Introduction to Time-Delay Systems

lecture no. 6

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Outline

Optimization-based design: introduction

Loop shifting for H^2 problem with loop delay

Loop shifting for H^∞ problem with loop delay

Preview control and estimation

Technical preliminaries

One-block example: L^2 optimization

One-block example: L^∞ optimization (Nehari problem)

Two-block example: L^2 optimization (self-study)

Two-block example: L^∞ optimization (self-study)

Some comparisons

Optimal control

Problem:

- minimize cost function (criterion) subject to constraints imposed by process dynamics

Hope:

- solution results in "good" (in whatever sense) control system

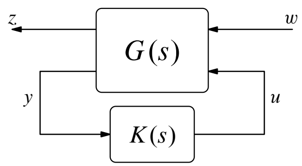
Advantages:

- analytic design method, with strong theoretic justification
- important byproducts (like stability, robustness, etc)

Things to remember:

- no criterion can ever reflect all our requirements
- more comprehensive cost functions result in less transparent solutions
- "optimal" might have nothing to do with "good"
- optimization should be used as a tool rather than as the goal

Generalized plant paradigm



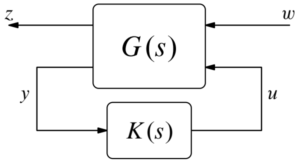
Systems:

- $G = \begin{bmatrix} G_{zw} & G_{zu} \\ G_{yw} & G_{yu} \end{bmatrix}$ is generalized plant (given components)
- K is controller (components we design)

Signals:

- w is exogenous input (reference, disturbances, noise, etc)
- u is control input (output of controller)
- z is regulated output (collection of signals we want to keep "small")
- y is measured output (input of controller)

Generalized plant paradigm (contd)



System-based performance measure:

- cost function is size (norm) of closed-loop system from w to z

Constraints imposed upon $K(s)$:

- proper (i.e., transfer function of causal system)
- stabilizing

Standard problem:

- given G, design proper and stabilizing $K(s)$ minimizing size of

$$T = \mathcal{F}_l(G, K) := G_{zw} + G_{zu}K(I - G_{yu}K)^{-1}G_{yw}$$

H^2 system norm

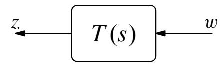
Define space

$$H^2 := \left\{ G(s) : G(s) \text{ analytic in } \mathbb{C}_0 \text{ and } \sup_{\sigma>0} \frac{1}{2\pi}\int_{-\infty}^{\infty} \|G(\sigma + j\omega)\|_F^2 d\omega < \infty \right\}$$

where $\|\cdot\|_F$ is Frobenius matrix norm. If $T \in H^2$, its H^2-norm is

$$\|T\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}[T^*(j\omega)T(j\omega)]d\omega$$

Signal interpretations:



In SISO case $\|T\|_2^2$ is

- energy of z when $w = \delta$ (energy of the impulse response of T)
- variance of z when w zero-mean unit intensity white noise

Example: LQR problem

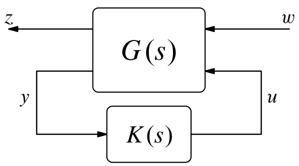
Given $\dot{x}(t) = Ax(t) + Bu(t)$ with initial condition $x(0) = x_0$, minimize

$$\mathcal{J} = \int_0^{\infty} \big(x'(t)Qx(t) + u'(t)Ru(t)\big)dt,$$

$Q \geq 0$ and $R > 0$, assuming that all state vector measured, i.e., $y(t) = x(t)$. Two things to notice:

1. $\mathcal{J} = \|z\|_2^2$, where $z := \begin{bmatrix} Q^{1/2}x \\ R^{1/2}u \end{bmatrix}$
2. system can be rewritten as $\dot{x}(t) = Ax(t) + x_0\delta(t) + Bu(t)$, $x(0) = 0$

Thus, LQR is H^2 standard problem



with $G(s) = \left[\begin{array}{c|c:c} A & x_0 & B \\ \hline Q^{1/2} & 0 & 0 \\ 0 & 0 & R^{1/2} \\ \hdashline I & 0 & 0 \end{array}\right]$

Example: Kalman filtering

Given $\dot{x}(t) = Ax(t) + v_x(t)$ and measurements $y(t) = Cx(t) + v_y(t)$, where v_x and v_y white Gaussian zero-mean stationary stochastic processes with

$$\mathcal{E}\{v_x(t)v_x'(\tau)\} = Q_x\delta(t-\tau) \quad \text{and} \quad \mathcal{E}\{v_y(t)v_y'(\tau)\} = Q_y\delta(t-\tau),$$

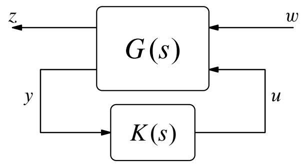
$Q_x \geq 0$ and $Q_y > 0$, estimate x so that estimation $\hat{x}$ minimizes cost function

$$\mathcal{J} = \mathrm{tr}\big[\mathcal{E}\big\{(x(\theta) - \hat{x}(\theta))(x(\theta) - \hat{x}(\theta))'\big\}\big].$$

One thing to notice:

- $v_x = Q_x^{1/2}w_1$ and $v_y = Q_y^{1/2}w_2$ for some white Gaussian zero-mean stationary unit intensity stochastic processes w_1 and w_2

Thus, Kalman filtering is H^2 standard problem



with $G(s) = \left[\begin{array}{c|cc:c} A & Q_x^{1/2} & 0 & 0 \\ \hline I & 0 & 0 & -I \\ \hdashline C & 0 & Q_y^{1/2} & 0 \end{array}\right]$

Outline

Preliminary: more on H^2 space

The H^2 space can be also thought of as

- the space of Laplace transforms of $L^2(\mathbb{R}^+)$ functions.

It is a Hilbert space with the inner product

$$\langle G_1, G_2 \rangle_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{tr}\{[G_2(\mathrm{j}\omega)]^* G_1(\mathrm{j}\omega)\}\, \mathrm{d}\omega.$$

By Parseval, the inner product has the following time-domain form as well:

$$\langle G_1, G_2 \rangle_2 = \int_0^{\infty} \operatorname{tr}\{g_2'(\tau) g_1(\tau)\}\, \mathrm{d}\tau$$

(the impulse response $g(\tau)$ of $G \in H^2$ must be zero in $\tau < 0$).

Preliminary: inner transfer function

Transfer function $G \in H^\infty$ is said to be inner if

$$G^\sim(s)G(s) = I \quad \text{or} \quad [G(\mathrm{j}\omega)]^* G(\mathrm{j}\omega) = I,$$

where conjugate system $G^\sim(s) := [G(-s)]'$. In the scalar case inner means stable with unit magnitude for all frequencies (as $G^\sim(\mathrm{j}\omega) = \overline{G(\mathrm{j}\omega)}$). Clearly

- delay e^{-sh} is inner.

Important property of inner functions is that they are

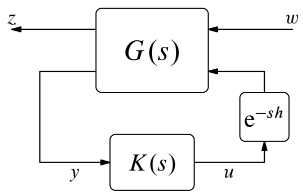
- energy preserving,

i.e., if $y = Gu$ for an inner G, then $\|y\|_2 = \|u\|_2$ for all $u \in L^2(\mathbb{R})$. Hence,

- multiplication by inner system preserves both L^2 and L^∞ norms,

i.e., if G inner, then both $\|GT\|_2 = \|T\|_2$ and $\|GT\|_\infty = \|T\|_\infty$.

Problem



Given G and $h \ge 0$, design proper stabilizing $K(s)$ minimizing

$$\|T\|_2,$$

where

$$\begin{aligned} T := \mathcal{F}_l(G, \mathrm{e}^{-sh}K) &= G_{zw} + G_{zu}\mathrm{e}^{-sh}K\big(I - G_{yu}\mathrm{e}^{-sh}K\big)^{-1}G_{yw} \\ &= G_{zw} + \mathrm{e}^{-sh}G_{zu}K\big(I - G_{yu}\mathrm{e}^{-sh}K\big)^{-1}G_{yw}. \end{aligned}$$

(as e^{-sh} commutes with G_{zu}).

Handling $G_{yu}\mathrm{e}^{-sh}$

Should be elementary by now (loop shifting). Indeed, the use of

$$K = \tilde{K}(I - \Pi\tilde{K})^{-1}, \quad \text{for } \Pi = \boldsymbol{\pi}_h\{G_{yu}\mathrm{e}^{-sh}\} = \tilde{G}_{yu} - G_{yu}\mathrm{e}^{-sh} \in H^\infty,$$

preserves internal stability and does the trick:

$$K(I - G_{yu}\mathrm{e}^{-sh}K)^{-1} = \tilde{K}(I - \tilde{G}_{yu}\tilde{K})^{-1}.$$

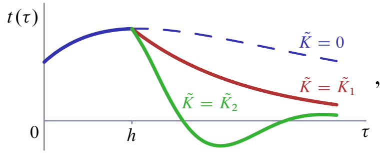
Hence,

$$T = G_{zw} + \mathrm{e}^{-sh}G_{zu}\tilde{K}(I - \tilde{G}_{yu}\tilde{K})^{-1}G_{yw}$$

and we have only one delay to handle.

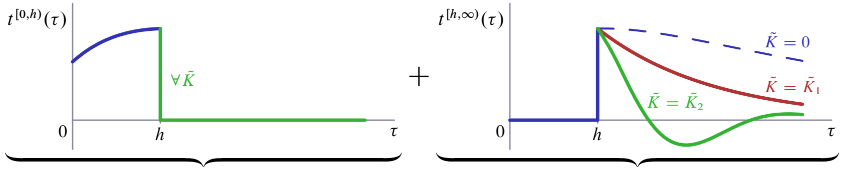
Structure of the impulse response of T

The impulse response of $T = G_{zw} + \mathrm{e}^{-sh}G_{zu}\tilde{K}(I - \tilde{G}_{yu}\tilde{K})^{-1}G_{yw}$,

$t(\tau) =$ 

,

can be split into two parts as

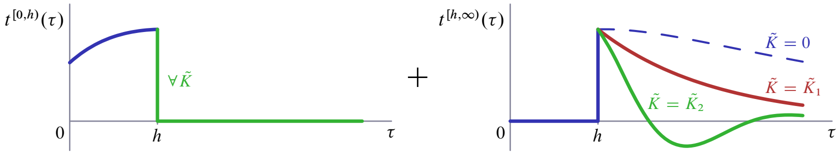
$t(\tau) =$ 

$t^{[0,h)}(\tau) = g_{zw}(\tau)\mathbb{1}_{[0,h)}(\tau)$ (independent of $\tilde{K}$) $\qquad$ $t^{[h,\infty)}(\tau)$ (depends on $\tilde{K}$)

where g_{zw} is the impulse response of G_{zw}.

Closer look at $t^{[0,h)}$

In

$t(\tau) =$ 

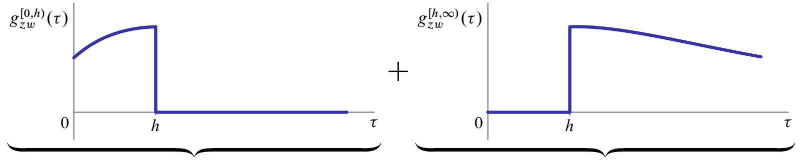
the term

- $t^{[0,h)}$ depends only on G_{zw}

(in fact it is merely the truncation of $g_{zw}(\tau)$ to $[0, h)$).

Decomposition of G_{zw}

Split

$g_{zw}(\tau) =$ 

$g_{zw}^{[0,h)}(\tau) := g_{zw}(\tau)\mathbb{1}_{[0,h)}(\tau)$ $\qquad$ $g_{zw}^{[h,\infty)}(\tau) := g_{zw}(\tau)\mathbb{1}_{[h,\infty)}(\tau)$

which corresponds to the decomposition

$$G_{zw}(s) = G_{zw}^{[0,h)}(s) + \mathrm{e}^{-sh}\hat{G}_{zw}(s)$$

where

- $G_{zw}^{[0,h)}$ is an FIR system the impulse response of which is $g_{zw}(\tau)\mathbb{1}_{[0,h)}(\tau)$
- $\hat{G}_{zw}$ is such that $G_{zw}^{[0,h)} = \boldsymbol{\pi}_h\{\mathrm{e}^{-sh}\hat{G}_{zw}\}$ (hence, $\hat{G}_{zw}(s)$ is rational)

We denote

- $\boldsymbol{\tau}_h\{G_{zw}(s)\} := G_{zw}^{[0,h)}(s)$ and call it *h-truncation of G_{zw}*.

Decomposition of T

Thus, we may write

$$T = \boldsymbol{\tau}_h\{G_{zw}\} + \underbrace{\mathrm{e}^{-sh}(\hat{G}_{zw} + G_{zu}\tilde{K}(I - \tilde{G}_{yu}\tilde{K})^{-1}G_{yw})}_{\hat{T}}$$

and then:

Lemma

Whenever $\tilde{K}$ is such that $\hat{T} \in H^2$, we have $\boldsymbol{\tau}_h\{G_{zw}\} \perp \hat{T}$.

Proof.

The inner product on H^2 is

$$\begin{aligned}\langle \boldsymbol{\tau}_h\{G_{zw}\}, \hat{T}\rangle_2 &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}\big([\boldsymbol{\tau}_h\{G_{zw}\}(\mathrm{j}\omega)]^*\hat{T}(\mathrm{j}\omega)\big)\mathrm{d}\omega \\ &= \int_0^{\infty} \operatorname{tr}\big(g_{zw}^{[0,h)}(\theta)'\hat{t}(\theta)\big)\mathrm{d}\theta \qquad \text{(Parseval)} \\ &= 0\end{aligned}$$

because impulse responses of $\boldsymbol{\tau}_h\{G_{zw}\}$ and $\hat{T}$ have disjoint supports. □

Norm of T

By Pythagoras, orthogonality implies that whenever $\hat{T} \in H^2$

$$\|T\|_2^2 = \|\boldsymbol{\tau}_h\{G_{zw}\}\|_2^2 + \|\hat{T}\|_2^2$$

where $\boldsymbol{\tau}_h\{G_{zw}\} \in H^2$ because $g_{zw}^{[0,h)} \in L^2(\mathbb{R}^+)$ (bounded and finite support). Moreover,

- e^{-sh} is inner

so that

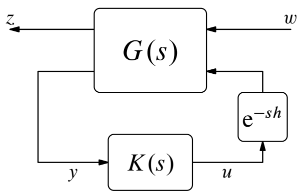
$$\begin{aligned}\|T\|_2^2 &= \|\boldsymbol{\tau}_h\{G_{zw}\}\|_2^2 + \|\hat{G}_{zw} + G_{zu}\tilde{K}(I - \tilde{G}_{yu}\tilde{K})^{-1}G_{yw}\|_2^2 \\ &= \|\boldsymbol{\tau}_h\{G_{zw}\}\|_2^2 + \|\mathcal{F}_l(\tilde{G}, \tilde{K})\|_2^2,\end{aligned}$$

where

$$\tilde{G} := \begin{bmatrix} \hat{G}_{zw} & G_{zu} \\ G_{yw} & \tilde{G}_{yu} \end{bmatrix}$$

is rational.

Solution of the standard H^2 problem with input delay



Summarizing, the following result can be formulated:

Theorem

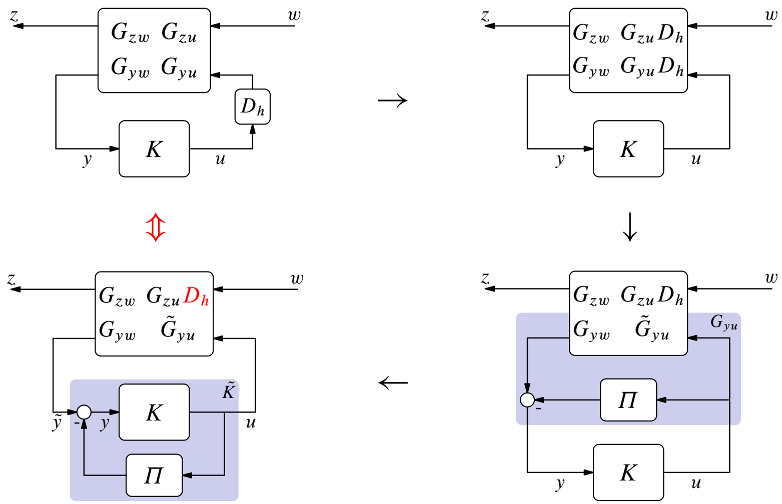
There exists a finite-dimensional $\tilde{G}$ such that the optimal

$$K_{opt} = \tilde{K}_{opt}(I - \boldsymbol{\pi}_h\{\mathrm{e}^{-sh}G_{yu}\}\tilde{K}_{opt})^{-1},$$

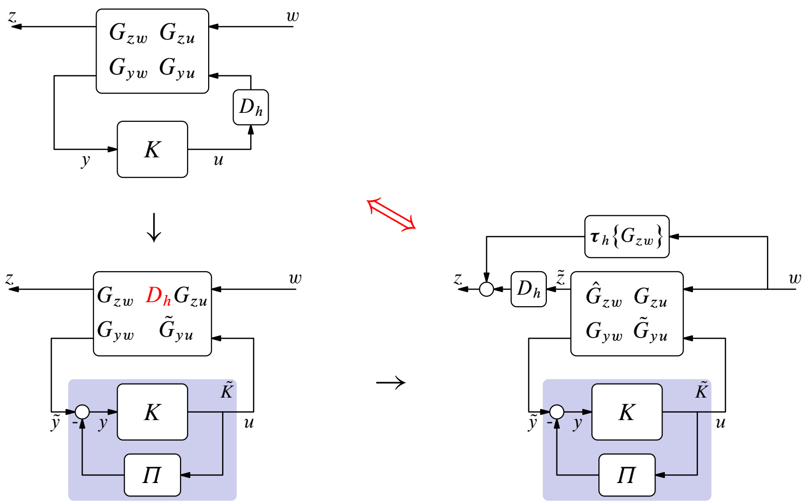
where $\tilde{K}_{opt}$ solves the standard delay-free H^2 problem for $\tilde{G}$. The optimal

$$\|T\|_2^2 = \|\boldsymbol{\tau}_h\{G_{zw}\}\|_2^2 + \|\mathcal{F}_l(\tilde{G}, \tilde{K}_{opt})\|_2^2.$$

Loop-shifting cartoon



Loop-shifting cartoon (contd)



- the optimal controller is a DTC (the modified Smith predictor).

Truncation in state space

If

$$g_{zw}(\tau)=\begin{cases}0 & \tau<0\\ C_z e^{A\tau}B_w & \tau\ge 0\end{cases} \quad\Rightarrow\quad g_{zw}^{[0,h)}(\tau)=\begin{cases}0 & \tau<0\ \&\ \tau\ge h\\ C_z e^{A\tau}B_w & 0\le\tau<h\end{cases}$$

and then

$$g_{zw}(\tau)-g_{zw}^{[0,h)}(\tau)=\begin{cases}0 & \text{if } \tau<h\\ C_z e^{A\tau}B_w = C_z e^{Ah}e^{A(\tau-h)}B_w & \text{if } \tau\ge h\end{cases}$$

Thus, if

$$G_{zw}(s)=\left[\begin{array}{c|c}A & B_w\\ \hline C_z & 0\end{array}\right] \quad\Longrightarrow\quad \hat G_{zw}(s)=\left[\begin{array}{c|c}A & B_w\\ \hline C_z e^{Ah} & 0\end{array}\right]$$

and then

$$\|\tau_h\{G_{zw}(s)\}\|_2^2=\operatorname{tr}\Bigl[C_z\int_0^h e^{A\theta}B_wB_w'e^{A'\theta}d\theta\, C_z'\Bigr].$$

State-space formula for $\tilde G$

If

$$G=\left[\begin{array}{c|cc}A & B_w & B_u\\ \hline C_z & 0 & D_{zu}\\ C_y & D_{yw} & 0\end{array}\right],$$

we have that

$$\tilde G_{yu}=\left[\begin{array}{c|c}A & e^{-Ah}B_u\\ \hline C_y & 0\end{array}\right] \quad\text{and}\quad \hat G_{zw}=\left[\begin{array}{c|c}A & B_w\\ \hline C_z e^{Ah} & 0\end{array}\right].$$

This yields (after similarity transformation with e^{Ah} for either G_{zu} or G_{yw})

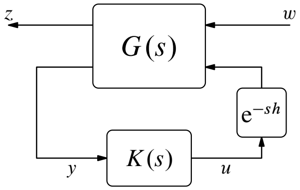
$$\tilde G=\left[\begin{array}{c|cc}A & B_w & e^{-Ah}B_u\\ \hline C_z e^{Ah} & 0 & D_{zu}\\ C_y & D_{yw} & 0\end{array}\right]=\left[\begin{array}{c|cc}A & e^{Ah}B_w & B_u\\ \hline C_z & 0 & D_{zu}\\ C_y e^{-Ah} & D_{yw} & 0\end{array}\right]$$

which has the same dimension and structure[1] as G.

[1] In the sense that standard assumptions hold for $\tilde G$ iff they hold for G.

Outline

Problem



Given G and $h \geq 0$, design proper stabilizing K such that

$$\|T\|_\infty < \gamma \qquad \text{for given } \gamma > 0,$$

where

$$T := \mathcal{F}_l\big(G, e^{-sh}K\big) = G_{zw} + G_{zu}e^{-sh}K\big(I - G_{yu}e^{-sh}K\big)^{-1}G_{yw}.$$

Loop shifting

We already know that if $K = \tilde{K}(I - \Pi\tilde{K})^{-1}$ for $\Pi = \boldsymbol{\pi}_h\{e^{-sh}G_{yu}\}$,

$$T = \boldsymbol{\tau}_h\{G_{zw}\} + e^{-sh}\big(\hat{G}_{zw} + G_{zu}\tilde{K}(I - \tilde{G}_{yu}\tilde{K})^{-1}G_{yw}\big).$$

In the H^2 case we just dropped the first term from the optimization process (as H^2 is Hilbert space and the Projection Theorem applied). Question is

- whether this policy is reasonable in the H^∞ case?

The answer is negative, because

- H^∞ is not a Hilbert space)-:

Example

Consider

$$T = \boldsymbol{\tau}_h\{\tfrac{1}{s}\} + e^{-sh}Q = \tfrac{1-e^{-sh}}{s} + e^{-sh}Q.$$

For $Q = 0$ (H^2-optimal) we have

$$\|T\|_\infty = \max_{\omega\in\mathbb{R}^+} \frac{|1 - e^{-j\omega h}|}{\omega} = \max_{\omega\in[0,2\pi/h]} \frac{\sqrt{2(1-\cos(\omega h))}}{\omega} = |T(0)| = h.$$

Now, let

$$Q = Q_\infty := \frac{1}{s} - \frac{(2h)^2s^2 + \pi^2}{\pi s(2hs + \pi e^{-sh})} \in H^\infty$$

so that

$$T = \frac{2h}{\pi}\frac{\pi - 2hse^{-sh}}{2hs + \pi e^{-sh}} = \frac{2h}{\pi}\underbrace{e^{-sh}\frac{(2hs + \pi e^{-sh})^\sim}{2hs + \pi e^{-sh}}}_{\text{inner}}$$

and $\|T\|_\infty = \frac{2}{\pi}h$, which is some 64% of what we achieved with $Q = 0$. As a matter of fact, $Q = Q_\infty$ is the optimal solution.

Sometimes it works

This happens in special case when $G_{zw} = 0$. Then

$$T = e^{-sh}\big(G_{zu}\tilde{K}(I - \tilde{G}_{yu}\tilde{K})^{-1}G_{yw}\big)$$

and

$$\|T\|_\infty = \|G_{zu}\tilde{K}(I - \tilde{G}_{yu}\tilde{K})^{-1}G_{yw}\|_\infty,$$

which is rational problem. Thus, original problem in this case

- solved by modified Smith predictor

too.

Application to robust stability analysis

Some robust stability problems cast as H^∞ problems with $G_{zw} \equiv 0$:

Additive uncertainty $P = P_0 + W_2 \Delta W_1 = \mathcal{F}_u(G, \Delta)$ with

$$G = \begin{bmatrix} 0 & W_1 \\ W_2 & P_0 \end{bmatrix}$$

Input multiplicative uncertainty $P = P_0(I + W_2 \Delta W_1) = \mathcal{F}_u(G, \Delta)$ with

$$G = \begin{bmatrix} 0 & W_1 \\ P_0 W_2 & P_0 \end{bmatrix}$$

Output multiplicative uncertainty $P = (I + W_2 \Delta W_1) P_0 = \mathcal{F}_u(G, \Delta)$ with

$$G = \begin{bmatrix} 0 & W_1 P_0 \\ W_2 & P_0 \end{bmatrix}$$

Closed loop of P with controller K robustly stable against all $\|\Delta\|_\infty \le \alpha$ iff $\|\mathcal{F}_l(G, K)\|_\infty < \frac{1}{\alpha}$ (this is application of the Small Gain Theorem).

Application to robust stability analysis (contd)

With the use of DTC-based controller,

$$\left\| \mathcal{F}_l\left(\begin{bmatrix} 0 & G_{zu} \\ G_{yw} & G_{yu} \end{bmatrix}, \mathrm{e}^{-sh} K \right) \right\|_\infty = \left\| \mathcal{F}_l\left(\begin{bmatrix} 0 & G_{zu} \\ G_{yw} & \tilde{G}_{yu} \end{bmatrix}, \tilde{K} \right) \right\|_\infty$$

for every $K = \tilde{K}(I - \Pi \tilde{K})^{-1}$ and $\tilde{G}_{yu}$ such that $\Pi := \tilde{G}_{yu} - \mathrm{e}^{-sh} G_{yu} \in H^\infty$.

If $G_{yu} \in H^\infty$ we can always choose $\tilde{G}_{yu} = G_{yu}$, which implies that

- Smith controller with primary part $\tilde{K}$ has same robustness level against additive / multiplicative uncertainty as delay-free loop with $K = \tilde{K}$.

If $G_{yu} \notin H^\infty$, $\tilde{G}_{yu} \ne G_{yu}$ and comparison is less tangible. Nevertheless, we can safely say that

- best robustness level brought about by DTC-based controllers,

which might appear counterintuitive (after all, DTCs cancel dynamics).

And what if $G_{zw} \ne 0$?

Solution is still a DTC, but now with

$$\Pi = \pi_h\{\mathrm{e}^{-sh}(G_{yu} + G_{yw}(\gamma^2 I - G_{zw}^\sim G_{zw})^{-1} G_{zw}^\sim G_{zu})\}$$

and can be interpreted as

- DTC under the worst-case disturbance for the open-loop system

after all, *the best way to predict the future is to invent it* (Alan Kay).

For example, the mixed sensitivity problem having the generalized plant

$$G(s) = \left[\begin{array}{c:c} W_\sigma(s) & -W_\sigma(s)P(s) \\ 0 & W_\varkappa(s) \\ \hdashline 1 & -P(s) \end{array}\right]$$

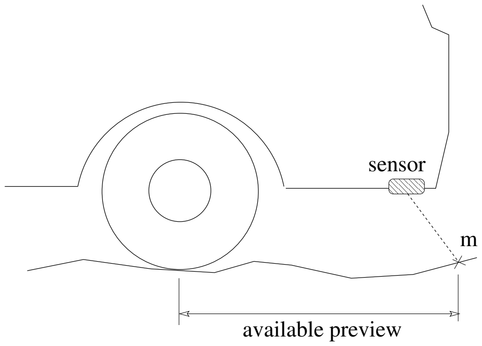
results in

$$\Pi(s) = -\pi_h\left\{ \frac{1}{1 - \gamma^{-2} W_\sigma^\sim(s) W_\sigma(s)} P(s)\,\mathrm{e}^{-sh} \right\},$$

which might have a complicated pattern of removable singularities.

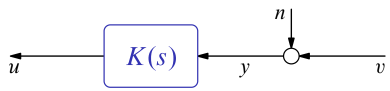
Outline

Active suspension



- road disturbances can be measured with preview

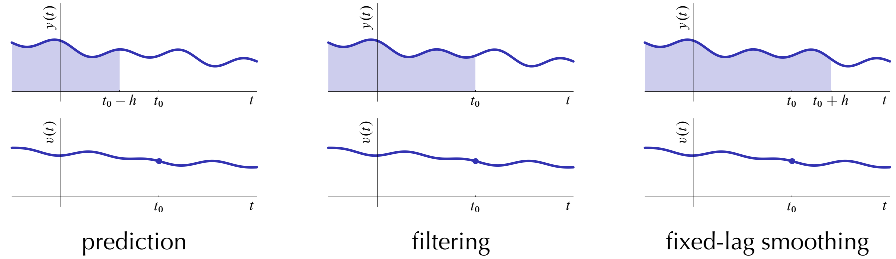
Estimation problems



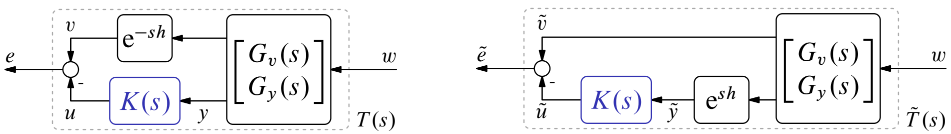
Problem:

- reconstruct v from noisy measurements y by a stable $K(s)$

Information patterns:



Fixed-lag smoothing setups



Error system:

$$T(s) = \mathrm{e}^{-sh} G_v(s) - K(s) G_y(s) \quad \text{or} \quad \tilde{T}(s) = G_v(s) - K(s)\mathrm{e}^{sh} G_y(s)$$

Because

$$T(s) = \mathrm{e}^{-sh}\tilde{T}(s) \quad \text{and } \mathrm{e}^{-sh} \text{ is inner,}$$

these two setups are essentially equivalent and

- fixed-lag smoothing is also a preview problem.

Outline

Two-sided Laplace transform

If $f(t) : \mathbb{R} \to \mathbb{C}$, its Laplace transform is defined as

$$F(s) = \mathcal{L}\{f\} := \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-st} \mathrm{d}t$$

for those $s \in \mathbb{C}$ for which this integral exists (region of convergence).

Control theory mainly studies causal systems, in which case signals may be assumed to satisfy

$$f(t) = 0, \quad \forall t < 0$$

and the one-sided transform ($\int_0^\infty \cdots$) is enough. But in studying non-causal systems we may no longer assume that.

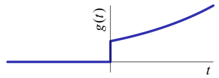
Unstable or non-causal ?

Consider a system $\mathcal{G}$ with the transfer function

$$G(s) = \frac{1}{s-1}.$$

We know that $G(s)$ is the Laplace transform of the impulse response $g(t)$ of $\mathcal{G}$. Can we safely say that $\mathcal{G}$ is causal and unstable with

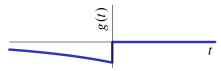
$$g(t) = \mathrm{e}^t \mathbb{1}_{[0,\infty)}(t) =$$



?

Not necessarily, as anti-causal and stable

$$g(t) = -\mathrm{e}^t \mathbb{1}_{(-\infty,0]}(t) =$$



also produces the same $G(s)$. The difference in the regions of convergence:

- the former exists in $\mathbb{C}_1$, whereas the latter—in $\mathbb{C} \setminus \bar{\mathbb{C}}_1$

Unstable or non-causal ? (contd)

If we have the transfer function

$$G(s) = \frac{1}{s-1},$$

we may (at least in open-loop settings) interpret it as the transfer function of either an

- unstable causal system with impulse response $g(t) = \mathrm{e}^t \mathbb{1}_{[0,\infty)}(t)$

or a

- stable anti-causal system with impulse response $g(t) = -\mathrm{e}^t \mathbb{1}_{(-\infty,0]}(t)$

$L^2(\mathbb{R})$ space

Consists of bounded-energy functions, i.e., such that

$$\|f\|_{L^2(\mathbb{R})} := \left(\int_{-\infty}^{\infty} \|f(t)\|^2 \mathrm{d}t\right)^{1/2} < \infty.$$

With some abuse of notation, by $L^2(\mathbb{R}^+)$ ($L^2(\mathbb{R}^-)$) we denote the subspace of $L^2(\mathbb{R})$ consisting of functions such that $f(t) = 0$ whenever $t < 0$ ($t > 0$).

$$L^2(\mathbb{R}) = L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^-)$$

$L^2(\mathrm{j}\mathbb{R})$ space

(or L^2) is the space of all functions $F : \mathrm{j}\mathbb{R} \to \mathbb{C}^n$ such that

$$\|F\|_2 := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\|F(\mathrm{j}\omega)\|_{\mathrm{F}}^2\,\mathrm{d}\omega\right)^{1/2} < \infty$$

It is a Hilbert space with the inner product

$$\langle F_1, F_2\rangle_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{tr}\big\{[F_2(\mathrm{j}\omega)]^* F_1(\mathrm{j}\omega)\big\}\,\mathrm{d}\omega.$$

It is also the space of Fourier transforms of $L^2(\mathbb{R})$ functions $f(t)$. Subspaces:

H^2: Fourier transforms of $L^2(\mathbb{R}^+)$ functions
(such functions are Laplace transformable, with the region of convergence in $\mathbb{C}_0$; hence, H^2 functions exist and analytic in $\mathrm{Re}\,s > 0$)

$H^2_\perp$: Fourier transforms of $L^2(\mathbb{R}^-)$ functions
(such functions are Laplace transformable, with the region of convergence in $\mathbb{C} \setminus \bar{\mathbb{C}}_0$; hence, $H^2_\perp$ functions exist and analytic in $\mathrm{Re}\,s < 0$)

$L^2(\mathrm{j}\mathbb{R})$ space (contd)

From definitions above,

$$L^2 = H^2 \oplus H^2_\perp.$$

Moreover, for any $F \in L^2$, its projections onto H^2 and $H^2_\perp$ are

$$\mathrm{proj}_{H^2} F = \mathcal{L}\{(I - \Pi_0)f\} \quad \text{and} \quad \mathrm{proj}_{H^2_\perp} F = \mathcal{L}\{\Pi_0 f\},$$

where Π_0 is the truncation operator defined in Lect. 1.

It is readily seen that if $F(\mathrm{j}\omega)$ is the frequency response of a system $\mathcal{F}$, then

- $\mathrm{proj}_{H^2} F$ yields the transfer function of its causal part;
- $\mathrm{proj}_{H^2_\perp} F$ yields the transfer function of its anti-causal part.

By Parseval,

$$\|F\|_2 = \|f\|_{L^2(\mathbb{R})}.$$

L^2 norm of $H^2_\perp$ systems

Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \in H^2_\perp$ (i.e., anti-causal and $-A$ is Hurwitz). Then,

$$\begin{aligned}\|G\|_2^2 &= \int_{-\infty}^{0}\mathrm{tr}\{g'(t)g(t)\}\mathrm{d}t = \int_{-\infty}^{0}\mathrm{tr}\{B'\mathrm{e}^{A't}C'C\mathrm{e}^{At}B\}\mathrm{d}t = \mathrm{tr}\{B'W_\mathrm{o}B\} \\ &= \int_{-\infty}^{0}\mathrm{tr}\{g(t)g'(t)\}\mathrm{d}t = \int_{-\infty}^{0}\mathrm{tr}\{C\mathrm{e}^{At}BB'\mathrm{e}^{A't}C'\}\mathrm{d}t = \mathrm{tr}\{CW_\mathrm{c}C'\}\end{aligned}$$

where W_c and W_o solve Lyapunov equations

$$-AW_\mathrm{c} - W_\mathrm{c}A' + BB' = 0 \quad \text{and} \quad -A'W_\mathrm{o} - W_\mathrm{o}A + C'C = 0.$$

In particular, if $a > 0$

$$\left\|\frac{b}{s-a}\right\|_2 = \frac{|b|}{\sqrt{2a}}.$$

$L^\infty(\mathrm{j}\mathbb{R})$ space

(or L^∞) is the space of all functions $F : \mathrm{j}\mathbb{R} \to \mathbb{C}^n$ such that

$$\|F\|_\infty := \sup_{\omega\in\mathbb{R}}\bar{\sigma}\{F(\mathrm{j}\omega)\} < \infty$$

It can be shown that a system $\mathcal{G}$ is a bounded operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ iff its frequency response $G \in L^\infty$. Moreover,

$$\|\mathcal{G}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \|F\|_\infty.$$

Thus, L^2 comprises frequency responses of all $L^2(\mathbb{R})$-stable systems.

It can also be shown that $H^\infty \subset L^\infty$ and comprises the transfer functions of all causal and $L^2(\mathbb{R})$-stable systems.

In the rational case[2] $H^2 \subset H^\infty$, i.e., all H^2 systems are stable.

[2] This is not true in general, i.e., the H^2 system with $g(t) = \mathrm{sinc}(t)\mathbb{1}_{\mathbb{R}^+}(t)$ is unstable.

Some relations

time domain: $L^2(\mathbb{R}) = L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^-)$

Fourier ↓ ↓ Laplace ↓ Laplace

frequency domain: $L^2 \;=\; H^2 \;\oplus\; H^2_\perp$

Also

- if $G \in L^\infty$, then $GL^2 \subset L^2$
- if $G \in H^\infty$, then $GH^2 \subset H^2$

Hankel norm

Let $G \in H^\infty$. Its Hankel norm is

$$\|G\|_{\mathrm{H}} := \sup_{u \in H^2_\perp} \frac{\|\mathrm{proj}_{H^2} Gu\|_2}{\|u\|_2} = \sup_{u \in H^2} \frac{\|\mathrm{proj}_{H^2_\perp} G^\sim u\|_2}{\|u\|_2}$$

i.e., it is its $L^2(\mathbb{R}^-) \to L^2(\mathbb{R}^+)$ induced norm. If $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$, then

$$\|G\|_{\mathrm{H}} = \sqrt{\rho(W_{\mathrm{c}} W_{\mathrm{o}})},$$

where W_{c} and W_{o} are controllability and observability Gramians verifying

$$AW_{\mathrm{c}} + W_{\mathrm{c}}A' + BB' = 0 \quad \text{and} \quad A'W_{\mathrm{o}} + W_{\mathrm{o}}A + C'C = 0$$

In particular, if $a > 0$,

$$\left\| \frac{b}{s+a} \right\|_{\mathrm{H}} = \frac{|b|}{2a}.$$

Outline

Optimization-based design: introduction

Loop shifting for H^2 problem with loop delay

Loop shifting for H^∞ problem with loop delay

Preview control and estimation

Technical preliminaries

One-block example: L^2 optimization

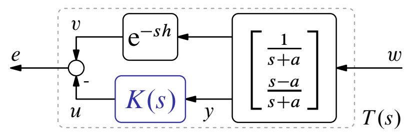
One-block example: L^∞ optimization (Nehari problem)

Two-block example: L^2 optimization (self-study)

Two-block example: L^∞ optimization (self-study)

Some comparisons

Setup



(i.e., $G_v(s) = \frac{1}{s+a}$ and $G_y(s) = \frac{s-a}{s+a}$) for some $a > 0$. Error system:

$$T(s) = \frac{e^{-sh}}{s+a} - K(s)\frac{s-a}{s+a}$$

and the error system is stable for every stable $K(s)$. The problem is to

- find stable and causal K minimizing L^2-norm $\|T\|_2$.

Conversion to a distance problem

Rewrite

$$T(s) = \underbrace{\left(\frac{1}{s-a}\mathrm{e}^{-sh} - K(s)\right)}_{T_{\mathrm{a}}(s)} \frac{s-a}{s+a}$$

As $\frac{s-a}{s+a}$ is inner,

$$\|T\|_2 = \|T_{\mathrm{a}}\|_2$$

so the problem becomes[3]

$$\min_{K \in H^2} \left\| \frac{1}{s-a}\mathrm{e}^{-sh} - K \right\|_2,$$

which is the problem of finding the distance from $\frac{1}{s-a}\mathrm{e}^{-sh} \in L^2$ to H^2.

[3] Should be done with some care as $H^2 \not\subset H^\infty$ in general (but it is for rational+delays).

Tadmor's reduction

We know that

$$\frac{1}{s-a}\mathrm{e}^{-sh} = \frac{\mathrm{e}^{-ah}}{s-a} - \boldsymbol{\pi}_h\left\{\frac{1}{s-a}\mathrm{e}^{-sh}\right\},$$

so that

$$T_{\mathrm{a}}(s) = \frac{\mathrm{e}^{-ah}}{s-a} - \left(K(s) + \boldsymbol{\pi}_h\left\{\frac{1}{s-a}\mathrm{e}^{-sh}\right\}\right).$$

Denoting

$$K_{\mathrm{a}}(s) := K(s) + \boldsymbol{\pi}_h\left\{\frac{1}{s-a}\mathrm{e}^{-sh}\right\}$$

and noting that $K_{\mathrm{a}} \in H^2$ iff $K \in H^2$, the distance problem can be cast as

$$\min_{K_{\mathrm{a}} \in H^2} \left\| \frac{\mathrm{e}^{-ah}}{s-a} - K_{\mathrm{a}} \right\|_2,$$

which is a delay-free distance problem from an $H_\perp^2$ function to H^2.

Solution

By the Projection Theorem, the optimal

$$K_{\mathrm{a}} = \operatorname*{proj}_{H^2} \frac{\mathrm{e}^{-ah}}{s-a} = 0 \quad \text{and} \quad \|T_{\mathrm{a}}\|_2 = \left\|\frac{\mathrm{e}^{-ah}}{s-a}\right\|_2 = \frac{\mathrm{e}^{-ah}}{\sqrt{2a}}.$$

Thus,

$$K_{\mathrm{opt}}(s) = -\boldsymbol{\pi}_h\left\{\frac{1}{s-a}\mathrm{e}^{-sh}\right\}$$

and the optimal performance

$$\|T\|_2 = \frac{\mathrm{e}^{-ah}}{\sqrt{2a}}$$

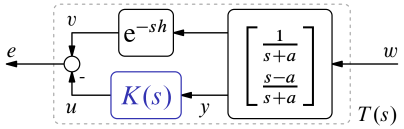
is an exponentially decreasing function of h, with $\lim_{h\to\infty}\|T\|_2 = 0$. I.e.,

- preview improves L^2 estimation performance,

alleviating the effect of the nonminimum-phase zero (canceling it if $h = \infty$).

Outline

Setup



(i.e., $G_v(s) = \frac{1}{s+a}$ and $G_y(s) = \frac{s-a}{s+a}$) for some $a > 0$. Error system:

$$T(s) = \frac{e^{-sh}}{s+a} - K(s)\frac{s-a}{s+a}$$

and the error system is stable for every stable $K(s)$. The problem is to

- find stable and causal K minimizing L^∞-norm $\|T\|_\infty$.

Conversion to a (delay-free) distance problem

Rewrite

$$T(s) = \underbrace{\left(\frac{1}{s-a}e^{-sh} - K(s)\right)}_{T_a(s)}\frac{s-a}{s+a}$$

As $\frac{s-a}{s+a}$ is inner,

$$\|T\|_\infty = \|T_a\|_\infty$$

so the problem becomes

$$\min_{K \in H^\infty}\left\|\frac{1}{s-a}e^{-sh} - K\right\|_\infty = \min_{K \in H^\infty}\left\|\frac{e^{-ah}}{s-a} - K_a\right\|_\infty,$$

where we used Tadmor's reduction procedure to end up with

- the problem of finding the distance from $\frac{e^{-ah}}{s-a} \in L^\infty$ to H^∞

known as the Nehari problem.

Delay-free Nehari problem

Let $G(s)$ be strictly proper rational transfer function of an anti-causal system (in particular, $G^\sim \in H^\infty$). Then

$$\min_{K \in H^\infty}\|G - K\|_\infty = \|G^\sim\|_H,$$

The optimal $K(s)$ is then an RH^∞ transfer function.

Proof (outline).

$$\|G-K\|_\infty = \sup_{u \in L^2(j\mathbb{R})}\frac{\|(G-K)u\|_2}{\|u\|_2} \geq \sup_{u \in H^2}\frac{\|(G-K)u\|_2}{\|u\|_2}$$
$$\geq \sup_{u \in H^2}\frac{\|\mathrm{proj}_{H^2_\perp}(G-K)u\|_2}{\|u\|_2} = \sup_{u \in H^2}\frac{\|\mathrm{proj}_{H^2_\perp} Gu\|_2}{\|u\|_2} = \|G^\sim\|_H$$

so that $\|G-K\|_\infty \geq \|G^\sim\|_H$ for any $K \in H^\infty$. Then $K \in H^\infty$ attaining the equality can be constructed. □

Solution

Hence,

$$\min_{K_a \in H^\infty}\left\|\frac{e^{-ah}}{s-a} - K_a\right\|_\infty = \left\|\frac{e^{-ah}}{s+a}\right\|_H = \frac{e^{-ah}}{2a}.$$

In fact, the optimal $K_{a,\mathrm{opt}}(s) = -\frac{e^{-ah}}{2a}$. This can be seen from the equality

$$\frac{e^{-ah}}{s-a} - K_{a,\mathrm{opt}}(s) = \frac{e^{-ah}}{s-a} + \frac{e^{-ah}}{2a} = \frac{s+a}{s-a}\,\frac{e^{-ah}}{2a},$$

which is all-pass and thus $\|T_a\|_\infty = e^{-ah}/(2a)$. Thus,

$$K_{\mathrm{opt}}(s) = -\frac{e^{-ah}}{2a} - \pi_h\left\{\frac{1}{s-a}e^{-sh}\right\},$$

and the optimal performance

$$\|T\|_\infty = \frac{e^{-ah}}{2a}$$

is an exponentially decreasing function of h, with $\lim_{h\to\infty}\|T\|_\infty = 0$. I.e.,

- preview improves L^∞ estimation performance,

alleviating the effect of the nonminimum-phase zero (canceling it if $h = \infty$).

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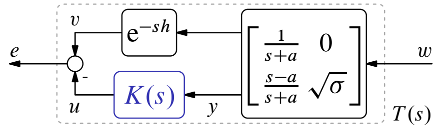


for some $a > 0$ and $\sigma \geq 0$ (measurement noise level). Error system:

$$T(s) = \begin{bmatrix} \frac{1}{s+a}e^{-sh} & 0 \end{bmatrix} - K(s)\begin{bmatrix} \frac{s-a}{s+a} & \sqrt{\sigma} \end{bmatrix}$$

and the error system is stable for every stable $K(s)$. The problem is to

- find stable and causal K minimizing L^2-norm $\|T\|_2$.

Reduction to a 1-block problem

Start with calculating

$$\begin{aligned} TT^\sim &= \left(\begin{bmatrix} \frac{1}{s+a}e^{-sh} & 0 \end{bmatrix} - K\begin{bmatrix} \frac{s-a}{s+a} & \sqrt{\sigma} \end{bmatrix}\right)\left(\begin{bmatrix} \frac{1}{-s+a}e^{sh} \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{s+a}{s-a} \\ \sqrt{\sigma} \end{bmatrix} K^\sim\right) \\ &= \frac{1}{-s^2+a^2} + K(1+\sigma)K^\sim + K\frac{1}{s+a}e^{sh} - e^{-sh}\frac{1}{s-a}K^\sim \\ &= \left(\frac{1}{1+\sigma}\frac{e^{-sh}}{s+a} - K\frac{s-a}{s+a}\right)(1+\sigma)(\cdot)^\sim + \frac{\sigma}{1+\sigma}\frac{1}{-s^2+a^2}. \end{aligned}$$

Thus,

$$\|T\|_2^2 = \underbrace{\left\|\sqrt{\frac{1}{1+\sigma}}\left(\frac{e^{-sh}}{s+a} - (1+\sigma)K\frac{s-a}{s+a}\right)\right\|_2^2}_{T_1(s),\text{ depends on } K} + \underbrace{\left\|\sqrt{\frac{\sigma}{1+\sigma}\frac{1}{s+a}}\right\|_2^2}_{T_0(s),\text{ independent of } K}$$

and

- minimizing T reduces to minimizing (1-block) T_1, whereas
- $\|T_0\|_2$ only adds to the optimal performance

Solution of the 1-block problem

As

$$T_1(s) = \frac{1}{\sqrt{1+\sigma}}\left(\frac{e^{-sh}}{s+a} - \underbrace{(1+\sigma)K(s)}_{K_\sigma(s)}\frac{s-a}{s+a}\right)$$

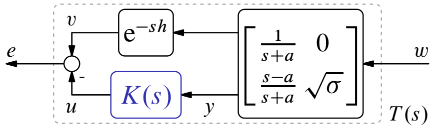
we already know that

$$K_{\text{opt}}(s) = \frac{1}{1+\sigma}K_{\sigma,\text{opt}}(s) = -\frac{1}{1+\sigma}\pi_h\left\{\frac{1}{s-a}e^{-sh}\right\}$$

and the optimal performance

$$\|T_1\|_2 = \frac{e^{-ah}}{\sqrt{2a(1+\sigma)}}.$$

Solution of the 2-block problem



Thus,

$$K_{\mathrm{opt}}(s) = -\frac{1}{1+\sigma}\boldsymbol{\pi}_h\left\{\frac{1}{s-a}\mathrm{e}^{-sh}\right\}$$

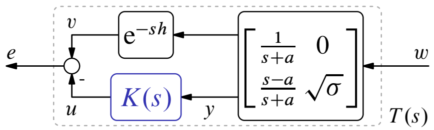
and the optimal performance

$$\|T\|_2 = \sqrt{\frac{\mathrm{e}^{-2ah}}{2a(1+\sigma)} + \|T_0\|_2^2} = \sqrt{\frac{\mathrm{e}^{-2ah}+\sigma}{2a(1+\sigma)}},$$

which exponentially decreases to $\|T_0\|_2 = \sqrt{\sigma/(2a(1+\sigma))}$.

Outline

Setup



for some $a > 0$ and $\sigma \geq 0$ (measurement noise level). Error system:

$$T(s) = \left[\tfrac{1}{s+a}\mathrm{e}^{-sh} \quad 0\right] - K(s)\left[\tfrac{s-a}{s+a} \quad \sqrt{\sigma}\right]$$

and the error system is stable for every stable $K(s)$. The problem is to

- find stable and causal K minimizing L^∞-norm $\|T\|_\infty$.

Reduction to a 1-block problem

We already know that

$$TT^{\sim} = \left(\frac{1}{1+\sigma}\frac{\mathrm{e}^{-sh}}{s+a} - K\frac{s-a}{s+a}\right)(1+\sigma)(\cdot)^{\sim} + \frac{\sigma}{1+\sigma}\frac{1}{-s^2+a^2}.$$

Thus, $\|T\|_\infty \leq \gamma$ iff

$$(1+\sigma)\left|\frac{1}{1+\sigma}\frac{\mathrm{e}^{-\mathrm{j}\omega h}}{\mathrm{j}\omega+a} - K(\mathrm{j}\omega)\frac{\mathrm{j}\omega-a}{\mathrm{j}\omega+a}\right|^2 + \frac{\sigma}{1+\sigma}\frac{1}{\omega^2+a^2} \leq \gamma^2, \quad \forall\omega\in\mathbb{R}$$

or, equivalently,

$$\left|\frac{\mathrm{e}^{-\mathrm{j}\omega h}}{\mathrm{j}\omega+a} - (1+\sigma)K(\mathrm{j}\omega)\frac{\mathrm{j}\omega-a}{\mathrm{j}\omega+a}\right|^2 \leq \gamma^2(1+\sigma) - \frac{\sigma}{\omega^2+a^2}, \quad \forall\omega\in\mathbb{R}$$

Reduction to a 1-block problem (contd)

This is possible *only if*

$$\gamma \ge \max_{\omega\in\mathbb{R}} \sqrt{\frac{\sigma}{1+\sigma}\frac{1}{\omega^2+a^2}} = \frac{1}{a}\sqrt{\frac{\sigma}{1+\sigma}} =: \gamma_\infty$$

and γ cannot be made smaller than γ_∞, no matter what K is chosen. Now, if we assume that $\gamma \ge \gamma_\infty$, we have that $\|T\|_\infty \le \gamma$ iff

$$\begin{aligned}\left|\frac{e^{-j\omega h}}{j\omega+a} - (1+\sigma)K(j\omega)\frac{j\omega-a}{j\omega+a}\right|^2 &\le \gamma^2(1+\sigma) - \frac{\sigma}{\omega^2+a^2}\\ &\le \left|\frac{\gamma\sqrt{1+\sigma}\,j\omega + \sqrt{\gamma^2a^2(1+\sigma)-\sigma}}{j\omega+a}\right|^2\end{aligned}$$

for all $\omega \in \mathbb{R}$. Equivalently,

$$\left|\frac{e^{-j\omega h}}{\alpha_1 j\omega+\alpha_2} - (1+\sigma)K(j\omega)\frac{j\omega-a}{\alpha_1 j\omega+\alpha_2}\right|^2 \le 1, \quad \forall\omega\in\mathbb{R}$$

where $\alpha_1 := \gamma\sqrt{1+\sigma} > 0$ and $\alpha_2 := \sqrt{\gamma^2a^2(1+\sigma)-\sigma} \ge 0$.

Reduction to a 1-block problem (contd)

Thus, $\gamma \ge \gamma_\infty$ and then

$$\|T\|_\infty \le \gamma \quad\iff\quad \Big\|\underbrace{\frac{1}{\alpha_1 s+\alpha_2}e^{-sh} - K_\sigma\frac{s-a}{\alpha_1 s+\alpha_2}}_{T_\gamma}\Big\|_\infty \le 1.$$

where $K_\sigma(s) := (1+\sigma)K(s)$. This is a 1-block problem reminiscent of what we studied before. The main nontrivial difference is that

- $T_\gamma(s)$ might contain unstable elements (if $\gamma = \gamma_\infty$, then $\alpha_2 = 0$).

In such a case, K_σ must stabilize T_γ first, by canceling the pole at $s = 0$.

Interpolation condition for stabilization

Consider the model-matching problem

$$T(s) = \frac{1}{s}G_1(s) + \frac{K(s)}{s}G_2(s),$$

where $G_1, G_2 \in H^\infty$ (in particular, such that $G_1(0)$ and $G_2(0)$ are finite). As the pole at the origin is the only instability, we have that

$$T \in H^\infty \iff \mathrm{Res}(T(s);0) = 0 \iff G_1(0) + K(0)G_2(0) = 0.$$

Thus, stabilization amounts to satisfying the interpolation constraint

$$K(0) = -\frac{G_1(0)}{G_2(0)}.$$

Resolving the interpolation condition

Lemma

The set of all $K \in H^\infty$ such that $K(0) = K_0$ is

$$K(s) = K_p(s) + Q(s)\frac{s}{s+a},$$

where $K_p \in H^\infty$ is any transfer function such that $K_p(0) = K_0$, $a > 0$, and $Q \in H^\infty$ but otherwise arbitrary.

Proof (outline).

"if": obvious (and a stable $Q(s)$ cannot cancel the zero at the origin)

"only if": let $K \in H^\infty$ be any t.f. such that $K(0) = K_0$. Then, for any K_p as above, $\frac{K(s)-K_p(s)}{s}$ is stable and strictly proper, i.e., that

$$Q(s) := (s+a)\frac{K(s)-K_p(s)}{s} \in H^\infty$$

for any $a > 0$. Hence, $K = K_p + Q\frac{s}{s+a}$ for $Q \in H^\infty$. □

Stabilizing T_γ when $\alpha_2 = 0$

Thus, if $\alpha_2 = 0$ (i.e., $\gamma = \gamma_\infty$),

$$T_\gamma(s) = \frac{1}{\alpha_1}\left(\frac{1}{s}\mathrm{e}^{-sh} - K_\sigma(s)\frac{s-a}{s}\right)$$

and the interpolation constraint $K_\sigma(0) = -\frac{1}{a}$ is resolved via

$$K_\sigma(s) = K_\mathrm{p}(s) + Q(s)\frac{s}{s+a} \quad \text{where } K_\mathrm{p}(0) = -\frac{1}{a}.$$

Then

$$T_\gamma(s) = \frac{1}{\alpha_1}\left(\frac{\mathrm{e}^{-sh} - K_\mathrm{p}(s)(s-a)}{s} - Q(s)\frac{s-a}{s+a}\right).$$

A particularly convenient choice (educated guess) is $K_\mathrm{p}(s) = -\frac{1}{s+a}\mathrm{e}^{-sh}$, in which case

$$T_\gamma(s) = \frac{1}{\alpha_1}\left(\frac{2\mathrm{e}^{-sh}}{s+a} - Q(s)\frac{s-a}{s+a}\right)$$

is practically in the form of the 1-block problem studied earlier.

General T_γ

Let

$$T_\gamma(s) = \frac{1}{\alpha_1 s + \alpha_2}\mathrm{e}^{-sh} - K_\sigma(s)\frac{s-a}{\alpha_1 s + \alpha_2}$$

Motivated by the stabilization problem, consider

$$K_\sigma(s) = -\frac{a\alpha_1 - \alpha_2}{a\alpha_1 + \alpha_2}\frac{1}{s+a}\mathrm{e}^{-sh} + Q(s)\frac{\alpha_1 s + \alpha_2}{s+a}$$

(stabilizing if $\alpha_2 = 0$ and non-restrictive if $\alpha_2 > 0$), in which case

$$T_\gamma(s) = \frac{2a}{a\alpha_1 + \alpha_2}\frac{1}{s+a}\mathrm{e}^{-sh} - Q(s)\frac{s-a}{s+a}$$

is again the 1-block form studied earlier.

Solvability conditions

Thus, $\exists K_\sigma$ such that $\|T_\gamma\| \le 1$ iff

$$\exists Q \in H^\infty \text{ such that } \left\|\frac{2a}{a\alpha_1 + \alpha_2}\frac{1}{s+a}\mathrm{e}^{-sh} - Q\frac{s-a}{s+a}\right\|_\infty \le 1$$

$$\Updownarrow$$

$$\min_{Q\in H^\infty}\left\|\frac{2a}{a\alpha_1 + \alpha_2}\frac{1}{s+a}\mathrm{e}^{-sh} - Q\frac{s-a}{s+a}\right\|_\infty \le 1$$

$$\Updownarrow$$

$$\left\|\frac{2a}{a\alpha_1 + \alpha_2}\frac{\mathrm{e}^{-ah}}{s+a}\right\|_\mathrm{H} = \frac{2a}{a\alpha_1 + \alpha_2}\frac{\mathrm{e}^{-ah}}{2a} = \frac{\mathrm{e}^{-ah}}{a\alpha_1 + \alpha_2} \le 1$$

Thus (remember, $\alpha_1 := \gamma\sqrt{1+\sigma} > 0$ and $\alpha_2 := \sqrt{\gamma^2 a^2(1+\sigma) - \sigma} \ge 0$),

$$\|T\|_\infty \le \gamma \iff \begin{cases} \gamma \ge \gamma_\infty = \frac{1}{a}\sqrt{\sigma/(1+\sigma)} \\ \mathrm{e}^{-ah} \le \sqrt{\gamma^2 a^2(1+\sigma)} + \sqrt{\gamma^2 a^2(1+\sigma) - \sigma} \end{cases}$$

Analysis of the solvability conditions

Note that $\mathrm{e}^{-ah} \le 1$ and $\sqrt{\gamma^2 a^2(1+\sigma)} + \sqrt{\gamma^2 a^2(1+\sigma) - \sigma} \ge \sqrt{\sigma}$. Hence,

- if $\sigma \ge 1$, then $\gamma = \gamma_\infty$ is attainable $\forall h \ge 0$
  (i.e., preview does not help us here at all)

This might be surprising. Then, even if $0 < \sigma < 1$, the inequality

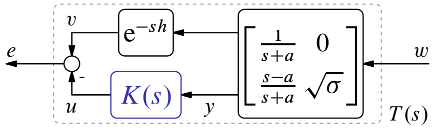
$$\mathrm{e}^{-ah} \le \sqrt{\gamma^2 a^2(1+\sigma)} + \sqrt{\gamma^2 a^2(1+\sigma) - \sigma} \xrightarrow{\gamma\to\gamma_\infty} \sqrt{\sigma}$$

holds whenever h is sufficiently long. Namely,

- if $\sigma < 1$, then $\gamma = \gamma_\infty$ is attainable $\forall h \ge -\frac{\ln\sigma}{2a}$
  (i.e., preview does not help us here after some finite value)

This might be surprising as well. In fact, only if $\sigma = 0$, then more preview is always advantageous from the L^∞ performance point of view.

Optimal performance



The minimal attainable $\gamma_{\min} = \|T\|_\infty$ is

$$\gamma_{\min} = \begin{cases} \dfrac{\sigma e^{ah} + e^{-ah}}{2a\sqrt{1+\sigma}} & \text{if } \sigma < 1 \text{ \& } h \le -\dfrac{\ln\sigma}{2a} \\ \dfrac{\sqrt{\sigma}}{a\sqrt{1+\sigma}} & \text{otherwise} \end{cases}$$

The central optimal estimators

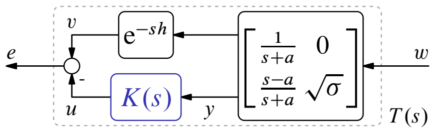
Now, whenever $\gamma \ge \gamma_{\min}$,

$$Q_{\mathrm{opt}}(s) := \arg\min_{Q \in H^\infty} \left\| \frac{2a}{a\alpha_1 + \alpha_2}\frac{1}{s+a} e^{-sh} - Q\frac{s-a}{s+a} \right\|_\infty = -\frac{2a}{a\alpha_1+\alpha_2}\left(\frac{e^{-ah}}{2a} + \boldsymbol{\pi}_h\left\{\frac{1}{s-a}e^{-sh}\right\}\right)$$

will solve the problem (although we don't need to minimize this norm if the minimum is < 1 and then there are infinitely many admissible Q's). Then,

$$\begin{aligned} K_{\sigma,\mathrm{opt}}(s) &= -\frac{a\alpha_1 - \alpha_2}{a\alpha_1+\alpha_2}\frac{1}{s+a}e^{-sh} \\ &\quad - \frac{2a}{a\alpha_1+\alpha_2}\left(\frac{e^{-ah}}{2a} + \boldsymbol{\pi}_h\left\{\frac{1}{s-a}e^{-sh}\right\}\right)\frac{\alpha_1 s + \alpha_2}{s+a} \\ &= -\frac{e^{-ah}\alpha_1}{a\alpha_1+\alpha_2} - \boldsymbol{\pi}_h\left\{\frac{1}{s-a}e^{-sh}\right\} \end{aligned}$$

The central optimal estimators (contd)



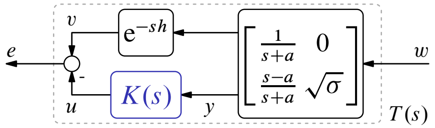
Thus, going back $Q \to K_\sigma \to K$, we end up with

$$\begin{aligned} K_{\mathrm{opt}}(s) &= -\frac{1}{1+\sigma}\left(\frac{e^{-ah}\gamma\sqrt{1+\sigma}}{a\gamma\sqrt{1+\sigma} + \sqrt{\gamma^2 a^2(1+\sigma) - \sigma}} + \boldsymbol{\pi}_h\left\{\frac{1}{s-a}e^{-sh}\right\}\right) \\ &= -\frac{1}{1+\sigma}\boldsymbol{\pi}_h\left\{\frac{1}{s-a}e^{-sh}\right\} - \begin{cases} \frac{\sigma e^{ah} + e^{-ah}}{2a(1+\sigma)} & \text{if } \sigma < 1 \text{ \& } h \le -\frac{\ln\sigma}{2a} \\ \frac{e^{-ah}}{a(1+\sigma)} & \text{otherwise} \end{cases} \end{aligned}$$

where the last equality is obtained by substituting $\gamma = \gamma_{\min}$.

Outline

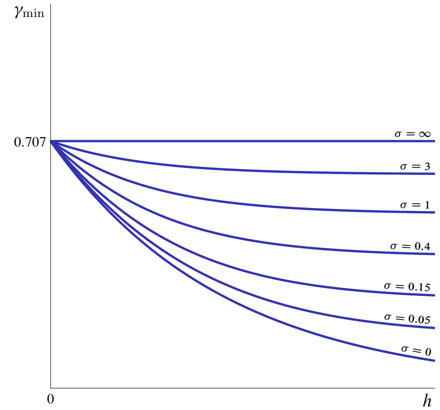
2-block problem



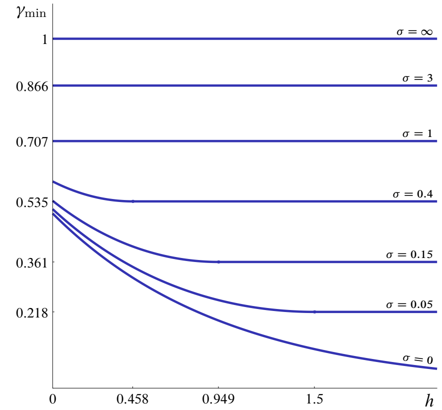
with

- preview length h
- σ representing intensity of measurement noise

Optimal performance



L^2 criterion ($a = 1$)



L^∞ criterion ($a = 1$)

Optimal estimators

L^2 criterion:

$$K_{\text{opt}}(s) = -\frac{1}{1+\sigma}\,\boldsymbol{\pi}_h\left\{\frac{1}{s-a}\mathrm{e}^{-sh}\right\}$$

and it vanishes as both $\sigma \to \infty$ and $h \to 0$.

L^∞ criterion: either

$$K_{\text{opt}}(s) = -\frac{\sigma \mathrm{e}^{ah} + \mathrm{e}^{-ah}}{2a(1+\sigma)} - \frac{1}{1+\sigma}\boldsymbol{\pi}_h\left\{\frac{1}{s-a}\,\mathrm{e}^{-sh}\right\}$$

(if $\sigma < 1$ & $h \le -\frac{\ln\sigma}{2a}$) or

$$K_{\text{opt}}(s) = -\frac{\mathrm{e}^{-ah}}{a(1+\sigma)} - \frac{1}{1+\sigma}\boldsymbol{\pi}_h\left\{\frac{1}{s-a}\,\mathrm{e}^{-sh}\right\}$$

(otherwise) and it vanishes as $\sigma \to \infty$, but not as $h \to 0$.

Optimal $|T(\mathrm{j}\omega)|$

L^2 criterion:

$$|T(\mathrm{j}\omega)|^2 = \frac{\mathrm{e}^{-2ah} + \sigma}{(1+\sigma)(\omega^2 + a^2)},$$

which is a low-pass function.

L^∞ criterion:

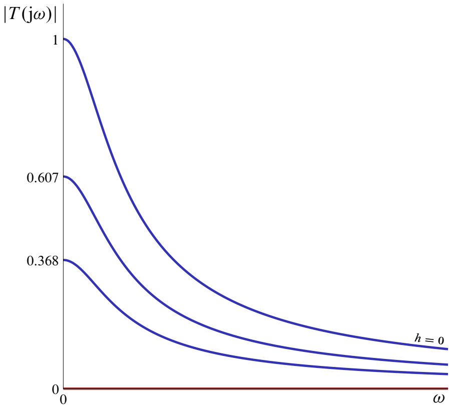
$$|T(\mathrm{j}\omega)|^2 = \begin{cases} \frac{(\sigma \mathrm{e}^{ah} + \mathrm{e}^{-ah})^2}{4a^2(1+\sigma)} & \text{if } \sigma < 1 \text{ \& } h \le -\frac{\ln\sigma}{2a} \\ \frac{\mathrm{e}^{-2ah}\omega^2 + a^2\sigma}{a^2(1+\sigma)(\omega^2+a^2)} & \text{otherwise} \end{cases}$$

which is

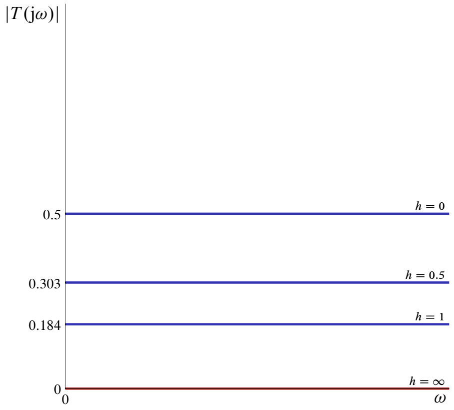
- all-pass if $\sigma < 1$ & $h \le -\frac{\ln\sigma}{2a}$
- a lag otherwise, with $|T(\mathrm{j}\infty)|^2 = \frac{\mathrm{e}^{-2ah}}{a^2(1+\sigma)} < \frac{\sigma}{a^2(1+\sigma)} = |T(0)|^2$

As $h \to \infty$, both frequency responses approach $\frac{\sigma}{1+\sigma}\frac{1}{\omega^2+a^2}$.

Optimal $|T(j\omega)|$ with $\sigma = 0$



L^2 criterion $(a = 1)$

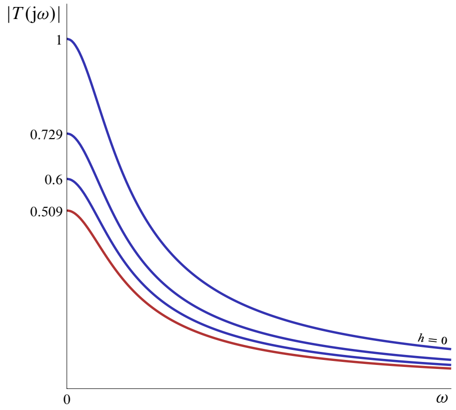


L^∞ criterion $(a = 1)$

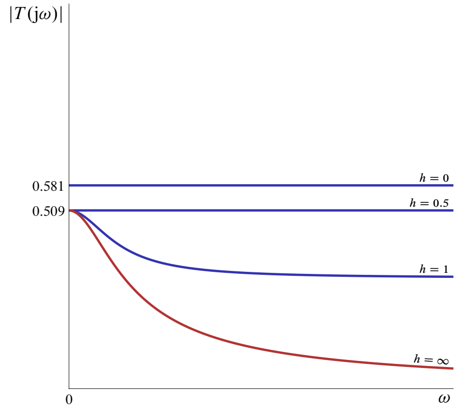
Here

- $\|T\|_\infty$ decreases whenever h decreases

Optimal $|T(j\omega)|$ with $\sigma = \frac{\ln 2}{2} \approx 0.35$



L^2 criterion $(a = 1)$

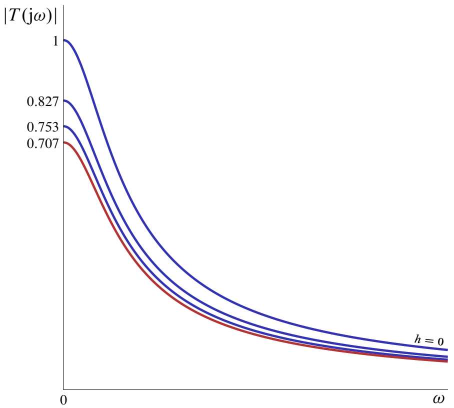


L^∞ criterion $(a = 1)$

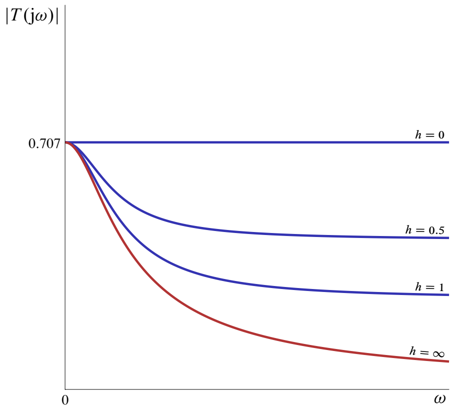
Here

- $\|T\|_\infty$ decreases up to $h = 0.5$ only

Optimal $|T(j\omega)|$ with $\sigma = 1$



L^2 criterion $(a = 1)$

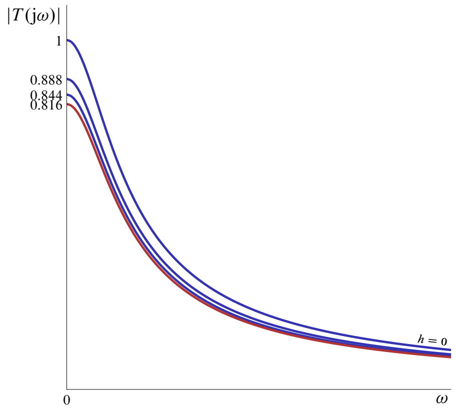


L^∞ criterion $(a = 1)$

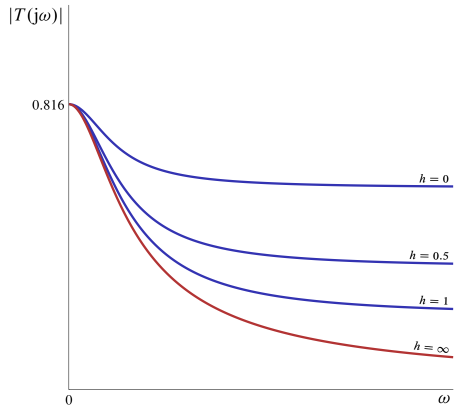
Here

- $\|T\|_\infty$ does not decrease as h increases

Optimal $|T(j\omega)|$ with $\sigma = 2$



L^2 criterion $(a = 1)$



L^∞ criterion $(a = 1)$

Here

- $\|T\|_\infty$ does not decrease as h increases