MTM^{-1}\|_\infty \to \frac{1}{2}$ as $\beta \to 0$).

Scaled H^∞ norm via LMI

Let

$$T(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \quad \text{so that } MT(s)M^{-1} = \left[\begin{array}{c|c} A & BM^{-1} \\ \hline MC & MDM^{-1} \end{array}\right].$$

Then $T \in H^\infty$ and $\|MTM^{-1}\|_\infty < 1$ iff $\exists X > 0$ such that

$$\begin{bmatrix} A'X + XA + C'M'MC & XBM^{-1} + C'M'MDM^{-1} \\ M^{-\prime}B'X + M^{-\prime}D'M'MC & M^{-\prime}D'M'MDM^{-1} - I \end{bmatrix} < 0.$$

Pre- and post-multiplying by $\left[\begin{smallmatrix} I & 0 \\ 0 & M' \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} I & 0 \\ 0 & M \end{smallmatrix}\right]$, resp., this condition reads

$$\begin{bmatrix} A'X + XA + C'YC & XB + C'YD \\ B'X + D'YC & D'YD - Y \end{bmatrix} < 0,$$

where $Y = M'M > 0$. Thus, $\exists M$ such that $\|MTM^{-1}\|_\infty < 1$ iff

$$\exists X, Y > 0 \quad \text{such that} \begin{bmatrix} A'X + XA + C'YC & XB + C'YD \\ B'X + D'YC & D'YD - Y \end{bmatrix} < 0.$$

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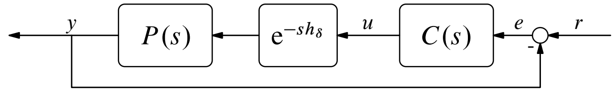
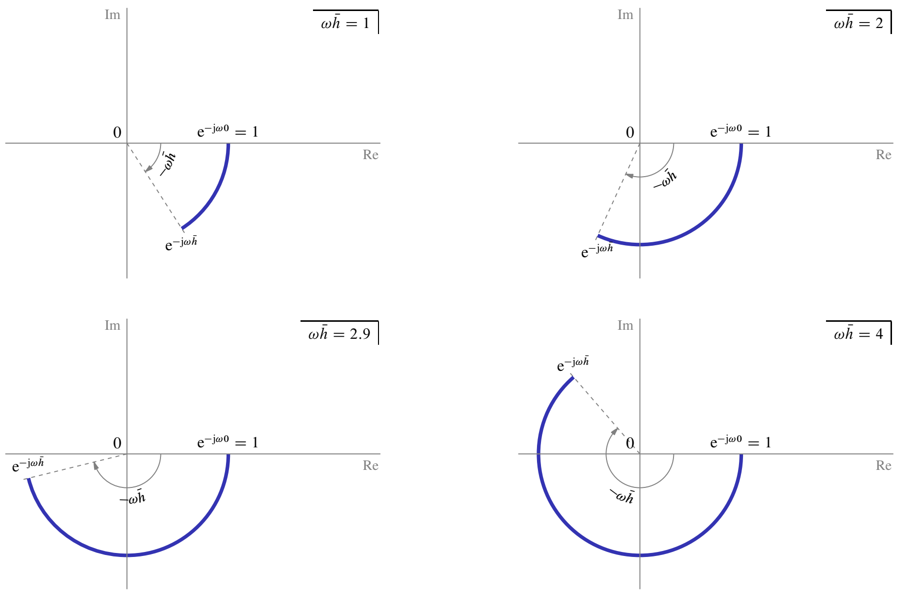
Simple setup



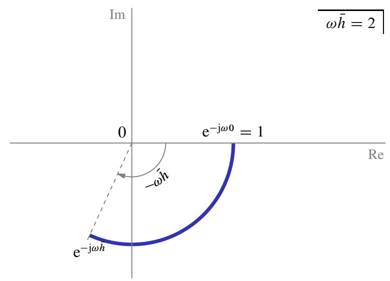
Problems:

- Given P and C, calculate μ_{d}
  i.e., the smallest h_δ destabilizing the system (we denote this smallest h_δ as $\bar{h}$)
- Given P, design C guaranteeing required μ_{d} (if possible)

Geometry of uncertain delays in $\mathbb{C}$



Arcs via disks



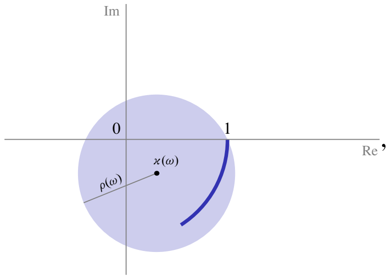
- arc uncertainties are hard to handle analytically
- disk uncertainties fall into the scope of SGT

We may try to cover arcs by disks. . .

Covering by disks

At each frequency, find $\varkappa(\omega) \in \mathbb{C}$ and (smallest possible) $\rho(\omega) \geq 0$ s.t.

$$\mathrm{e}^{-\mathrm{j}\omega h_\delta} \subseteq \varkappa(\omega) + \rho(\omega)\bar{\mathbb{D}} =$$

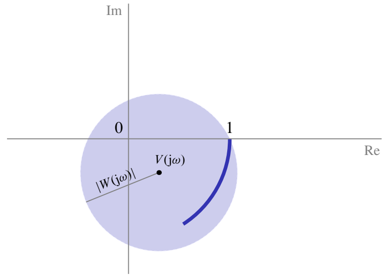


$$\forall h_\delta \in [0, \bar{h}]$$

As rational approximations are preferable, we may look for stable rational $V(s)$ and $W(s)$ s.t.

$$\mathrm{e}^{-\mathrm{j}\omega h_\delta} \subseteq V(\mathrm{j}\omega) + |W(\mathrm{j}\omega)|\bar{\mathbb{D}}, \quad \forall h_\delta \in [0, \bar{h}].$$

Covering by disks (contd)



The disk can be described as all possible values of $D(\mathrm{j}\omega)$, where

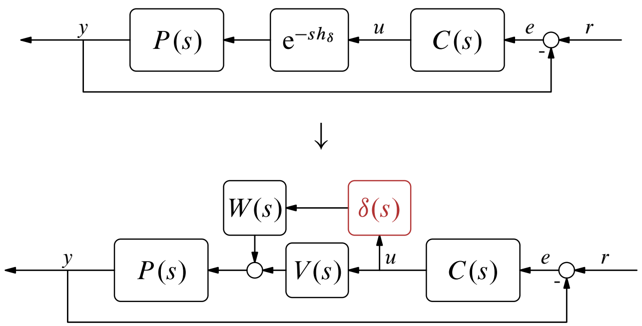
$$D(s) := V(s) + W(s)\delta(s),$$

where

- $\delta(s)$ is arbitrarily transfer function from the unit ball in H^∞
  in this case $|D(\mathrm{j}\omega) - V(\mathrm{j}\omega)| = |W(\mathrm{j}\omega)| \cdot |\delta(\mathrm{j}\omega)| \leq |W(\mathrm{j}\omega)|$ for every ω

Covering by disks: implication

The replacement $\mathrm{e}^{-sh} \to D(s) = V(s) + W(s)\delta(s)$ results in

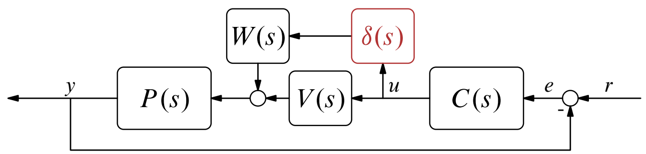


where

- $V(s)$ is "nominal delay"
- $W(s)$ reflects the "size" of uncertainty ($|W(\mathrm{j}\omega)|$ is uncertainty radius)
- $\delta(s)$ unstructured disk uncertainty (stable and such that $\|\delta\|_\infty \leq 1$)

Covering by disks: implication (contd)

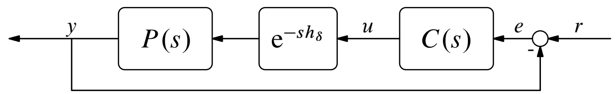
Then,



stable for all $\|\delta\|_\infty \leq 1$

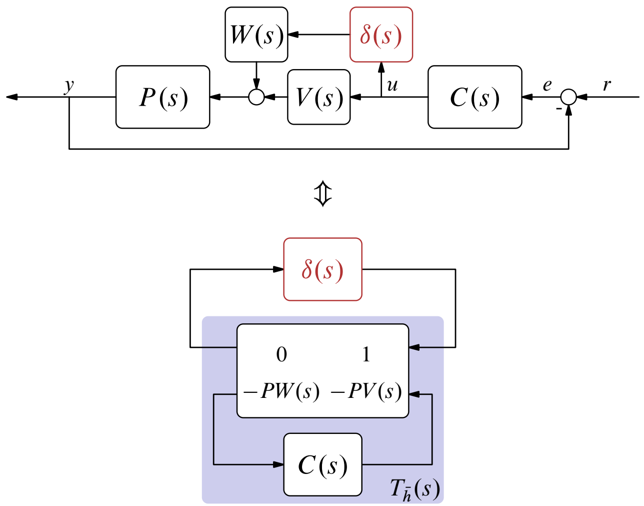
$\Downarrow$

stable for all $h_\delta \in [0, \bar{h}]$



:-(but not necessarily vice versa).

Reduction to standard robust stability problem



and the system is robustly stable iff $\|T_{\bar{h}}\|_\infty < 1$, with $T_{\bar{h}}(s) = -\frac{P(s)W(s)C(s)}{1+P(s)V(s)C(s)}$

So, what did we gain / loose ?

We gained

- simplicity

  μ_{d} calculation / design reduce to a standard H^∞ problem, which is readily solvable

We payed by

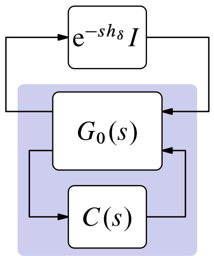
- conservatism

  introduced when we replace an arc with a disk;

  if the H^∞ problem isn't solvable, the system may still be stable for all $h_\delta \in [0, \bar{h}]$

Abstract setup

In many problems delay can be isolated as



Examples:

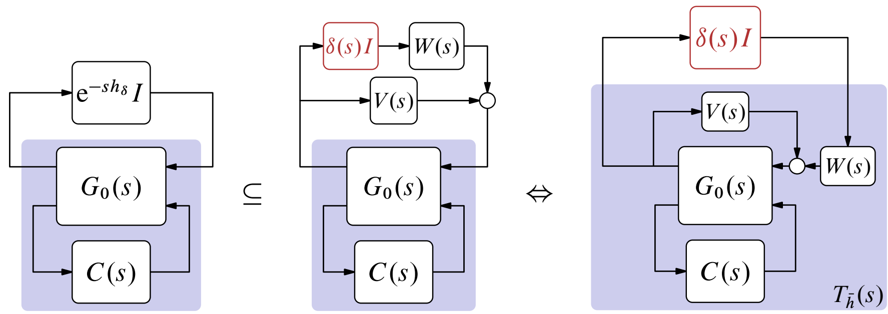
- For the problem above, $G_0(s) = \left[\begin{smallmatrix} 0 & 1 \\ -P(s) & 0 \end{smallmatrix}\right]$
- If $\dot{x}(t) = A_0 x(t) + A_h x(t - h_\delta) + Bu(t)$ with output $y(t) = Cx(t)$,

$$G_0(s) = \begin{bmatrix} C_z \\ C \end{bmatrix} (sI - A_0)^{-1} \begin{bmatrix} B_w & B \end{bmatrix}.$$

  where C_z and B_w are any matrices such that $A_h = B_w C_z$.

Abstract setup: reduction to robust stability

Embedding e^{-sh_δ} into the wider class (disk) $D(s) = V(s) + W(s)\delta(s)$:



and the system is stable for all $h_\delta \in [0, \bar{h}]$ if

$$\exists M \quad \text{such that } \|M T_{\bar{h}} M^{-1}\|_\infty < 1.$$

If $T_{\bar{h}}(s)$ is proper and rational, the problem is LMI able (hence, computable).
To compute μ_{d}, a binary search in $\bar{h}$ can be carried out.

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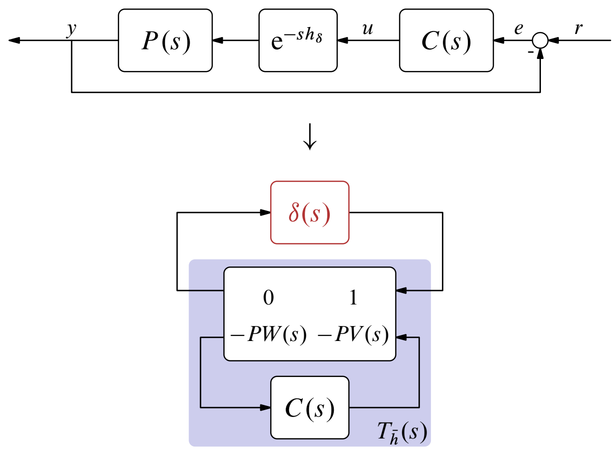
Setup

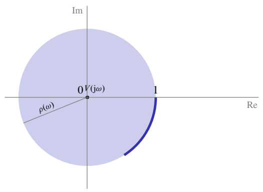


and the system is robustly stable if $\|T_{\bar h}\|_\infty < 1$, with $T_{\bar h}(s) = -\frac{P(s)W(s)C(s)}{1+P(s)V(s)C(s)}$.

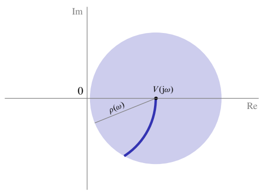
How to choose $V(s)$?

Basically, tradeoff between tightness and complexity. Common choices:

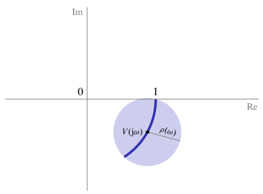
$V(s) = V_0(s) := 0$



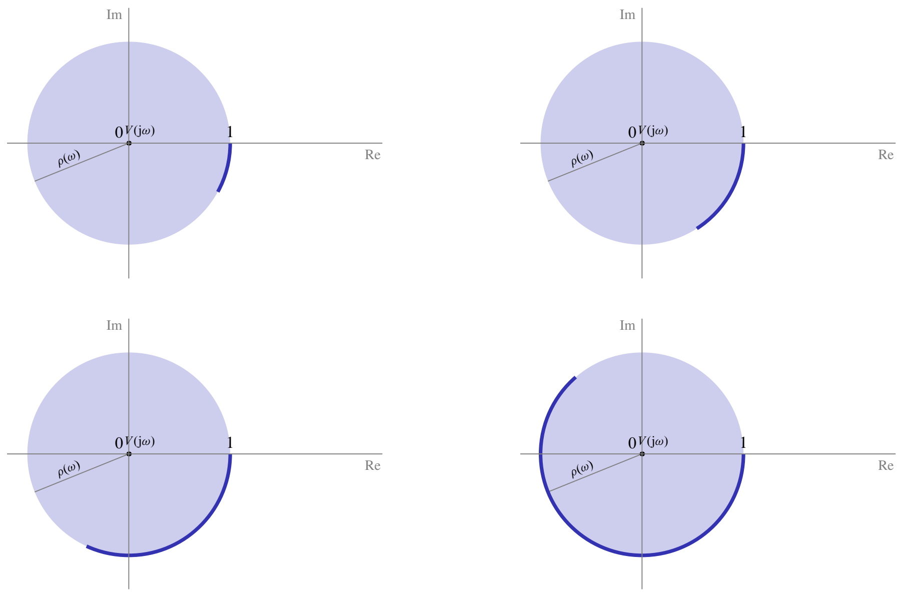
$V(s) = V_1(s) := 1$ (zero nominal delay)



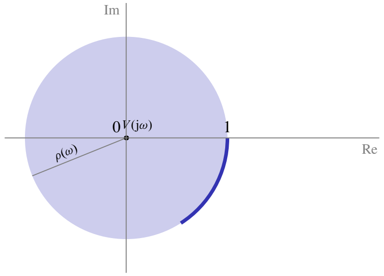
$V(s) = V_2(s) := e^{-s\bar h/2}$ (nominal delay of $\frac{\bar h}{2}$)



$V = V_0$



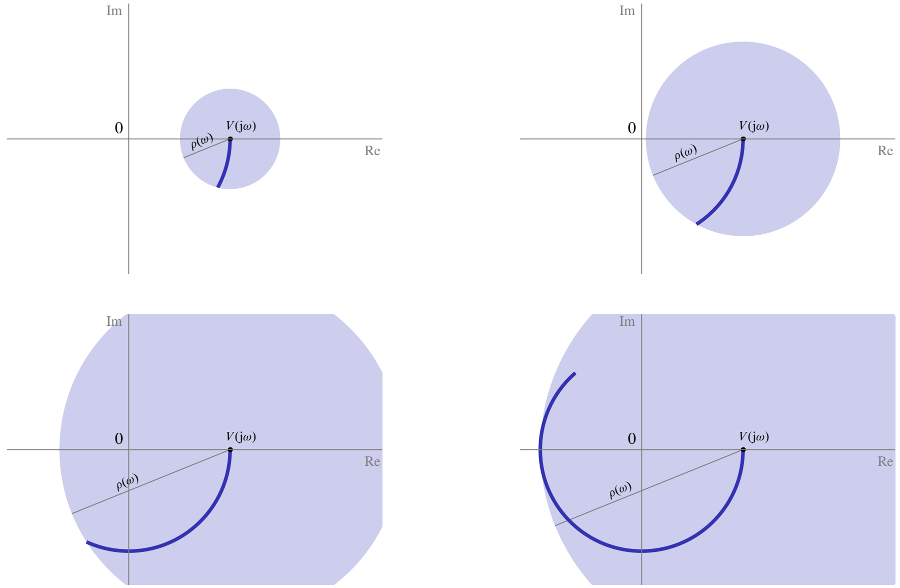
$V = V_0$: delay-independent conditions



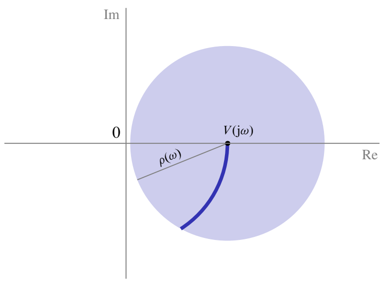
$$\rho(\omega) \equiv 1$$
$$\downarrow$$
$$W(s) = 1$$
$$\downarrow$$
$$\|PC\|_\infty < 1$$

- if holds, guarantees stability for all $h_\delta > 0$
- disregards phase information about e^{-sh_δ}
- very conservative if h_δ known to be bounded (realistic assumption)

$V = V_1$



$V = V_1$: delay-dependent conditions

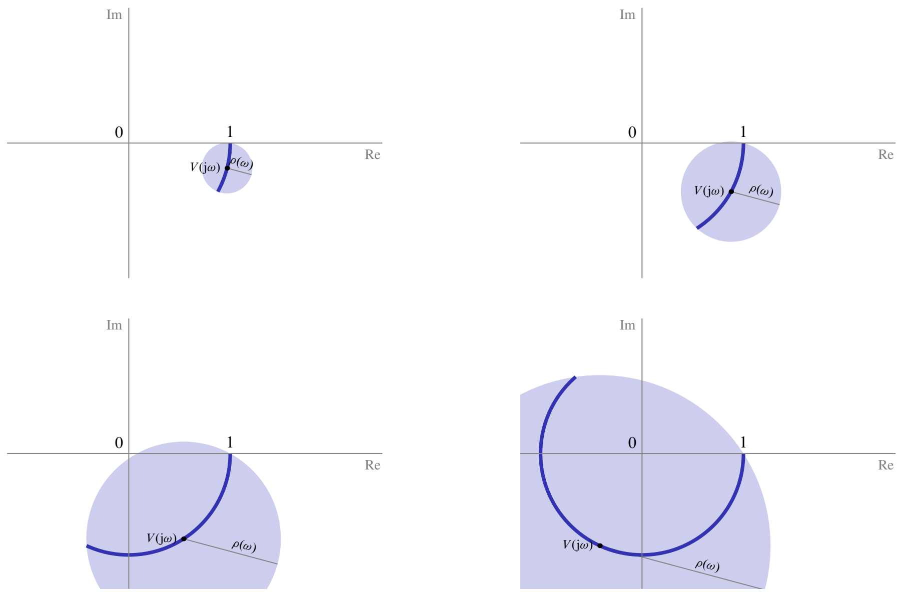


$$\rho(\omega) = \rho_1(\omega) := \begin{cases} 2\sin\frac{\omega\bar{h}}{2} & \omega\bar{h} \le \pi \\ 2 & \text{otherwise} \end{cases}$$
$$\downarrow$$
$$W(s) \text{ any rational s.t. } |W(\mathrm{j}\omega)| \ge \rho_1(\omega)$$
$$\downarrow$$
$$\left\| \frac{PCW}{1+PC} \right\|_\infty < 1$$

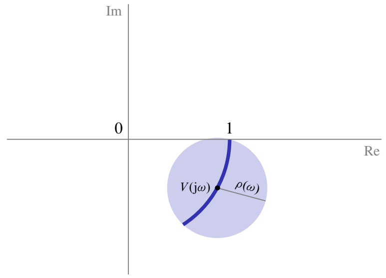
- does take phase information about e^{-sh_δ} into account

  $\rho_1(\omega)$ is HPF; reflects the fact that delay uncertainty more harmful at high frequencies

$V = V_2$



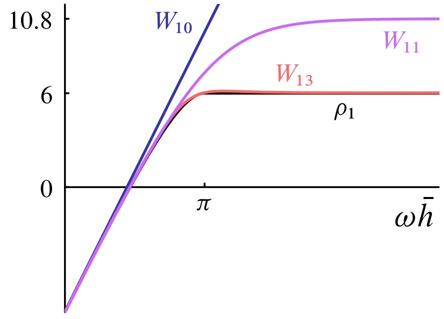
$V = V_2$: delay-dependent conditions



$$\rho(\omega) = \rho_2(\omega) := \begin{cases} 2\sin\frac{\omega\bar{h}}{4} & \omega\bar{h} \le 2\pi \\ 2 & \text{otherwise} \end{cases}$$
$$\downarrow$$
$W(s)$ any rational s.t. $|W(\mathrm{j}\omega)| \ge \rho_2(\omega)$
$$\downarrow$$
$$\left\| \frac{PCW}{1 + P\mathrm{e}^{-s\bar{h}/2}C} \right\|_\infty < 1$$

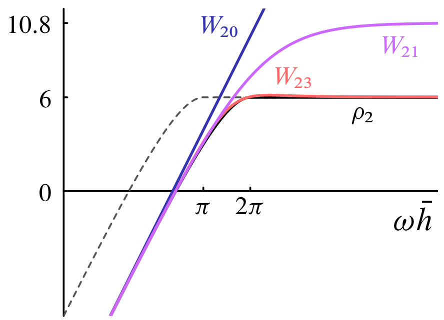
- ☺ does take phase information about e^{-sh_s} into account
  $\rho_2(\omega)$ is HPF; reflects the fact that delay uncertainty more harmful at high frequencies
- ☺ $\rho_2(\omega) = \rho_1(\omega/2)$, so the radius is smaller
- ☹ $T(s)$ is irrational (contains $\mathrm{e}^{-s\bar{h}/2}$), which might complicate the analysis
  although in the design of $C(s)$ the use of DTC reduces the problem to the rational one

Rational upper bounds of $\rho_1(\omega)$



- $W(s) = W_{10}(s) := \bar{h}s$
- $W(s) = W_{11}(s) := \dfrac{2\sqrt{3}\,\bar{h}s}{\bar{h}s + 2\sqrt{3}}$ $\quad(|W_{11}(\mathrm{j}\omega)| < |W_{10}(\mathrm{j}\omega)|, \forall\omega > 0)$
- $W(s) = W_{13}(s) := \dfrac{2.007\bar{h}s}{\bar{h}s + 2}\,\dfrac{\bar{h}^2s^2 + 3.695\bar{h}s + 5.56}{\bar{h}^2s^2 + 3.026\bar{h}s + 5.56}$

Rational upper bounds of $\rho_2(\omega)$

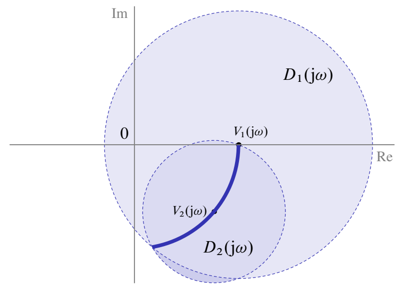


- $W(s) = W_{20}(s) := \bar{h}s/2$
- $W(s) = W_{21}(s) := \dfrac{2\sqrt{3}\,\bar{h}s}{\bar{h}s + 4\sqrt{3}}$ $\quad(|W_{21}(\mathrm{j}\omega)| < |W_{20}(\mathrm{j}\omega)|, \forall\omega > 0)$
- $W(s) = W_{23}(s) := \dfrac{2.007\bar{h}s}{\bar{h}s + 4}\,\dfrac{\bar{h}^2s^2 + 7.39\bar{h}s + 22.24}{\bar{h}^2s^2 + 6.052\bar{h}s + 22.24}$

Is D_2 less conservative than D_1 ?

Although $\rho_2(\omega) \le \rho_1(\omega)$ for all ω, what matters is that

$$D_2(\mathrm{j}\omega) \not\subseteq D_1(\mathrm{j}\omega) :$$



It might happen that destabilizing $\delta(\mathrm{j}\omega)$ corresponds to the darkest region of $D_2(\mathrm{j}\omega)$, which doesn't belong to $D_1(\mathrm{j}\omega)$. If this is the case covering with the smaller D_2 will be more conservative than that with the larger D_1.

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Approach

Consider system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h)$$

where matrices A_0 and A_1 given and delay h uncertain in $[0, \bar{h}]$, $\bar{h} > 0$. We want to find maximal $\bar{h}$ for which this system stable. To this end, we

- construct a Lyapunov-Krasovskiĭ functional $V(x_\tau)$ and find conditions under which its derivative along system trajectory is non-negative

for all $[0, \bar{h}]$.

Solution steps typically involve

1. transform system equation into a Lyapunov-Krasovskiĭ-friendly form
2. choose a Lyapunov-Krasovskiĭ functional
3. upper-bound cross terms in its derivative

Below we provide a flavor of these steps.

Descriptor model transformation

Rewrite

$$\begin{aligned}\dot{x}(t) &= A_0 x(t) + A_1 x(t-h) = (A_0 + A_1)x(t) - A_1\big(x(t) - x(t-h)\big) \\ &= (A_0 + A_1)x(t) - A_1 \int_{-h}^{0} \dot{x}(t+\theta)\mathrm{d}\theta\end{aligned}$$

It turns out to be advantageously to rewrite this equation in descriptor form

$$\underbrace{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix}}_{\dot{\eta}(t)} = \underbrace{\begin{bmatrix} 0 & I \\ A_0 + A_1 & -I \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}}_{\eta(t)} - \underbrace{\begin{bmatrix} 0 \\ A_1 \end{bmatrix}}_{B} \int_{-h}^{0} y(t+\theta)\mathrm{d}\theta$$

by adding auxiliary variable

$$\dot{x}(t) =: y(t) = \underbrace{\begin{bmatrix} 0 & I \end{bmatrix}}_{C} \eta(t)$$

State vector here is $(x(t), y_\tau(t))$.

Lyapunov-Krasovskiĭ functional

Let

$$V = V_1 + V_2 := \eta'(t)P'E\eta(t) + \int_{-h}^{0}\int_{t+\tau}^{t} y'(\theta)Ry(\theta)\mathrm{d}\theta\mathrm{d}\tau$$

for $R > 0$ and $P = \left[\begin{smallmatrix} P_1 & 0 \\ P_2 & P_3 \end{smallmatrix}\right]$, $P_1 > 0$. Note that $\eta'(t)EP\eta(t) = x'(t)P_1x(t)$, so that V is indeed positive function of state. Now,

$$\begin{aligned}\dot{V}_1 &= 2\eta'(t)P'E\dot{\eta}(t) = 2\eta'(t)P'\left(A\eta(t) - B\int_{-h}^{0} y(t+\theta)\mathrm{d}\theta\right) \\ &= \eta'(t)\big(P'A + A'P\big)\eta(t) - 2\int_{-h}^{0} \eta'(t)P'By(t+\theta)\mathrm{d}\theta\end{aligned}$$

and

$$\begin{aligned}\dot{V}_2 &= \int_{-h}^{0} \big(y'(t)Ry(t) - y'(t+\tau)Ry(t+\tau)\big)\,\mathrm{d}\tau \\ &= h\eta'(t)C'RC\eta(t) - \int_{-h}^{0} y'(t+\tau)Ry(t+\tau)\mathrm{d}\tau\end{aligned}$$

Covering cross-term

Thus, taking into account that $h \le \bar{h}$,

$$\dot{V} \le \eta'(t)\big(P'A + A'P + \bar{h}C'RC\big)\eta(t) - \int_{-h}^{0} y'(t+\theta)Ry(t+\theta)\mathrm{d}\theta - \phi(t),$$

where $\phi(t) := 2\int_{-h}^{0} \eta'(t)P'By(t+\theta)\mathrm{d}\theta$ is cross term. To handle $\phi(t)$, note that for any $Q > 0$ and vectors v_1 and v_2,

$$0 \le (v_1 + Q^{-1}v_2)'Q(v_1 + Q^{-1}v_2) = v_1'Qv_1 + v_2'Q^{-1}v_2 + 2v_2'v_1$$

or, equivalently, $-2v_2'v_1 \le v_1'Qv_1 + v_2'Q^{-1}v_2$. Thus

$$\begin{aligned} -\phi(t) &\le \int_{-h}^{0} \big(\eta'(t)P'BQ^{-1}B'P\eta'(t) + y'(t+\theta)Qy(t+\theta)\big)\,\mathrm{d}\theta \\ &\le \bar{h}\eta'(t)P'BQ^{-1}B'P\eta'(t) + \int_{-h}^{0} y'(t+\theta)Qy(t+\theta)\mathrm{d}\theta, \end{aligned}$$

which is true for all $Q > 0$, in particular, for $Q = R$.

Fitting things together

Thus,

$$\dot{V} \le \eta'(t)\big(P'A + A'P + \bar{h}C'RC + \bar{h}P'BR^{-1}B'P\big)\eta(t)$$

and $\dot{V} < 0$ for all $\eta \ne 0$ iff

$$P'A + A'P + \bar{h}C'RC + \bar{h}P'BR^{-1}B'P < 0$$

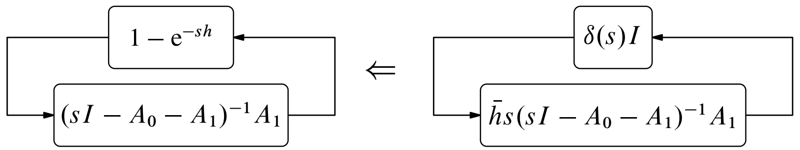
or, equivalently (via Schur complement arguments), iff LMI

$$\begin{bmatrix} P'A + A'P + \bar{h}C'RC & \bar{h}P'B \\ \bar{h}B'P & -\bar{h}R \end{bmatrix} < 0$$

solvable in some $R > 0$ and $\left[\begin{smallmatrix} P_1 & 0 \\ P_2 & P_3 \end{smallmatrix}\right]$ with $P_1 > 0$.

Addressing the same problem via UUE

Choosing $V(s) = 1$, we reduce problem to



Covering $|1 - \mathrm{e}^{-\mathrm{j}\omega h}|$ by $W(s) = \bar{h}s$ (loosest cover), we end up with robust stability problem for scalar uncertainty $\delta(s)$ satisfying $\|\delta\|_\infty \le 1$.

This problem, in turn, solvable if

$$\exists M = M' > 0 \quad \text{such that } \|s\bar{h}M(sI - A_0 - A_1)^{-1}A_1M^{-1}\|_\infty < 1,$$

which is LMI-able.

Finding connections

To make simple things complicated, note that

$$\begin{aligned} s(sI - A_0 - A_1)^{-1}A_1 &= \begin{bmatrix} 0 & I \end{bmatrix}\left(s\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & I \\ A_0 + A_1 & -I \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ A_1 \end{bmatrix} \\ &= C(sE - A)^{-1}B \end{aligned}$$

It is known that $\|C(sE - A)^{-1}B\|_\infty < 1$ iff $\exists P$ such that

$$E'P = P'E \ge 0 \quad \text{and} \quad P'A + A'P + P'BB'P + C'C < 0.$$

First inequality equivalent to

$$P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix} \quad \text{with} \quad P_1 \ge 0$$

and second inequality and structure of C yield that $\det P_1 \ne 0$, i.e., $P_1 > 0$.

Finding connections (contd)

Thus, robust stability condition reads

$$\exists M > 0 \quad \text{and} \quad P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix} \quad \text{with } P_1 > 0$$

such that

$$P'A + A'P + \bar{h}P'BM^{-2}B'P + \bar{h}C'M^2C < 0.$$

Denoting $R = M^2 > 0$ and using Schur complement arguments, stability conditions reduce to solvability of LMI

$$\begin{bmatrix} P'A + A'P + \bar{h}C'RC & \bar{h}P'B \\ \bar{h}B'P & -\bar{h}R \end{bmatrix} < 0$$

in some $R > 0$ and $\left[\begin{smallmatrix} P_1 & 0 \\ P_2 & P_3 \end{smallmatrix}\right]$ with $P_1 > 0$. Haven't we already seen this?

Reducing conservatism of Lyapunov-Krasovskiĭ results

Several possibilities:

- alternative model transformations
  (perhaps, those going beyond $W(s) = \bar{h}s$ covering)
- more complete Lyapunov-Krasovskiĭ functionals
  (complete Lyapunov-Krasovskiĭ from lecture 2, discretized Lyapunov-Krasovskiĭ, etc)
- tighter cross-term covering
  (like
  $$-2v_2'v_1 \le \begin{bmatrix} v_1' & v_2' \end{bmatrix} \begin{bmatrix} Q & QS \\ S'Q & (I + S'Q)Q^{-1}(I + QS) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
  instead of $-2v_2'v_1 \le v_1'Qv_1 + v_2'Q^{-1}v_2$, which corresponds to $S = 0$ here)

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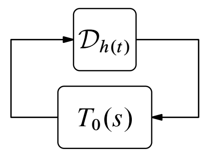
Time-varying delay



Here $\mathcal{D}_{h(t)}$ is time-varying delay operator for

$$0 \le h(t) \le \bar{h}, \quad \text{for some given } \bar{h}.$$

Might be inspired by

- networked control

Interesting fact (remember Homework 1):

- $\mathcal{D}_{h(t)}$ is not bounded as an operator $L^2 \mapsto L^2$,

no matter how small $\bar{h}$ is.

Key result

Theorem
Let $h(t) \in [0, \bar{h}]$, then linear operator

$$O : \zeta \mapsto \eta = \int_{t-h(t)}^{t} \zeta(\theta)\mathrm{d}\theta = \int_{-h(t)}^{0} \zeta(t+\theta)\mathrm{d}\theta$$

is bounded L^2 operator with $\|O\|_{L^2(\mathbb{R}^+)\mapsto L^2(\mathbb{R}^+)} = \bar{h}$.

Proof (scalar case). By Cauchy-Schwartz inequality,

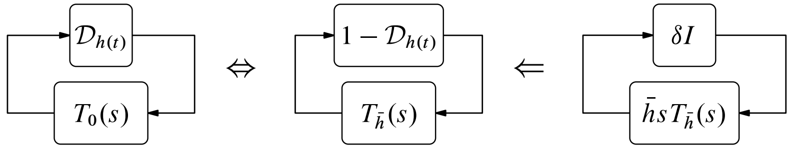
$$\eta^2(t) = \left(\int_{-h(t)}^{0} \zeta(t+\theta)\mathrm{d}\theta\right)^2 \le h(t)\int_{-h(t)}^{0} \zeta^2(t+\theta)\mathrm{d}\theta \le \bar{h}\int_{-\bar{h}}^{0} \zeta^2(t+\theta)\mathrm{d}\theta$$

so that

$$\|\eta\|_2^2 \le \int_0^\infty \bar{h}\int_{-\bar{h}}^{0} \zeta^2(t+\theta)\mathrm{d}\theta\mathrm{d}t = \bar{h}\int_{-\bar{h}}^{0}\int_0^\infty \zeta^2(t+\theta)\mathrm{d}t\mathrm{d}\theta = \bar{h}\int_{-\bar{h}}^{0} \|\zeta\|_2^2\mathrm{d}\theta$$

Thus, $\|\eta\|_2^2 \le \bar{h}^2\|\zeta\|_2^2$. □

Robust stability conditions



Reduces to robust stability for all time-varying δ such that

$$\|\delta\|_{L^2\mapsto L^2} \le 1.$$

Then original system stable for all $h(t) \in [0, \bar{h}]$ if

1. $T_{\bar{h}}(s) := -T_0(s)\big(I - T_0(s)\big)^{-1} \in H^\infty$
2. $\|\bar{h}sT_{\bar{h}}\|_\infty < 1$

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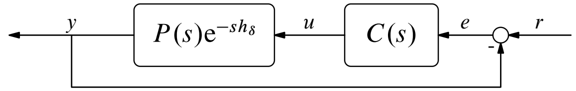
Achievable μ_d for $h_0 = 0$

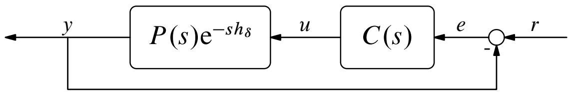


- What μ_d can be achieved by an appropriate choice of $C(s)$?

Equivalently, we'd like to know whether there is an upper bound[1] on μ_d.

[1] This part is based on (Middleton & Miller, 2007), *IEEE TAC*, **52**, pp. 1194–1207.

Stable plant

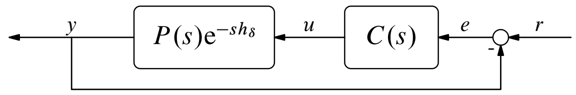


If $P \in H^\infty$, then

- μ_d is unbounded,

any $C \in H^\infty$ with $\|C\|_\infty < \frac{1}{\|P\|_\infty}$ does the job (by the Small Gain Theorem).

Examples (some from Homework 2 and Lecture 3)

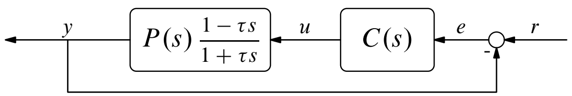


with $P(s) = \frac{1}{s-1}$ and

- PI controller $C(s) = k_\mathrm{p}\big(1 + \frac{k_\mathrm{i}}{s}\big)$, the achievable $\mu_\mathrm{d} < 1$
- PD controller $C(s) = k_\mathrm{p} + k_\mathrm{d}s$, the achievable $\mu_\mathrm{d} < 2$

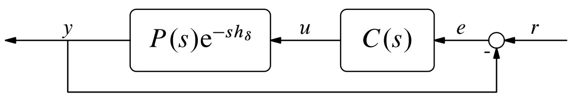
Can we do better?

Preliminaries: auxiliary system



Let $C(s)$ stabilize rational $P(s)$, then:

1. $C(s)$ stabilizes the closed-loop system for all sufficiently small τ;
2. if $C(s)$ does *not* stabilize the closed-loop system for all $\tau > 0$, $\exists\,\bar\tau > 0$ such that the closed-loop system is stable $\forall \tau \in [0, \bar\tau)$ and unstable for $\tau = \bar\tau$ with poles at $\pm j\bar\omega$ for some $\bar\omega > 0$;
3. the closed-loop system



unstable if $h_\delta = \dfrac{2\arctan(\bar\omega\bar\tau)}{\bar\omega} > 0$.

Real unstable pole

Theorem

Let $P(s)$ have a real pole at $s = a > 0$. Then

$$\mu_d < \frac{2}{a}.$$

Moreover, if $P(s)$ is minimum-phase and has no other unstable poles, then this upper bound is tight.

Proof (outline).

Auxiliary system unstable if $\tau = \frac{1}{a}$ (unstable cancellations). Hence, $\exists\,\bar\tau < \frac{1}{a}$ and $\bar\omega > 0$ such that $\pm j\bar\omega$ is a closed-loop pole. Hence,

$$\mu_\mathrm{d} < \frac{2\arctan(\bar\omega\bar\tau)}{\bar\omega} < \lim_{\bar\omega\to 0}\frac{2\arctan(\bar\omega\bar\tau)}{\bar\omega} = 2\bar\tau < \frac{2}{a}.$$

If $P(s) = \frac{1}{s-a}P_0(s)$ for some stable and minimum-phase $P_0(s)$, controller $C(s) = P_0^{-1}(s)(k_\mathrm{p} + k_\mathrm{d}s)$ (or its proper modification) does the job. □

Real unstable pole: bad news

It follows from the proof that

- the highest μ_d requires $\omega_\mathrm{c} \to 0$,

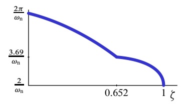
which renders the resulting design meaningless (no closed-loop bandwidth, zero μ_g and μ_ph). If bandwidth requirements are accounted for, even the $\frac{2}{a}$ bound might be very conservative :(

Pair of complex unstable poles

Theorem

Let $P(s)$ have a pair of poles at $s = (\zeta \pm \mathrm{j}\sqrt{1-\zeta^2})\omega_n$ for $\zeta \in [0,1)$, $\omega_n \geq 0$. Then

$$\mu_d < \frac{\sqrt{1-\zeta^2}}{\omega_n}\left(\pi + 2\max\left\{\frac{\zeta}{\sqrt{1-\zeta^2}}, \arctan\frac{\sqrt{1-\zeta^2}}{\zeta}\right\}\right) =$$



Proof (outline).

If $\zeta > 0$, similarly to the real pole case, modulo complex τ and lengthier technicalities.

If $\zeta = 0$, there must be a crossover $\omega_\mathrm{c} > \omega_\mathrm{n}$ with some $\mu_\mathrm{ph}^+ \in (0, 2\pi)$. Then

$$\mu_\mathrm{d} \leq \frac{\mu_\mathrm{ph}^+}{\omega_\mathrm{c}} < \frac{2\pi}{\omega_\mathrm{n}}. \qquad \square$$

Unstable poles only at the origin

Theorem

Let the only $\bar{\mathbb{C}}_0$ poles of $P(s)$ be those at the origin. Then μ_d can be made arbitrarily large.

Proof.

Exploits the fact that such systems can be stabilized with an arbitrarily low crossover... $\square$