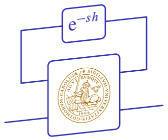
# Introduction to Time-Delay Systems




## lecture no. 4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## Outline



## Discrete case: problem statement

Consider

$$\bar{\Sigma}_h : \begin{cases} \bar{x}[k+1] = \bar{A}\bar{x}[k] + \bar{B}\bar{u}[k-h] \\ \bar{y}[k] = \bar{C}\bar{x}[k] \end{cases}$$

We look for $\bar{u}[t]$ stabilizing $\bar{\Sigma}_h$ for any given $h$.

## Luenberger observer: delay-free case

Consider the problem of reconstructing $\bar{x}$ in

$$\bar{\Sigma}_0 : \begin{cases} \bar{x}[k+1] = \bar{A}\bar{x}[k] + \bar{B}\bar{u}[k] \\ \bar{y}[k] = \bar{C}\bar{x}[k] \end{cases}$$

To this end, construct *observer*

$$\bar{x}_o[k+1] = \bar{A}\bar{x}_o[k] + \bar{B}\bar{u}[k] - \bar{L}(\bar{y}[k] - \bar{C}\bar{x}_o[k])$$

which has two inputs ($\bar{u}$ and $\bar{y}$). The observation error $\bar{\epsilon}[k] := \bar{x}[k] - \bar{x}_o[k]$ satisfies then autonomous (input free) equation

$$\bar{\epsilon}[k+1] = (\bar{A} + \bar{L}\bar{C})\bar{\epsilon}[k],$$

which can be made stable provided $(\bar{C}, \bar{A})$ is detectable. Stability implies that $\lim_{k\to\infty} \bar{\epsilon}[k] = 0$ from any initial condition.

## Observer-based feedback: delay-free case

Combining plant, observer, and observer-based control law $\bar u[k]=\bar F\bar x_o[k]$, we get the following closed-loop system:

$$\bar\Sigma_{cl}:\begin{cases}\begin{bmatrix}\bar x[k+1]\\ \bar x_o[k+1]\end{bmatrix}=\begin{bmatrix}\bar A & \bar B\bar F\\ -\bar L\bar C & \bar A+\bar L\bar C+\bar B\bar F\end{bmatrix}\begin{bmatrix}\bar x[k]\\ \bar x_o[k]\end{bmatrix}\\ \bar y[k]=\begin{bmatrix}\bar C & 0\end{bmatrix}\begin{bmatrix}\bar x[k]\\ \bar x_o[k]\end{bmatrix}\end{cases}$$

As the dynamics of $\bar\epsilon$ are autonomous, rewrite (by similarity transformation)

$$\bar\Sigma_{cl}:\begin{cases}\begin{bmatrix}\bar x[k+1]\\ \bar\epsilon[k+1]\end{bmatrix}=\begin{bmatrix}\bar A+\bar B\bar F & \bar B\bar F\\ 0 & \bar A+\bar L\bar C\end{bmatrix}\begin{bmatrix}\bar x[k]\\ \bar\epsilon[k]\end{bmatrix}\\ \bar y[k]=\begin{bmatrix}\bar C & 0\end{bmatrix}\begin{bmatrix}\bar x[k]\\ \bar\epsilon[k]\end{bmatrix}\end{cases}$$

whence $\operatorname{spec}(\bar\Sigma_{cl})=\operatorname{spec}(\bar A+\bar B\bar F)\bigcup\operatorname{spec}(\bar A+\bar L\bar C)$ (separation).

## Reduced-order observer: simple special case

If

$$\bar\Sigma_0:\begin{cases}\begin{bmatrix}\bar x_1[k+1]\\ \bar x_2[k+1]\end{bmatrix}=\begin{bmatrix}\bar A_{11} & 0\\ \bar A_{21} & \bar A_{22}\end{bmatrix}\begin{bmatrix}\bar x_1[k]\\ \bar x_2[k]\end{bmatrix}+\begin{bmatrix}\bar B_1\\ \bar B_2\end{bmatrix}\bar u[k]\\ \begin{bmatrix}\bar y_1[k]\\ \bar y_2[k]\end{bmatrix}=\begin{bmatrix}I & 0\\ 0 & \bar C\end{bmatrix}\begin{bmatrix}\bar x_1[k]\\ \bar x_2[k]\end{bmatrix}\end{cases}$$

i.e., we measure the whole $\bar x_1$ and a part of $\bar x_2$, we need to observe only $\bar x_2$. From the second state equation:

$$\begin{cases}\bar x_2[k+1]=\bar A_{22}\bar x_2[k]+\underbrace{\bar A_{21}\bar y_1[k]+\bar B_2\bar u[k]}_{\text{“known input”}}\\ \bar y_2[k]=\bar C\bar x_2[k]\end{cases}$$

and then the reduced-order observer

$$\bar x_o[k+1]=\bar A_{22}\bar x_o[k]+\bar A_{21}\bar y_1[k]+\bar B_2\bar u[k]-\bar L(\bar y_2[k]-\bar C\bar x_o[k]).$$

results in autonomous $\bar\epsilon_2[k]:=\bar x_2[k]-\bar x_o[k]$ again:

$$\bar\epsilon_2[k+1]=(\bar A_{22}+\bar L\bar C)\bar\epsilon_2[k].$$

## Reduced-order observer-based feedback

Combining plant, reduced-order observer, and observer-based control law

$$\bar u[k]=\begin{bmatrix}\bar F_1 & \bar F_2\end{bmatrix}\begin{bmatrix}\bar x_1[k]\\ \bar x_o[k]\end{bmatrix},$$

we get the following closed-loop system:

$$\bar\Sigma_{cl}:\begin{cases}\begin{bmatrix}\bar x_1[k+1]\\ \bar x_2[k+1]\\ \bar\epsilon_2[k+1]\end{bmatrix}=\begin{bmatrix}\bar A_{11}+\bar B_1\bar F_1 & \bar B_1\bar F_2 & \bar B_1\bar F_2\\ \bar A_{21}+\bar B_2\bar F_1 & \bar A_{22}+\bar B_2\bar F_2 & \bar B_2\bar F_2\\ 0 & 0 & \bar A_{22}+\bar L\bar C\end{bmatrix}\begin{bmatrix}\bar x_1[k]\\ \bar x_2[k]\\ \bar\epsilon_2[k]\end{bmatrix}\\ \begin{bmatrix}\bar y_1[k]\\ \bar y_2[k]\\ \bar\epsilon_2[k]\end{bmatrix}=\begin{bmatrix}I & 0 & 0\\ 0 & \bar C & 0\end{bmatrix}\begin{bmatrix}\bar x_1[k]\\ \bar x_2[k]\\ \bar\epsilon_2[k]\end{bmatrix}\end{cases}$$

## State observer for input-delay systems

Let now

$$\bar\Sigma_h:\begin{cases}\bar x[k+1]=\bar A\bar x[k]+\bar B\bar u[k-h]\\ \bar y[k]=\bar C\bar x[k]\end{cases}$$

Remember, the true state model looks like

$$\bar\Sigma_h:\begin{cases}\begin{bmatrix}\bar u[k]\\ \bar u[k-1]\\ \vdots\\ \bar u[k-h+1]\\ \hline \bar x[k+1]\end{bmatrix}=\left[\begin{array}{cccc:c}0 & \cdots & 0 & 0 & 0\\ I & \cdots & 0 & 0 & 0\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & \cdots & I & 0 & 0\\ \hline 0 & \cdots & 0 & \bar B & \bar A\end{array}\right]\begin{bmatrix}\bar u[k-1]\\ \bar u[k-2]\\ \vdots\\ \bar u[k-h]\\ \hline \bar x[k]\end{bmatrix}+\begin{bmatrix}I\\ 0\\ \vdots\\ 0\\ \hline 0\end{bmatrix}\bar u[k]\\ \bar y[k]=\left[\begin{array}{cccc:c}I & 0 & \cdots & 0 & 0\\ 0 & I & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & I & 0\\ \hline 0 & 0 & \cdots & 0 & \bar C\end{array}\right]\begin{bmatrix}\bar u[k-1]\\ \bar u[k-2]\\ \vdots\\ \bar u[k-h]\\ \hline \bar x[k]\end{bmatrix}\end{cases}$$

## State observer for input-delay systems (contd)

Of state vector, $\bar{x}_{\mathrm{a}}$, we measure all $\bar{u}$'s, hence we need only reduced-order observer

$$\bar{x}_{\mathrm{o}}[k+1] = \bar{A}\bar{x}_{\mathrm{o}}[k] + \bar{B}\bar{u}[k-h] - \bar{L}(\bar{y}[k] - \bar{C}\bar{x}_{\mathrm{o}}[k])$$

The error equation reads then

$$\bar{\epsilon}[k+1] = (\bar{A} + \bar{L}\bar{C})\bar{\epsilon}[k]$$

and can be made stable iff $(\bar{C}, \bar{A})$ detectable (exactly as in delay-free case).

## Discrete output feedback: input delay

Mechanical amalgamation of

- reduced-order discrete observer
- state-feedback shifting only the modes of $\bar{A}$

yields

$$\begin{cases} \bar{x}_{\mathrm{o}}[k+1] = \bar{A}\bar{x}_{\mathrm{o}}[k] + \bar{B}\bar{u}[k-h] - \bar{L}(\bar{y}[k] - \bar{C}\bar{x}_{\mathrm{o}}[k]) \\ \bar{u}[k] = \bar{F}\left(\bar{A}^h\bar{x}_{\mathrm{o}}[k] + \sum_{i=1}^{h} \bar{A}^{i-1}\bar{B}\bar{u}[k-i]\right) \end{cases}$$

which is called observer-predictor.

As this is a special case of the reduced-order observer-based feedback,

- $\mathrm{spec}(\bar{\Sigma}_{\mathrm{cl}}) = \mathrm{spec}(\bar{A} + \bar{B}\bar{F}) \bigcup \{0\}^{mh} \bigcup \mathrm{spec}(\bar{A} + \bar{L}\bar{C})$.

## Continuous output feedback: input delay

If

$$\Sigma_h : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t-h) \\ y(t) = Cx(t) \end{cases}$$

the observer-predictor controller

$$\begin{cases} \dot{x}_{\mathrm{o}}(t) = (A + LC)x_{\mathrm{o}}(t) + Bu(t-h) - Ly(t) \\ u(t) = F\left(\mathrm{e}^{Ah}x_{\mathrm{o}}(t) + \int_0^h \mathrm{e}^{A\theta}Bu(t-\theta)\mathrm{d}\theta\right) \end{cases}$$

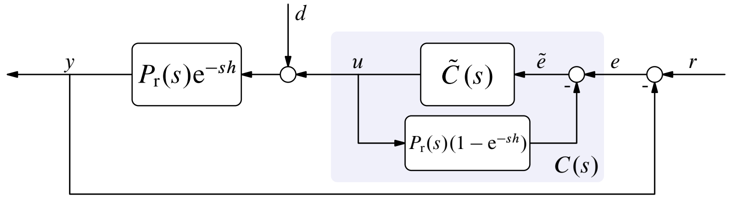
assigns

- $\mathrm{spec}(\Sigma_{\mathrm{cl}}) = \mathrm{spec}(A + BF) \bigcup \mathrm{spec}(A + LC)$

(this can be verified by standard arguments of FSA).

## Outline

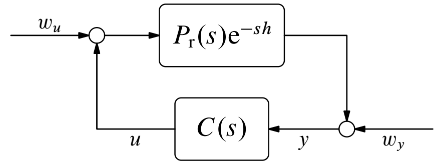
## Smith controller: preliminary conclusions




Remember, Smith controller

- works if $P_r(s)$ is stable
  (stabilization problem reduces then to that for delay-free plant)
- does not necessarily work if $P_r(s)$ is unstable
  (might lead to unstable loop)

## More rigorous analysis




System is said to be internally stable if

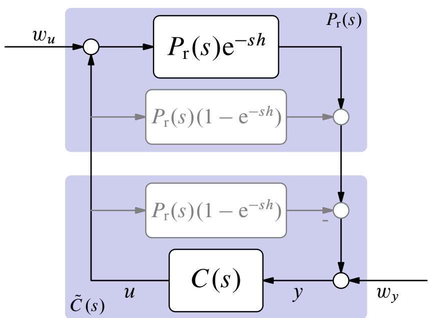
- transfer matrix from $\left[\begin{smallmatrix} w_y \\ w_u \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$,

$$T_{\mathrm{cl}} := \frac{1}{1 - P_r C e^{-sh}} \begin{bmatrix} 1 & P_r e^{-sh} \\ C & P_r C e^{-sh} \end{bmatrix} =: \begin{bmatrix} S & T_d \\ T_u & T \end{bmatrix} \in H^\infty.$$

Let's

- analyze Smith controller from internal stability perspectives.

## Loop shifting



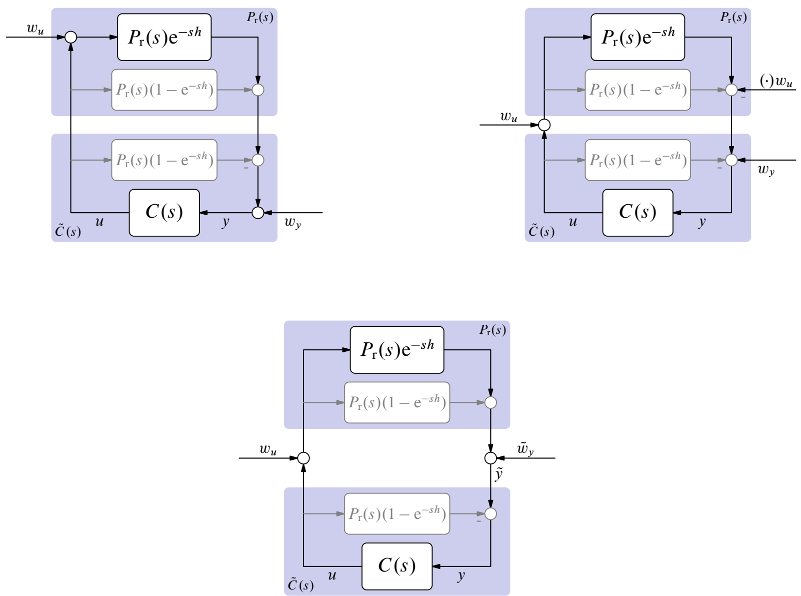

Adding and subtracting block $P_r(1 - e^{-sh})$ we

- redistribute loop components w/o changing the whole system.

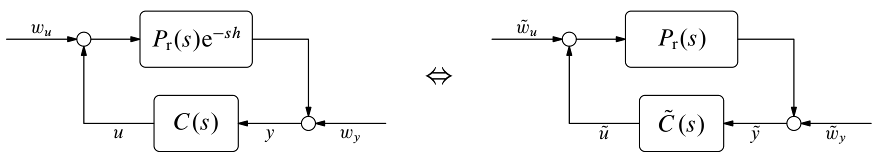
We end up with a new loop with the plant $P_r$ and the controller

$$\tilde{C} := \frac{C}{1 + C P_r (1 - e^{-sh})} \qquad \left(\text{so that} \quad C = \frac{\tilde{C}}{1 - \tilde{C} P_r (1 - e^{-sh})}\right)$$

## Loop shifting: signal transformations



## Loop shifting (contd)




The new system is delay-free yet with

- different signals.

To complete[1] the picture, we have to calculate them:

$$\begin{bmatrix} \tilde{y} \\ \tilde{u} \end{bmatrix} = \begin{bmatrix} y + P_r(1-e^{-sh})u \\ u \end{bmatrix} = \begin{bmatrix} I & P_r(1-e^{-sh}) \\ 0 & I \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}$$

and

$$\begin{bmatrix} \tilde{w}_y \\ \tilde{w}_u \end{bmatrix} = \begin{bmatrix} w_y - P_r(1-e^{-sh})w_u \\ w_u \end{bmatrix} = \begin{bmatrix} I & -P_r(1-e^{-sh}) \\ 0 & I \end{bmatrix} \begin{bmatrix} w_y \\ w_u \end{bmatrix}.$$

[1] Well, we also have to guarantee that internal "$\tilde{C}$" loop is well posed.

## Loop shifting (contd)

Thus,

$$T_{cl} = \begin{bmatrix} I & -P_r(1-e^{-sh}) \\ 0 & I \end{bmatrix} \tilde{T}_{cl} \begin{bmatrix} I & -P_r(1-e^{-sh}) \\ 0 & I \end{bmatrix}.$$

1. if $P_r(1-e^{-sh}) \in H^\infty$, then $\tilde{T}_{cl} \in H^\infty$ implies $T_{cl} \in H^\infty$
2. if $P_r(1-e^{-sh}) \notin H^\infty$, then $\tilde{T}_{cl} \in H^\infty$ not necessarily implies $T_{cl} \in H^\infty$
   (in fact, never; this can be verified by making use of explicit form of $\tilde{T}_{cl}$)

Key question:

- when does $P_r(1-e^{-sh}) \in H^\infty$ ?

Obviously, this is true if $P_r \in H^\infty$. Yet this also true if

- $P_r(s)$ proper and its only unstable poles are single poles at $j\frac{2\pi}{h}k$, $k \in \mathbb{Z}$,

which are simple zeros of $1 - e^{-sh}$.

## Outline

Output feedback for input-delay systems: adding observers

Smith controller revised

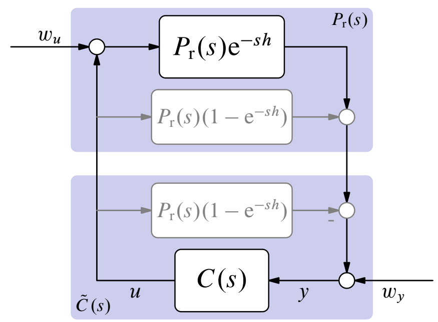
Modified Smith predictor and dead-time compensation

Modified Smith predictor vs. observer-predictor

Coprime factorization over $H^\infty$ and Youla parametrization

Two-stage design of dead-time compensators

## Loop shifting: idea



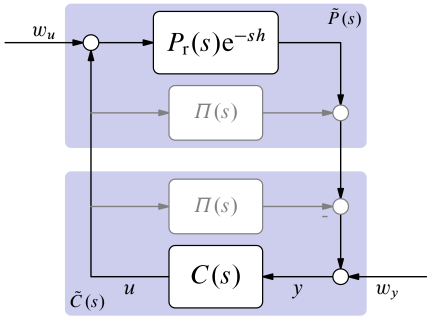

Driving idea here is to

- cancel (compensate) delay via $P_r(s)e^{-sh} + P_r(s)(1-e^{-sh}) = P_r(s)$.

The question:

- can we do it with stable compensation element ?

## Dead-time compensation question



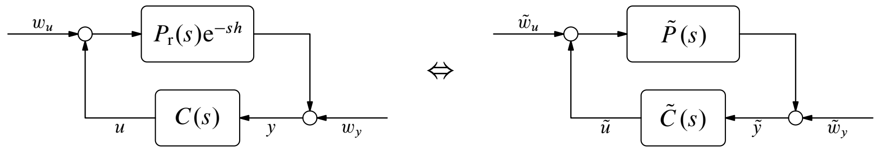

Technicaly speaking, we are looking for $\Pi(s) \in H^\infty$ such that

$$P_r(s)e^{-sh} + \Pi(s) = \tilde{P}(s)$$

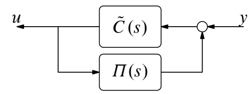
for some proper and rational $\tilde{P}(s)$.

## Dead-time compensation: aspirations

If we obtained required $\Pi(s) \in H^\infty$, we would transform




for rational (delay-free) $\tilde{P}(s)$. Resulting $C(s)$,

$$C(s) = \tilde{C}(s)\big(I - \Pi(s)\tilde{C}(s)\big)^{-1} =$$  ,


called dead-time compensator (DTC) and

- $\tilde{C}(s)$ internally stabilizes $\tilde{P}(s)$ iff $C(s)$ internally stabilizes $P_r(s)e^{-sh}$ as

$$T_{cl}(s) = \begin{bmatrix} I & -\Pi(s) \\ 0 & I \end{bmatrix} \tilde{T}_{cl}(s) \begin{bmatrix} I & -\Pi(s) \\ 0 & I \end{bmatrix}$$

(provided DTC loop well posed).

## DTC question: stable $P_r(s)$

Choice of $\Pi(s)$ apparent and non-unique:

- $\Pi(s) = \tilde{P}(s) - P_r(s)e^{-sh}$ for any $\tilde{P}(s) \in H^\infty$ does the job.

Some standard choices:

- $\tilde{P}(s) = P_r(s)$ results in Smith predictor
- $\tilde{P}(s) = 0$ results in internal model controller (IMC)

## DTC question: unstable $P_r(s)$ in Example 2 (Lect 3)

In this case Smith predictor

$$\Pi(s) = \frac{1 - e^{-sh}}{s} = \int_0^h e^{-s\theta} d\theta \in H^\infty$$

indeed (integrand analytic and bounded in $\mathbb{C}_0$ and integration path finite). In this case

$$\tilde{P}(s) = P_r(s) = \frac{1}{s}$$

can be interpreted as unstable rational part of partial fraction expansion[2] of

$$P_r(s)e^{-sh} = \frac{\operatorname{Res}\big(\frac{e^{-sh}}{s}; 0\big)}{s} + \Delta(s)$$

for some entire $\Delta(s)$ ($P_r(s)$ has only one pole, that at the origin).

[2] Here $\operatorname{Res}(f(s); c) := \lim_{s \to c}(s - c) f(s)$ stands for residue of $f(s)$ at $c$.

## DTC question: unstable $P_r(s)$ in Example 1 (Lect 3)

We're looking for stable $\tilde{P}(s)$ such that

$$\Pi(s) = \tilde{P}(s) - \frac{1}{s-1}\mathrm{e}^{-sh} \in H^\infty.$$

As $P_r(s)\mathrm{e}^{-sh}$ has one singularity, pole at $s = 1$, we have that

$$\frac{1}{s-1}\mathrm{e}^{-sh} = \frac{\operatorname{Res}\left(\frac{1}{s-1}\mathrm{e}^{-sh}; 1\right)}{s-1} + \Delta(s),$$

for some entire $\Delta(s)$. This suggest choice

$$\tilde{P}(s) = \frac{\operatorname{Res}\left(\frac{1}{s-1}\mathrm{e}^{-sh}; 1\right)}{s-1} = \frac{\mathrm{e}^{-h}}{s-1},$$

for which $\Pi(s)$ is distributed-delay system

$$\Pi(s) = -\Delta(s) = \frac{\mathrm{e}^{-h} - \mathrm{e}^{-sh}}{s-1} = \mathrm{e}^{-h}\int_0^h \mathrm{e}^{-(s-1)\theta}\mathrm{d}\theta \in H^\infty.$$

## DTC question: first-order unstable $P_r(s)$

Likewise,

$$\frac{1}{s-a}\mathrm{e}^{-sh} = \frac{\operatorname{Res}\left(\frac{1}{s-a}\mathrm{e}^{-sh}; a\right)}{s-a} + \Delta(s) = \frac{\mathrm{e}^{-ah}}{s-a} - \frac{\mathrm{e}^{-ah} - \mathrm{e}^{-sh}}{s-a},$$

so that if $P_r(s) = \frac{1}{s-a}$, the choice

$$\tilde{P}(s) = \frac{\mathrm{e}^{-ah}}{s-a}$$

gives us required

$$\Pi(s) = \frac{\mathrm{e}^{-ah}}{s-a} - \frac{\mathrm{e}^{-sh}}{s-a} = \mathrm{e}^{-ah}\int_0^h \mathrm{e}^{-(s-a)\theta}\mathrm{d}\theta \in H^\infty$$

again.

## First-order unstable $P_r(s)$: time-domain interpretation

Impulse responses of $P(s) := P_r(s)\mathrm{e}^{-sh} = \frac{1}{s-a}\mathrm{e}^{-sh}$ and $\tilde{P}(s) = \frac{1}{s-a}\mathrm{e}^{-ah}$ are

$$p(t) = \begin{cases} 0 & \text{if } t < h \\ \mathrm{e}^{a(t-h)} & \text{if } t \ge h \end{cases} \quad \text{and} \quad \tilde{p}(t) = \begin{cases} 0 & \text{if } t < 0 \\ \mathrm{e}^{a(t-h)} & \text{if } t \ge 0 \end{cases}$$

respectively. Thus,

$$p(t) \equiv \tilde{p}(t), \quad \forall t \ge h.$$

Impulse response of $\Pi(s) = \tilde{P}(s) - P_r(s)\mathrm{e}^{-sh}$ then

$$\pi(t) = \tilde{p}(t) - p(t) = \begin{cases} 0 & \text{if } t < 0 \text{ or } t \ge h \\ \mathrm{e}^{a(t-h)} & \text{if } 0 \le t < h \end{cases}$$

i.e., $\Pi(s)$ is FIR (finite impulse response).

## Completion operator

Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$. Impulse response of $G_h(s) := G(s)\mathrm{e}^{-sh}$ is

$$g_h(t) = \begin{cases} 0 & \text{if } t < h \\ C\mathrm{e}^{A(t-h)}B & \text{if } t \ge h \end{cases}$$

We are looking for rational $\tilde{G}(s)$ such that $\tilde{g}(t) \equiv g_h(t)$, $\forall t \ge h$. Obviously,

$$\tilde{g}(t) = \begin{cases} 0 & \text{if } t < 0 \\ C\mathrm{e}^{-Ah}\mathrm{e}^{At}B & \text{if } t \ge 0 \end{cases} \quad \text{so that} \quad \tilde{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C\mathrm{e}^{-Ah} & 0 \end{array}\right].$$

LTI system

$$\pi_h\{G(s)\mathrm{e}^{-sh}\} := \tilde{G}(s) - G(s)\mathrm{e}^{-sh} := \left[\begin{array}{c|c} A & B \\ \hline C\mathrm{e}^{-Ah} & 0 \end{array}\right] - \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]\mathrm{e}^{-sh}$$

completes, in a sense, $G(s)\mathrm{e}^{-sh}$ to rational $\tilde{G}(s)$, so we call it completion of $G(s)\mathrm{e}^{-sh}$.

## Completion operator (contd)

For any rational $G(s)$, $\boldsymbol{\pi}_h\{G(s)\mathrm{e}^{-sh}\}$ is FIR with (bounded) impulse response

$$\pi_h(t) = \begin{cases} C\mathrm{e}^{A(t-h)}B & \text{if } 0 \le t < h \\ 0 & \text{otherwise} \end{cases}$$

Hence, transfer function

$$\boldsymbol{\pi}_h\{G(s)\mathrm{e}^{-sh}\} = \int_0^h \pi_h(t)\mathrm{e}^{-st}\mathrm{d}t = \int_0^h C\mathrm{e}^{A(t-h)}B\mathrm{e}^{-st}\mathrm{d}t$$

is bounded in $\mathbb{C}_0$ and, as it also entire, belongs to $H^\infty$. Thus,

- $\boldsymbol{\pi}_h\{G(s)\mathrm{e}^{-sh}\}$ stable for every proper rational $G(s)$.

In time domain, $y = \boldsymbol{\pi}_h\{G(s)\mathrm{e}^{-sh}\}u$ writes

$$y(t) = C\int_{t-h}^{t} \mathrm{e}^{A(t-\theta-h)}Bu(\theta)\mathrm{d}\theta = C\int_0^h \mathrm{e}^{A(\theta-h)}Bu(t-\theta)\mathrm{d}\theta.$$

## DTC question: general $P_\mathrm{r}(s)$

Let

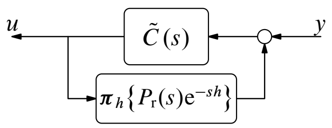
$$P_\mathrm{r}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right].$$

We can always choose $\Pi(s) = \boldsymbol{\pi}_h\{P_\mathrm{r}(s)\mathrm{e}^{-sh}\} \in H^\infty$, for which

$$\tilde{P}(s) = P_\mathrm{r}(s)\mathrm{e}^{-sh} + \boldsymbol{\pi}_h\{P_\mathrm{r}(s)\mathrm{e}^{-sh}\} = \left[\begin{array}{c|c} A & B \\ \hline C\mathrm{e}^{-Ah} & 0 \end{array}\right]$$

is indeed rational.

## Modified Smith predictor



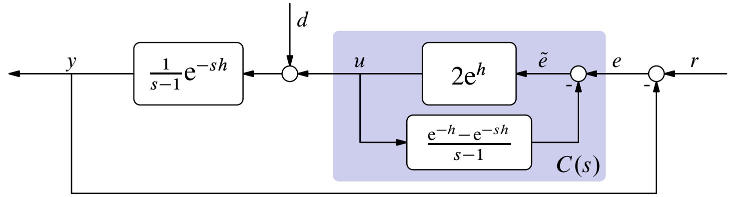

Transfer function

$$C(s) = \tilde{C}(s)\big(I - \boldsymbol{\pi}_h\{P_\mathrm{r}(s)\mathrm{e}^{-sh}\}\tilde{C}(s)\big)^{-1}$$

Then,

- if $P_\mathrm{r}(s)$ is strictly proper, internal loop is well-posed for every proper $\tilde{C}$,
- $C(s)$ stabilizes $P_\mathrm{r}(s)\mathrm{e}^{-s}$ iff $\tilde{C}$ stabilizes $\tilde{P}(s)$

## Example 1




This $\tilde{C}$ stabilizes $\tilde{P}(s) = \frac{\mathrm{e}^{-h}}{s-1}$ and then

$$T_u(s) = \frac{2\mathrm{e}^h(s-1)}{s+1}, \quad T(s) = \frac{2\mathrm{e}^h}{s+1}\mathrm{e}^{-sh},$$

and

$$T_d(s) = \frac{1}{s+1}\left(1 + \frac{2(1-\mathrm{e}^{-(s-1)h})}{s-1}\right)\mathrm{e}^{-sh}$$

are all stable, as expected.

## Outline

Output feedback for input-delay systems: adding observers

Smith controller revised

Modified Smith predictor and dead-time compensation

Modified Smith predictor vs. observer-predictor

Coprime factorization over $H^\infty$ and Youla parametrization

Two-stage design of dead-time compensators

## Observer-predictor revised

Let $P_r(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ and consider observer-predictor control law

$$\begin{cases} \dot{x}_o(t) = (A + LC)x_o(t) + Bu(t-h) - Ly(t) \\ u(t) = F\left(e^{Ah}x_o(t) + \displaystyle\int_{t-h}^{t} e^{A(t-\theta)}Bu(\theta)d\theta\right) \end{cases}$$

where $F$ and $L$ are matrices making $A + BF$ and $A + LC$ Hurwitz. Denote

$$\eta(t) := e^{Ah}x_o(t) + \int_{t-h}^{t} e^{A(t-\theta)}Bu(\theta)d\theta.$$

Then

$$\begin{aligned} \dot{\eta}(t) &= e^{Ah}\big((A + LC)x_o(t) + Bu(t-h) - Ly(t)\big) \\ &\qquad + \big(Bu(t) - e^{Ah}Bu(t-h)\big) + A\int_{t-h}^{t} e^{A(t-\theta)}Bu(\theta)d\theta \\ &= A\eta(t) + Bu(t) - e^{Ah}L\big(y(t) - Cx_o(t)\big). \end{aligned}$$

## Observer-predictor revised (contd)

In other words, observer-predictor control law writes as

$$\begin{cases} \dot{\eta}(t) = A\eta(t) + Bu(t) - e^{Ah}L\big(y(t) - Cx_o(t)\big) \\ u(t) = F\eta(t) \end{cases}$$

Substituting $x_o(t) = e^{-Ah}\eta(t) - \int_{t-h}^{t} e^{A(t-h-\theta)}Bu(\theta)d\theta$, we get

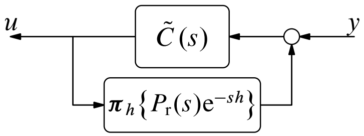
$$\begin{cases} \dot{\eta}(t) = (A + e^{Ah}LCe^{-Ah})\eta(t) + Bu(t) \\ \qquad\qquad - e^{Ah}LC\displaystyle\int_{t-h}^{t} e^{A(t-h-\theta)}Bu(\theta)d\theta - e^{Ah}Ly(t) \\ u(t) = F\eta(t) \end{cases}$$

## Observer-predictor revised (contd)

Thus, we end up with control law

$$\begin{cases} \dot{\eta} = (A + BF + e^{Ah}LCe^{-Ah})\eta - e^{Ah}L\left(y + C\displaystyle\int_{t-h}^{t} e^{A(t-h-\theta)}Bu(\theta)d\theta\right) \\ u = F\eta \end{cases}$$

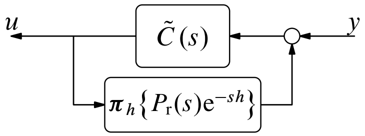
Now, noting that the last term above is the output of $\boldsymbol{\pi}_h\{P_r(s)e^{-sh}\}$ when $u$ is its input, control law above is actually




for

$$\tilde{C}(s) = \left[\begin{array}{c|c} A + BF + e^{Ah}LCe^{-Ah} & -e^{Ah}L \\ \hline F & 0 \end{array}\right].$$

## Connections




This is clearly MSP with primary controller,

$$\tilde{C}(s) = \left[\begin{array}{c|c} A + BF + e^{Ah}LCe^{-Ah} & -e^{Ah}L \\ \hline F & 0 \end{array}\right],$$

which is observer-based controller for

$$\tilde{P}(s) = \left[\begin{array}{c|c} A & B \\ \hline Ce^{-Ah} & 0 \end{array}\right]$$

(note that $A + e^{Ah}LCe^{-Ah} = e^{Ah}(A + LC)e^{-Ah}$ is Hurwitz). Thus,

- observer-predictor is MSP when primary controller $\tilde{C}$ is observer-based controller for $\tilde{P}$

## Outline



## Coprime factorization over $H^\infty$

We say that transfer function $P(s)$ has (strongly) coprime factorization over $H^\infty$ if there are transfer functions

$$M(s), N(s), \tilde{M}(s), \tilde{N}(s), X(s), Y(s), \tilde{X}(s), \tilde{Y}(s) \in H^\infty$$

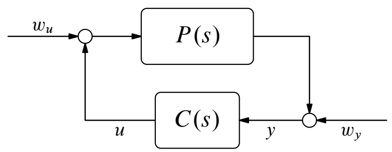
such that

$$P(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$$

and

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix}\begin{bmatrix} M(s) & -\tilde{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

## Coprime factorization and stabilizability




### Theorem

*There is controller $C(s)$ internally stabilizing this system iff*[3] *$P(s)$ has strong coprime factorization over $H^\infty$. In this case all stabilizing controllers can be parametrized as (Youla parametrization)*

$$\begin{aligned} C(s) &= \big(-\tilde{Y}(s) + M(s)Q(s)\big)\big(\tilde{X}(s) + N(s)Q(s)\big)^{-1} \\ &= \big(X(s) + Q(s)\tilde{N}(s)\big)^{-1}\big(-Y(s) + Q(s)\tilde{M}(s)\big) \end{aligned}$$

*for some $Q(s) \in H^\infty$ but otherwise arbitrary.*

[3] Most nontrivial part here, only if, was proved by Malcolm C. Smith (1989).

## Coprime factorization for rational systems

Let $P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with $(A, B)$ stabilizable and $(C, A)$ detectable. Then

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} = \left[\begin{array}{c|c:c} A + LC & B + LD & -L \\ \hline -F & I & 0 \\ \hdashline -C & -D & I \end{array}\right]$$

and

$$\begin{bmatrix} M(s) & -\tilde{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix} = \left[\begin{array}{c|c:c} A + BF & B & -L \\ \hline F & I & 0 \\ \hdashline C + DF & D & I \end{array}\right],$$

where $F$ and $L$ are any matrices such that $A + BF$ and $A + LC$ Hurwitz.

## Reduction to rational factorization

Let $P(s)$ be (not necessarily rational) proper transfer function such that

$$P(s) = P_{\mathrm{a}}(s) - \Delta(s)$$

for some $\Delta(s) \in H^\infty$ and rational $P_{\mathrm{a}}(s)$ with coprime factorization

$$\begin{bmatrix} X_{\mathrm{a}}(s) & Y_{\mathrm{a}}(s) \\ -\tilde{N}_{\mathrm{a}}(s) & \tilde{M}_{\mathrm{a}}(s) \end{bmatrix} \begin{bmatrix} M_{\mathrm{a}}(s) & -\tilde{Y}_{\mathrm{a}}(s) \\ N_{\mathrm{a}}(s) & \tilde{X}_{\mathrm{a}}(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

We're looking for

- strongly coprime factorization of $P(s)$ in terms of that of $P_{\mathrm{a}}(s)$.

## Reduction to rational factorization (contd)

**Lemma**
*$P(s) = P_a(s) - \Delta(s)$ has strongly coprime factorization*

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} = \begin{bmatrix} X_a(s) & Y_a(s) \\ -\tilde{N}_a(s) & \tilde{M}_a(s) \end{bmatrix} \begin{bmatrix} I & 0 \\ \Delta(s) & I \end{bmatrix} \in H^\infty$$

*and*

$$\begin{bmatrix} M(s) & -\tilde{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ -\Delta(s) & I \end{bmatrix} \begin{bmatrix} M_a(s) & -\tilde{Y}_a(s) \\ N_a(s) & \tilde{X}_a(s) \end{bmatrix} \in H^\infty$$

**Proof.**
By direct substitution. □

## Resulting stabilizing controllers

Youla parametrization in this case is

$$\begin{aligned} C &= (-\tilde{Y} + MQ)(\tilde{X} + NQ)^{-1} \\ &= (-\tilde{Y}_{\mathrm{a}} + M_{\mathrm{a}}Q)\big(\tilde{X}_{\mathrm{a}} + N_{\mathrm{a}}Q - \Delta(-\tilde{Y}_{\mathrm{a}} + M_{\mathrm{a}}Q)\big)^{-1} \end{aligned}$$

Hence,

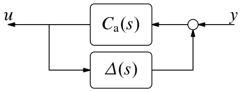
$$C(\tilde{X}_{\mathrm{a}} + N_{\mathrm{a}}Q) - C\Delta(-\tilde{Y}_{\mathrm{a}} + M_{\mathrm{a}}Q) = -\tilde{Y}_{\mathrm{a}} + M_{\mathrm{a}}Q$$

or, equivalently,

$$C(\tilde{X}_{\mathrm{a}} + N_{\mathrm{a}}Q) = (I + C\Delta)(-\tilde{Y}_{\mathrm{a}} + M_{\mathrm{a}}Q).$$

Thus, denoting $C_{\mathrm{a}} := (-\tilde{Y}_{\mathrm{a}} + M_{\mathrm{a}}Q)(\tilde{X}_{\mathrm{a}} + N_{\mathrm{a}}Q)^{-1}$, we end up with

$$(I + C\Delta)^{-1}C = C_{\mathrm{a}} \quad \Longleftrightarrow \quad C = C_{\mathrm{a}}(I - \Delta C_{\mathrm{a}})^{-1} =$$




Hmm, looks familiar...

## Dead-time systems

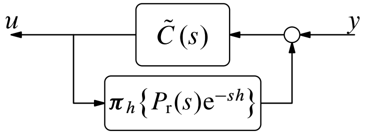
If

$$P(s) = P_{\mathrm{r}}(s)\mathrm{e}^{-sh} = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]\mathrm{e}^{-sh}$$

we already know how to present it in form

$$P(s) = \tilde{P}(s) - \boldsymbol{\pi}_h\{P(s)\} = \left[\begin{array}{c|c} A & B \\ \hline C\mathrm{e}^{-Ah} & 0 \end{array}\right] - \boldsymbol{\pi}_h\left\{\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]\mathrm{e}^{-sh}\right\}$$

Thus, any stabilizing controller for $P(s)$ is of the form




where $\tilde{C}(s)$ is stabilizing controller for $\tilde{P}(s)$. Thus, we end up with

- modified Smith predictor yet again.

## Outline

Output feedback for input-delay systems: adding observers

Smith controller revised

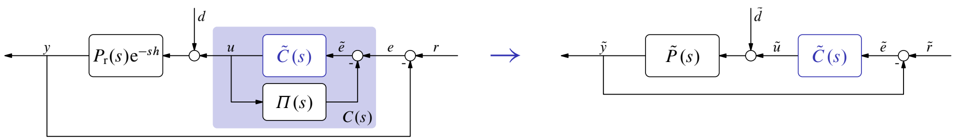
Modified Smith predictor and dead-time compensation

Modified Smith predictor vs. observer-predictor

Coprime factorization over $H^\infty$ and Youla parametrization

Two-stage design of dead-time compensators

## Idea




$\mathcal{S}_1$: design $\tilde{C}$ for (delay-free) $\tilde{P}$

and then

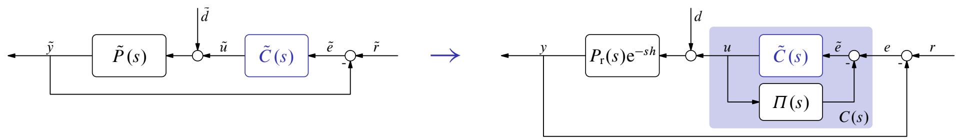
$\mathcal{S}_2$: implement $\tilde{C}$ in combination with DTC $\Pi$.

Hence, two-stage design.

Clear advantages:

- design delay-free
- stability preserved
- resulting controller implementable

## Good question




Assume that $\tilde{C}$ is designed well (for $\tilde{P}$). This probably means that

- responses of $\tilde{y}$ and $\tilde{u}$ to "reasonable" $\tilde{d}$ and $\tilde{r}$ are "good".

Does it imply that

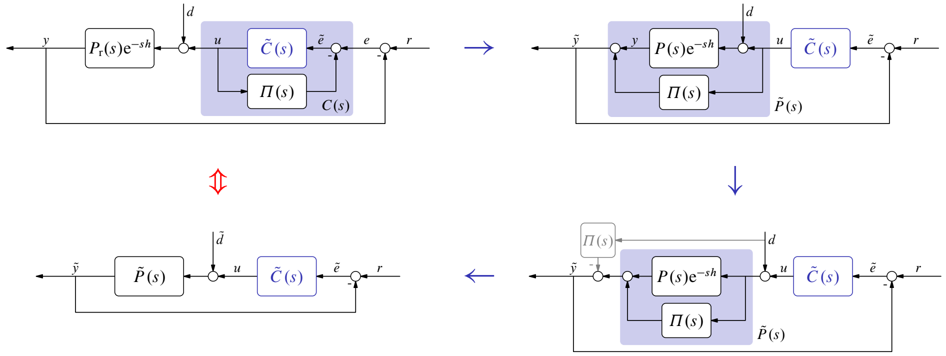
- responses of $y$ and $u$ to "reasonable" $d$ and $r$ are also "good"

for the original system ?

Not necessarily, just because

- $\{\tilde{y}, \tilde{u}, \tilde{r}, \tilde{d}\} \not\Leftrightarrow \{y, u, r, d\}$
- loop $\tilde{P} \circlearrowleft \tilde{C}$ is different from loop $P_{\mathrm{r}}\mathrm{e}^{-sh} \circlearrowleft C$

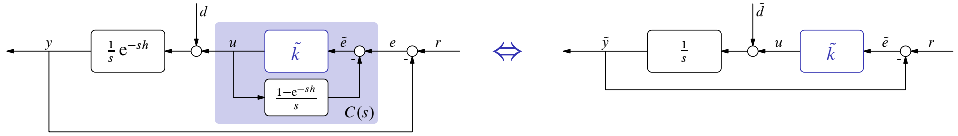
## Signal tracing



with

$$\tilde{y} = y + \Pi u \quad \text{and} \quad \tilde{d} = \left(1 - \frac{\Pi}{\tilde{P}}\right) d = \frac{P_{\mathrm{r}} \mathrm{e}^{-sh}}{\tilde{P}} d$$

## Example 1: static gain with P primary controller



Control goal: reduce steady-state effect of step $d$ on $y$ or, equivalently,

- end up with high static gain of controller, $|C(0)|$.

Stability is equivalent to $\tilde{k} > 0$ and for system in $\mathcal{S}_1$

- we can end up with arbitrarily large (primary) controller static gain, $\tilde{k}$.

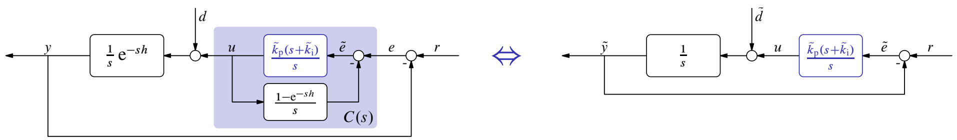
Looks perfect... yet (mind that $\Pi(0) = h$)

$$|C(0)| = \frac{\tilde{k}}{1 + \tilde{k}h} = \frac{1}{1/\tilde{k} + h} < \frac{1}{h},$$

which is not necessarily what we expect. Thus, in this case

- two-stage design might be misleading.

## Example 2: static gain with PI primary controller



Now, stability is guaranteed for all $\tilde{k}_{\mathrm{p}} > 0$ and $\tilde{k}_{\mathrm{i}} \geq 0$ and

- primary controller has infinite static gain

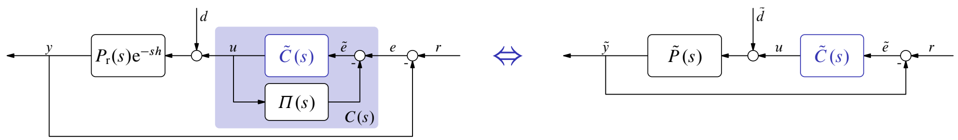
because of its integral action. Still,

$$|C(0)| = \frac{1}{h},$$

which is far from what we expect. Thus, in this case

- two-stage design is even more misleading.

## Comparing loop gains



Let $\tilde{L} := \tilde{P}\tilde{C}$ be loop gain in $\mathcal{S}_1$. Then resulting loop gain

$$L = P_{\mathrm{r}} \mathrm{e}^{-sh} \frac{\tilde{C}}{1 + \tilde{C}\Pi} = \frac{(\tilde{P} - \Pi)\tilde{C}}{1 + \tilde{C}\Pi} = \frac{\tilde{P}\tilde{C} + 1 - (1 + \Pi\tilde{C})}{1 + \tilde{C}\Pi},$$

from which[4]

$$1 + L = \frac{1}{1 + \tilde{C}\Pi}(1 + \tilde{L}).$$

Thus, we need to

- keep $|\tilde{C}(\mathrm{j}\omega)\Pi(\mathrm{j}\omega)|$ small at frequencies of interest

to preserve loop gain properties achieved in $\mathcal{S}_1$ at these frequencies.

[4] $1 + L(s)$ called return difference t.f. and plays an important role in feedback analysys.

## How to achieve $|\tilde{C}(j\omega)\Pi(j\omega)| \ll 1$

We can

1. decrease $|\tilde{C}(j\omega)|$
   (only makes sense at frequencies where a low loop gain is required)
2. decrease $|\Pi(j\omega)|$
   (our only choice at frequencies where a hight loop gain is required)

Option 2 is effectively an

- additional requirement for the choice of (rational) $\tilde{P}$,

apart from $\Pi \in H^\infty$. Small $|\Pi(j\omega)|$ implies that at these frequencies

- $\tilde{P}(j\omega)$ is a good approximation of $P_r(j\omega)\,e^{-j\omega h}$,

which is a reasonable strategy justifying two-stage design (note that we then also have $C(j\omega) \approx \tilde{C}(j\omega)$).

## Static gain of MSP

Consider the requirement

$$C(0) = \tilde{C}(0)$$

implying that we want to guarantee that static gain of $\tilde{C}$ is preserved in the two-stage design. This clearly is equivalent to

$$\Pi(0) = 0.$$

In the MSP case,

$$\Pi(0) = C \int_0^h e^{A(t-h)} dt B \neq 0$$

generically. To render it zero we may

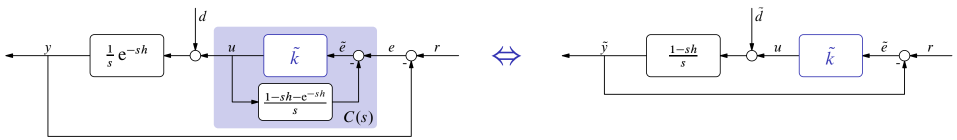
- subtract from $\Pi$ any $RH^\infty$ transfer function with static gain $\Pi(0)$.

Apparently, the easiest example is a static gain $\Pi(0)$, resulting in

$$\Pi_{WI}(s) = C \int_0^h e^{A(t-h)}(e^{-st} - 1) dt B$$

(clearly $\Pi_{WI}(0) = 0$), which sometimes called Watanabe-Ito DTC.

## Example 1: static gain with P primary controller (contd)




Now $\Pi_{WI}(0) = 0$, which implies that

$$|C(0)| = |\tilde{C}(0)| = \tilde{k}$$

We still cannot reach high-gain $C$ because stability in $\mathcal{S}_1$ requires

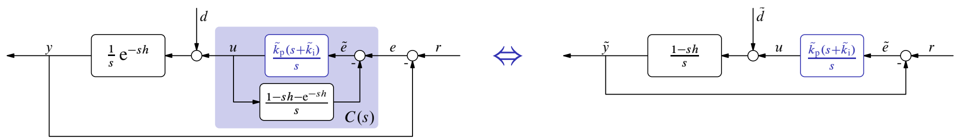
$$0 < \tilde{k} < \frac{1}{h}$$

(if $\tilde{k} = \frac{1}{h}$, controller loop is ill posed). Advantage here is that

- this limitation shows up already in $\mathcal{S}_1$,

thus helping us to be under no illusions when $\tilde{k}$ is designed in $\mathcal{S}_1$.

## Example 2: static gain with PI primary controller (contd)




Again, $\Pi_{WI}(0) = 0$ guarantees (provided $0 < \tilde{k}_p < \frac{1}{h}$ and $0 < \tilde{k}_i < \frac{1}{h}$) that

$$|C(0)| = |\tilde{C}(0)| = \infty,$$

exactly what we need.

Note that $\tilde{P}(s) = \frac{1-sh}{s}$ can be thought of as

- approximation of $P(s) = \frac{1}{s}e^{-sh}$

(in fact, $\tilde{P}(s)$ is the $[1,1]$-Padé approximation of $P(s)$).