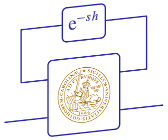
Introduction to Time-Delay Systems



lecture no. 3

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Outline

Smith controller

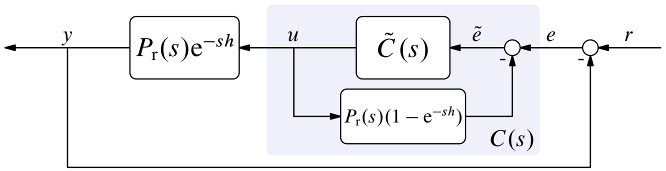
Finite spectrum assignment

Alternative viewpoint on FSA: Kwon-Pearson-Artstein reduction

Fiagbedzi-Pearson reduction

"Just because you can explain it doesn't mean it's not still a miracle."

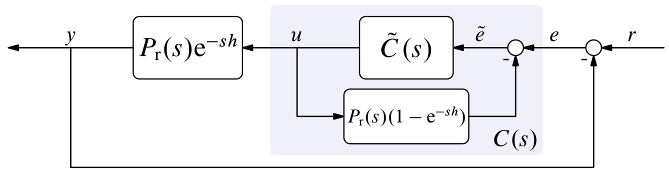
The scheme



Introduced by Otto J. M. Smith more than half a century ago (in 1957).

- $\tilde{C}(s)$ called primary controller
- $P_r(s)(1-e^{-sh})$ called Smith predictor
  (note that $\tilde{e} = (r-y) - P_r(1-e^{-sh})u = r - P_r u$, which is "prediction" of e with $h = 0$)

The magic



Closed-loop signal trailing:

$$u = \tilde{C}\big((r - P_r e^{-sh}u) - P_r(1-e^{-sh})u\big) = \tilde{C}(r - P_r u)$$

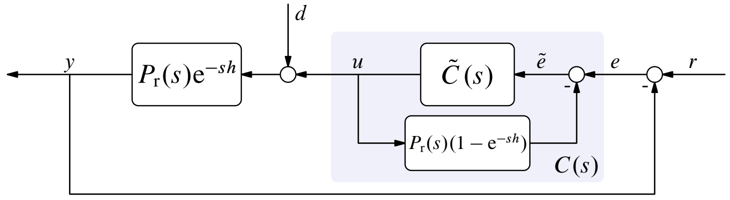
from which the closed-loop systems from r to u and y are

$$T_u(s) = \frac{\tilde{C}(s)}{1 + P_r(s)\tilde{C}(s)} \qquad \text{and} \qquad T(s) = \frac{P_r(s)\tilde{C}(s)}{1 + P_r(s)\tilde{C}(s)} e^{-sh},$$

respectively. Thus, delay is eliminated from the characteristic equation and

- stabilization of $(P_r e^{-sh}, C)$ reduces to delay-free stabilization of $(P_r, \tilde{C})$.

Is it that simple ?



Consider the effect of the input disturbance d:

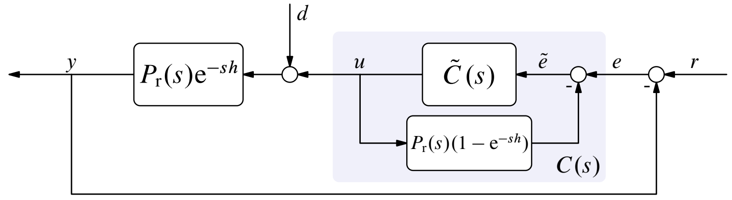
$$u = \tilde{C}\big((r - P_r e^{-sh}u - P_r e^{-sh}d) - P_r(1-e^{-sh})u\big) = \tilde{C}(r - P_r u - P_r e^{-sh}d)$$

from which the transfer function from d to y is

$$T_d(s) = \frac{P_r(s)}{1 + P_r(s)\tilde{C}(s)}\big(1 + \tilde{C}(s)P_r(s)(1-e^{-sh})\big)e^{-sh},$$

which might be unstable if so is P_r.

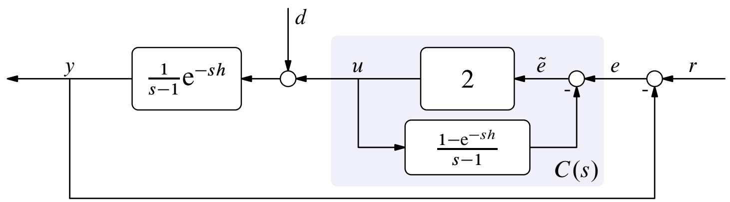
Preliminary conclusions



Smith controller

- works if $P_r(s)$ is stable
  (stabilization problem reduces then to that for delay-free plant)
- does not necessarily work if $P_r(s)$ is unstable
  (might lead to unstable loop)

Example 1



In this case $\tilde{C}$ does stabilize P_r, so that

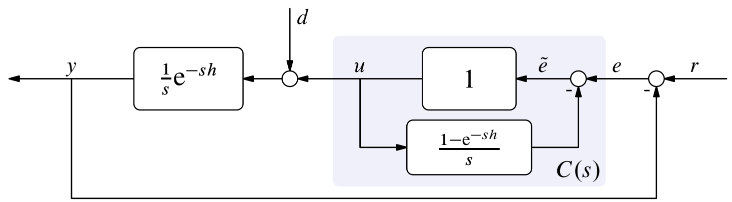
$$T_u(s) = \frac{2(s-1)}{s+1} \quad \text{and} \quad T(s) = \frac{2}{s+1}e^{-sh}$$

are stable. Yet

$$T_d(s) = \frac{2}{s+1}\left(1 + \frac{2(1-e^{-sh})}{s-1}\right)e^{-sh} = \frac{2}{s+1}\,\frac{s+1-2e^{-sh}}{s-1}\,e^{-sh}$$

is unstable.

Example 2



In this case $\tilde{C}$ does stabilize P_r, so that

$$T_u(s) = \frac{s}{s+1} \quad \text{and} \quad T(s) = \frac{1}{s+1}e^{-sh}$$

are stable. Oddly enough,

$$T_d(s) = \frac{1}{s+1}\left(1 + \frac{1-e^{-sh}}{s}\right)e^{-sh}$$

is stable too (since $\frac{1-e^{-sh}}{s} \in H^\infty$).

Extensions

The idea of Smith can be extended to

- more general class of i/o delays
- unstable systems (e.g., the modified Smith predictor)

We'll study some of these extensions in due course...

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Preliminary: state feedback revised

Let

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

and $u(t) = Fx(t)$. Rewrite this set of equations in s-domain as

$$\begin{bmatrix} sI - A & -B \\ -F & I \end{bmatrix} \begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}.$$

The characteristic equation of this system is[1]

$$\chi(s) = \det \begin{bmatrix} sI - A & -B \\ -F & I \end{bmatrix} = \det(sI - A - BF) = 0,$$

so the closed-loop system stable iff $A + BF$ is Hurwitz.

[1] Remember, $\det \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix} \right] = \det(A_{22}) \det(A_{11} - A_{12} A_{22}^{-1} A_{21})$ whenever $\det(A_{22}) \neq 0$.

Preliminary: Laplace transform of distributed delay

Let

$$\zeta(t) = \int_{t-h}^{t} \Phi(t - \theta) \xi(\theta) \mathrm{d}\theta$$

for some function Φ. This can be seen as the convolution $\zeta = \tilde{\Phi} * \xi$, where

$$\tilde{\Phi}(t) := \Phi(t) \mathbb{1}_{[0,h]}(t), \quad \text{where } \mathbb{1}_{\mathbb{A}} \text{ is the indicator function of } \mathbb{A} \subset \mathbb{R}^+.$$

Then

$$\begin{aligned} \mathcal{L}\left\{ \int_{t-h}^{t} \Phi(t - \theta) \xi(\theta) \mathrm{d}\theta \right\} &= \mathcal{L}\{\tilde{\Phi}\} \mathcal{L}\{\xi\} = \int_{0}^{\infty} \tilde{\Phi}(t) \mathrm{e}^{-st} \mathrm{d}t \, \Xi(s) \\ &= \int_{0}^{h} \Phi(t) \mathrm{e}^{-st} \mathrm{d}t \, \Xi(s) \end{aligned}$$

Problem statement

Let

$$\Sigma_h : \dot{x}(t) = Ax(t) + Bu(t-h), \quad (x(0), u_\tau(0)) = (x_0, 0)$$

and we measure $x(t)$. We look for $u(t)$ stabilizing Σ_h for any given h.

Motivation

If at each $t \in \mathbb{R}$ we measured $x(t+h)$, the control law

$$u(t) = Fx(t+h)$$

would stabilize Σ_h whenever $A + BF$ is Hurwitz. Although $x(t+h)$ cannot be measured, we may try to use its calculated (predicted) values

$$\begin{aligned} x_h(t) &:= \mathrm{e}^{Ah}x(t) + \int_t^{t+h} \mathrm{e}^{A(t+h-\theta)}Bu(\theta-h)\mathrm{d}\theta \\ &= \mathrm{e}^{Ah}x(t) + \int_{t-h}^{t} \mathrm{e}^{A(t-\theta)}Bu(\theta)\mathrm{d}\theta \\ &= \mathrm{e}^{Ah}x(t) + \int_0^{h} \mathrm{e}^{A\theta}Bu(t-\theta)\mathrm{d}\theta \end{aligned}$$

instead (think of observer-based feedback). Will it work?

Characteristic equation

In s-domain,

$$\begin{aligned} sX(s) - x_0 &= AX(s) + B\mathrm{e}^{-sh}U(s) \\ U(s) &= F\mathrm{e}^{Ah}X(s) + F\int_0^h \mathrm{e}^{-(sI-A)\theta}\mathrm{d}\theta B\, U(s) \end{aligned}$$

or

$$\begin{bmatrix} sI - A & -B\mathrm{e}^{-sh} \\ -F\mathrm{e}^{Ah} & I - F\int_0^h \mathrm{e}^{-(sI-A)\theta}\mathrm{d}\theta\, B \end{bmatrix} \begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}.$$

Hence, the characteristic quasi-polynomial is

$$\chi_h(s) = \det \begin{bmatrix} sI - A & -B\mathrm{e}^{-sh} \\ -F\mathrm{e}^{Ah} & I - F\int_0^h \mathrm{e}^{-(sI-A)\theta}\mathrm{d}\theta\, B \end{bmatrix}.$$

Characteristic equation (contd)

Now,

$$\begin{aligned} &\begin{bmatrix} sI - A & -B\mathrm{e}^{-sh} \\ -F\mathrm{e}^{Ah} & I - F\int_0^h \mathrm{e}^{-(sI-A)\theta}\mathrm{d}\theta\, B \end{bmatrix} \begin{bmatrix} I & -\mathrm{e}^{-Ah}\int_0^h \mathrm{e}^{-(sI-A)\theta}\mathrm{d}\theta\, B \\ 0 & I \end{bmatrix} \\ &\quad = \begin{bmatrix} sI - A & -B\mathrm{e}^{-sh} - \mathrm{e}^{-Ah}(sI-A)\int_0^h \mathrm{e}^{-(sI-A)\theta}\mathrm{d}\theta\, B \\ -F\mathrm{e}^{Ah} & I \end{bmatrix} \\ &\quad = \begin{bmatrix} sI - A & -\mathrm{e}^{-Ah}B \\ -F\mathrm{e}^{Ah} & I \end{bmatrix} \end{aligned}$$

because

$$-\mathrm{e}^{-Ah}(sI-A)\int_0^h \mathrm{e}^{-(sI-A)\theta}\mathrm{d}\theta = \mathrm{e}^{-Ah}(\mathrm{e}^{-(sI-A)h} - I) = \mathrm{e}^{-sh}I - \mathrm{e}^{-Ah}.$$

Hence,

$$\begin{aligned} \chi_h(s) &= \det(sI - A - \mathrm{e}^{-Ah}BF\mathrm{e}^{Ah}) = \det\big(\mathrm{e}^{-Ah}(sI - A - BF)\mathrm{e}^{Ah}\big) \\ &= \det(sI - A - BF). \end{aligned}$$

Characteristic equation (contd)

Thus,

- characteristic quasi-polynomial is actually a polynomial

and the characteristic equation,

$$\det(sI - A - BF) = 0,$$

is finite dimensional. In other words, (predictive) control law

$$u(t) = F x_h(t)$$

keeps the closed-loop spectrum finite. Hence the terms

- finite spectrum assignment (FSA).

Problem solution

Theorem

The FSA control law

$$u(t) = F\left(e^{Ah}x(t) + \int_0^h e^{A\theta}Bu(t-\theta)d\theta\right)$$

stabilizes

$$\dot{x}(t) = Ax(t) + Bu(t-h)$$

iff $A + BF$ is Hurwitz.

Note that

- stabilizability of Σ_h is equivalent to that of its delay-free version Σ_0.

Extensions

Was extended to the cases when

- we measure $y(t) = Cx(t)$
  In this case *observer-predictor* of the form

$$\begin{cases} \dot{\hat{x}}(t) = (A + LC)\hat{x}(t) + Bu(t-h) - Ly(t) \\ u(t) = F\left(e^{Ah}\hat{x}(t) + \int_0^h e^{A\theta}Bu(t-\theta)d\theta\right) \end{cases}$$

  should be used.
- there are multiple / distributed input delays
- there are limited classes of state delays

Historical remarks

- First proposed by Mayne (1968) for single input delay, $y = Cx$
  (effectively forgotten; as of April 2012, is cited *once* according to Google Scholar; as of November 2012, 4 more citations were added)
- Independently derived by Kleinman (1969)
  (as a solution to delayed LQG, not credited with the invention of FSA either)
- Solving the case of measured x is credited to Manitius & Olbrot (1979)
  (general input delays and some classes of state delay)
- Observer-predictor proposed by Furukawa & Shimemura (1983)

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Let

$$\Sigma_h : \dot{x}(t) = Ax(t) + Bu(t-h), \quad (x(0), u_\tau(0)) = (x_0, 0)$$

and we measure $x(t)$. We look for $u(t)$ stabilizing Σ_h for any given h.

Preliminary: Leibniz integral rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(\theta,t)\mathrm{d}\theta = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(\theta,t)\mathrm{d}\theta + \frac{\mathrm{d}b(t)}{\mathrm{d}t} f(b(t),t) - \frac{\mathrm{d}a(t)}{\mathrm{d}t} f(a(t),t)$$

Reduced system

Introduce

$$\tilde{x}(t) := x(t) + \int_0^h \mathrm{e}^{A(\theta-h)} Bu(t-\theta)\mathrm{d}\theta = x(t) + \int_{t-h}^{t} \mathrm{e}^{A(t-\theta-h)} Bu(\theta)\mathrm{d}\theta$$

(note that $\tilde{x} = \mathrm{e}^{-Ah} x_h$). Then, using Leibniz rule, we have:

$$\begin{aligned}\dot{\tilde{x}}(t) &= \dot{x}(t) + A\int_{t-h}^{t} \mathrm{e}^{A(t-\theta-h)} Bu(\theta)\mathrm{d}\theta + \mathrm{e}^{-Ah} Bu(t) - Bu(t-h) \\ &= Ax(t) + A\int_{t-h}^{t} \mathrm{e}^{A(t-\theta-h)} Bu(\theta)\mathrm{d}\theta + \mathrm{e}^{-Ah} Bu(t) \\ &= \underbrace{A}_{\tilde{A}}\tilde{x}(t) + \underbrace{\mathrm{e}^{-Ah}B}_{\tilde{B}} u(t)\end{aligned}$$

i.e.,

- $\tilde{x}(t)$ satisfies ordinary (delay-free) differential equation.

We call this finite-dimensional system reduced system and denote it as $\tilde{\Sigma}$.

Stabilization via reduced system

It can be proved that

- $\tilde{\Sigma}$ is controllable iff Σ_h is (absolutely[2]) controllable
- if the control law

$$u(t) = \tilde{F}\tilde{x}(t) = \tilde{F}\left(x(t) + \int_0^h \mathrm{e}^{A(\theta-h)} Bu(t-\theta)\mathrm{d}\theta\right)$$

stabilizes $\tilde{\Sigma}$, then if stabilizes Σ_h as well.

[2] Absolute controllability means the controllability of the whole $(x(t), u_\tau(t))$.

FSA property

The transformation

$$\tilde{x}(t) = x(t) + \int_{t-h}^{t} \mathrm{e}^{A(t-\theta-h)} Bu(\theta)\mathrm{d}\theta$$

can be rewritten in the s-domain as

$$\begin{bmatrix} \tilde{X}(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} I & \mathrm{e}^{-Ah}\int_0^h \mathrm{e}^{-(sI-A)\theta}\mathrm{d}\theta\, B \\ 0 & I \end{bmatrix} \begin{bmatrix} X(s) \\ U(s) \end{bmatrix}$$

(haven't we already seen it?). If $u = \tilde{F}\tilde{x}$, the closed-loop reduced system is

$$\begin{bmatrix} sI - A & -\mathrm{e}^{-Ah}B \\ -\tilde{F} & I \end{bmatrix} \begin{bmatrix} \tilde{X}(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$$

so that

$$\begin{bmatrix} sI - A & -\mathrm{e}^{-Ah}B \\ -\tilde{F} & I \end{bmatrix} \begin{bmatrix} I & \mathrm{e}^{-Ah}\int_0^h \mathrm{e}^{-(sI-A)\theta}\mathrm{d}\theta\, B \\ 0 & I \end{bmatrix} \begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}.$$

FSA property (contd)

Thus, using the equality

$$\mathrm{e}^{-Ah}(sI - A)\int_0^h \mathrm{e}^{-(sI-A)\theta}\mathrm{d}\theta = \mathrm{e}^{-Ah} - \mathrm{e}^{-sh}I$$

again, we have the following closed-loop equations for Σ_h:

$$\begin{bmatrix} sI - A & -\mathrm{e}^{-sh}B \\ -\tilde{F} & I - \tilde{F}\mathrm{e}^{-Ah}\int_0^h \mathrm{e}^{-(sI-A)\theta}\mathrm{d}\theta\, B \end{bmatrix} \begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}.$$

In other words, the control law $u = \tilde{F}\tilde{x}$ applied to Σ_h yields a closed-loop system with finite spectrum. Moreover,

$$\mathrm{spec}(\Sigma_{h,\mathrm{cl}}) = \mathrm{spec}(\tilde{\Sigma}_{\mathrm{cl}}) = \mathrm{spec}(A + \mathrm{e}^{-Ah}B\tilde{F})$$

and the choice $\tilde{F} = F\mathrm{e}^{Ah}$ returns us to FSA.

Extension to distributed input delay

Let now

$$\Sigma_h : \dot{x}(t) = Ax(t) + \int_{-h}^{0} \beta(\tau)u(t+\tau)\mathrm{d}\tau.$$

Introduce

$$\tilde{x}(t) := x(t) + \int_{t-h}^{t}\int_{-h}^{\theta-t} \mathrm{e}^{A(t-\theta+\tau)}\beta(\tau)\mathrm{d}\tau\, u(\theta)\mathrm{d}\theta$$

Then

$$\begin{aligned} \dot{\tilde{x}}(t) &= \dot{x}(t) + \int_{t-h}^{t}\Bigl(A\int_{-h}^{\theta-t} \mathrm{e}^{A(t-\theta+\tau)}\beta(\tau)\mathrm{d}\tau - \beta(\theta-t) - 0\Bigr)u(\theta)\mathrm{d}\theta \\ &\qquad + \int_{-h}^{0} \mathrm{e}^{A\tau}\beta(\tau)\mathrm{d}\tau\, u(t) - 0 \\ &= A\tilde{x}(t) + \int_{-h}^{0} \mathrm{e}^{A\tau}\beta(\tau)\mathrm{d}\tau\, u(t) \end{aligned}$$

and we only need to stabilize $\tilde{\Sigma}$ with $\tilde{A} = A$ and $\tilde{B} = \int_{-h}^{0} \mathrm{e}^{A\tau}\beta(\tau)\mathrm{d}\tau$.

Historical remarks

- Roots in optimal control, e.g., (Bate, 1969; Slater & Wells, 1972)
- First explicitly proposed by Kwon & Pearson (1980) for systems of the form $\dot{x}(t) = Ax(t) + B_0u(t) + B_hu(t-h)$
  (motivated by finite-horizon minimum energy control with zero final constraints)
- Extended by Artstein (1982) to rather general class of input delays and to time-varying systems
- Extended by Fiagbedzi & Pearson (1986, 1990) to state / measurement delays
  (although details of the algorithm are less elegant in general)

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Nontrivial complication

So far, we studied problems in which Σ_h has finite spectrum. Let now

$$\Sigma_h : \dot{x}(t) = A_0 x(t) + A_h x(t-h) + Bu(t),$$

which might have

- infinite number of open-loop poles $\quad\longleftarrow\quad$ complicates matters

Workaround:

- move only part of these eigenvalues by feedback,

namely, unstable ones (we know that there is only a finite number of them).

Yet another transformation

Consider

$$\tilde{x}(t) := Qx(t) + \int_{t-h}^{t} \mathrm{e}^{\tilde{A}(t-\theta-h)} Q A_h x(\theta) \mathrm{d}\theta$$

for some $\tilde{A} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ and $Q \in \mathbb{R}^{\tilde{n}\times n}$. Then,

$$\begin{aligned}
\dot{\tilde{x}}(t) &= Q\dot{x}(t) + \tilde{A}\int_{t-h}^{t} \mathrm{e}^{\tilde{A}(t-\theta-h)} Q A_h x(\theta)\mathrm{d}\theta + \mathrm{e}^{-\tilde{A}h} Q A_h x(t) - Q A_h x(t-h) \\
&= Q A_0 x(t) + QBu(t) + \tilde{A}\int_{t-h}^{t} \mathrm{e}^{\tilde{A}(t-\theta-h)} Q A_h x(\theta)\mathrm{d}\theta + \mathrm{e}^{-\tilde{A}h} Q A_h x(t) \\
&= \tilde{A}\tilde{x}(t) + QBu(t) - \big(\tilde{A}Q - QA_0 - \mathrm{e}^{-\tilde{A}h} Q A_h\big)x(t).
\end{aligned}$$

If we can choose $\tilde{A}$ satisfying the left characteristic matrix equation

$$\tilde{A}Q = QA_0 + \mathrm{e}^{-\tilde{A}h} Q A_h, \qquad \text{(l.c.m.e.)}$$

Σ_h reduces to

$$\tilde{\Sigma} : \dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + QBu(t).$$

Properties of solutions of l.c.m.e. $\tilde{A}Q = QA_0 + \mathrm{e}^{-\tilde{A}h} Q A_h$

Let $\tilde{\lambda} \in \operatorname{spec}(\tilde{A})$ and $\tilde{\eta}$ be the corresponding left eigenvector. Then

$$\tilde{\lambda}\tilde{\eta}Q = \tilde{\eta}\tilde{A}Q = \tilde{\eta}QA_0 + \tilde{\eta}\,\mathrm{e}^{-\tilde{A}h} Q A_h = \tilde{\eta}Q(A_0 + \mathrm{e}^{-\tilde{\lambda}h} A_h).$$

In other words, $\tilde{\eta}Q(\tilde{\lambda}I - A_0 - \mathrm{e}^{-\tilde{\lambda}h} A_h) = 0$, so that

- whenever $\eta := \tilde{\eta}Q \neq 0$, every eigenvalue of $\tilde{A}$ is a pole of Σ_h

(remember, those poles are the roots of $\chi_h(s) = \det(sI - A_0 - A_h \mathrm{e}^{-sh})$).

Closed-loop spectrum

Suppose we can solve (l.c.m.e.) in $\tilde{A}$ and Q. Then

$$u(t) = \tilde{F}\tilde{x}(t) = \tilde{F}\biggl(Qx(t) + \int_{t-h}^{t} \mathrm{e}^{\tilde{A}(t-\theta-h)} QA_h x(\theta)\mathrm{d}\theta\biggr)$$

for some $\tilde{F}$ leads to the closed-loop system

$$\begin{bmatrix} sI - A_0 - A_h\mathrm{e}^{-sh} & -B \\ -\tilde{F}Q - \tilde{F}\mathrm{e}^{-\tilde{A}h}\int_0^h \mathrm{e}^{-(sI-\tilde{A})\theta}\mathrm{d}\theta QA_h & I \end{bmatrix}\begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = [\text{I.C.}].$$

and the characteristic quasi-polynomial $\chi_{\mathrm{cl}}(s) = \det \Delta(s)$, where

$$\Delta(s) := \begin{bmatrix} sI - A_0 - A_h\mathrm{e}^{-sh} & -B \\ -\tilde{F}\bigl(Q + \mathrm{e}^{-\tilde{A}h}\int_0^h \mathrm{e}^{-(sI-\tilde{A})\theta}\mathrm{d}\theta QA_h\bigr) & I \end{bmatrix}.$$

Closed-loop spectrum (contd)

Next,

$$\begin{aligned}(sI-\tilde{A})\biggl(Q + \mathrm{e}^{-\tilde{A}h}\int_0^h \mathrm{e}^{-(sI-\tilde{A})\theta}\mathrm{d}\theta\, QA_h\biggr) \\ = sQ - \tilde{A}Q + (\mathrm{e}^{-Ah} - \mathrm{e}^{-sh}I)QA_h = Q(sI - A_0 - A_h\mathrm{e}^{-sh}),\end{aligned}$$

so that

$$Q + \mathrm{e}^{-\tilde{A}h}\int_0^h \mathrm{e}^{-(sI-\tilde{A})\theta}\mathrm{d}\theta QA_h = (sI-\tilde{A})^{-1}Q(sI - A_0 - A_h\mathrm{e}^{-sh}).$$

Then

$$\Delta(s) = \begin{bmatrix} sI - A_0 - A_h\mathrm{e}^{-sh} & -B \\ -\tilde{F}(sI-\tilde{A})^{-1}Q(sI - A_0 - A_h\mathrm{e}^{-sh}) & I \end{bmatrix}$$

and

$$\begin{aligned}\chi_{\mathrm{cl}}(s) &= \det\bigl((sI - A_0 - A_h\mathrm{e}^{-sh}) - B\tilde{F}(sI-\tilde{A})^{-1}Q(sI - A_0 - A_h\mathrm{e}^{-sh})\bigr) \\ &= \det\bigl(I - B\tilde{F}(sI-\tilde{A})^{-1}Q\bigr)\det(sI - A_0 - A_h\mathrm{e}^{-sh}).\end{aligned}$$

Closed-loop spectrum (contd)

Because

$$\left[\begin{array}{c|c} \tilde{A} & Q \\ \hline -B\tilde{F} & I \end{array}\right] = \left[\begin{array}{c|c} \tilde{A} + QB\tilde{F} & Q \\ \hline B\tilde{F} & I \end{array}\right]^{-1},$$

we have that

$$\det\bigl(I - \tilde{F}(sI-\tilde{A})^{-1}QB\bigr) = \frac{\det(sI - \tilde{A} - QB\tilde{F})}{\det(sI-\tilde{A})}$$

and thus

$$\chi_{\mathrm{cl}}(s) = \frac{\det(sI - A_0 - A_h\mathrm{e}^{-sh})}{\det(sI-\tilde{A})}\det(sI - \tilde{A} - QB\tilde{F}).$$

In other words,

- $\operatorname{spec}(\Sigma_{h,\mathrm{cl}}) = [\operatorname{spec}(\Sigma_h) \setminus \operatorname{spec}(\tilde{A})] \bigcup \operatorname{spec}(\tilde{\Sigma}_{\mathrm{cl}})$

(remember, $\operatorname{spec}(\tilde{A}) \subset \operatorname{spec}(\Sigma_h)$).

Implications

Thus, if we can

- solve (l.c.m.e.) so that $\tilde{A}$ contains all unstable modes of Σ_h,
- find $\tilde{F}$ so that $\tilde{A} + QB\tilde{F}$ is Hurwitz (requires stabilizability of $(\tilde{A}, QB)$),

the control law

$$u(t) = \tilde{F}\biggl(Qx(t) + \int_{t-h}^{t} \mathrm{e}^{\tilde{A}(t-\theta-h)} QA_h x(\theta)\mathrm{d}\theta\biggr)$$

stabilizes Σ_h by moving all its unstable modes—those in $\operatorname{spec}(\tilde{A})$—to the eigenvalues of $\tilde{A} + QB\tilde{F}$ and keeping the other modes of Σ_h untouched.

Distributed state / input delays

Let

$$\Sigma_h : \dot{x}(t) = \int_{-h}^{0} \big(\alpha(\tau)x(t+\tau) + \beta(\tau)u(t+\tau)\big)\mathrm{d}\tau.$$

Then transformation

$$\tilde{x}(t) := Qx(t) + \int_{t-h}^{t}\int_{-h}^{\theta-t} \mathrm{e}^{\tilde{A}(t-\theta+\tau)} Q\big(\alpha(\tau)x(\theta) + \beta(\tau)u(\theta)\big)\mathrm{d}\tau\mathrm{d}\theta$$

with l.c.m.e.

$$\tilde{A}Q = \int_{-h}^{0} \mathrm{e}^{\tilde{A}\tau} Q\alpha(\tau)\mathrm{d}\tau \qquad \text{(l.c.m.e.')}$$

yields reduced system

$$\tilde{\Sigma} : \dot{\tilde{x}}(t) = \tilde{A}\tilde{x} + \tilde{B}u(t), \quad \text{where } \tilde{B} := \int_{-h}^{0} \mathrm{e}^{\tilde{A}\tau} Q\beta(\tau)\mathrm{d}\tau$$

and

- $\operatorname{spec}(\Sigma_{h,\mathrm{cl}}) = \big[\operatorname{spec}(\Sigma_h) \setminus \operatorname{spec}(\tilde{A})\big] \bigcup \operatorname{spec}(\tilde{\Sigma}_{\mathrm{cl}})$.

Is it that simple ?

Not quite, solving $\tilde{A}Q = \int_{-h}^{0} \mathrm{e}^{\tilde{A}\tau} Q\alpha(\tau)\mathrm{d}\tau$ is highly nontrivial. Specifically,

- we have to find all troublesome modes of Σ_h
  (in most cases, have to rely on numerical approaches)
- solve (l.c.m.e.) / (l.c.m.e.')
  (solution is non-unique and not especially elegant)

Only a handful of cases where the steps above can be solved analytically. One example is

$$\alpha(\tau) = \begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & * & * \\ 0 & 0 & \cdots & 0 & * \end{bmatrix} \delta(\tau) + \sum_i \begin{bmatrix} 0 & * & \cdots & * & * \\ 0 & 0 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & * \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \delta(\tau + h_i)$$

in which case $\operatorname{spec}(\Sigma_h)$ is finite and (l.c.m.e.') is solvable with $Q = I$.

Example

Let

$$\Sigma_h : \dot{x}(t) = -x(t) + x(t-h) + u(t),$$

whole characteristic quasi-polynomial is (here $s = \sigma + \mathrm{j}\omega$)

$$\chi_h(s) = s + 1 - \mathrm{e}^{-sh} = \sigma + 1 + \mathrm{j}\omega - \mathrm{e}^{-\sigma h}\mathrm{e}^{-\mathrm{j}\omega h}.$$

Solutions of $\chi_h(s) = 0$ must satisfy the magnitude condition

$$(\sigma + 1)^2 + \omega^2 = \mathrm{e}^{-2\sigma h}.$$

If $\sigma > 0$, this equation is unsolvable. If $\sigma = 0$, then $\omega = 0$ is the only option. Indeed, $s = 0$ is a root. Then, by L'Hôpital's rule,

$$\lim_{s\to 0} \frac{\chi_h(s)}{s} = 1 + \lim_{s\to 0} \frac{1 - \mathrm{e}^{-sh}}{s} = 1 + \lim_{s\to 0} \frac{h}{1} = 1 + h,$$

which implies that $s = 0$ is a single root.

Example (contd)

Thus, we have only one unstable pole to shift and may pick $\tilde{n} = 1$, $\tilde{A} = 0$. Eqn. (l.c.m.e.) then reads $0 = -q + q$, so we may pick $q = 1$. Then

$$\tilde{\Sigma} : \dot{\tilde{x}}(t) = u(t),$$

which is stabilized by $u(t) = -k\tilde{x}(t)$ for any $k > 0$. Thus

$$u(t) = -k\Big(x(t) + \int_{t-h}^{t} x(\theta)\mathrm{d}\theta\Big)$$

stabilizes Σ_h and renders its closed-loop characteristic polynomial

$$\chi_{h,\mathrm{cl}}(s) = \frac{s+k}{s}(s + 1 - \mathrm{e}^{-sh}).$$

In fact, the controller above has the transfer function

$$C(s) = -k\Big(1 + \frac{1 - \mathrm{e}^{-sh}}{s}\Big) \in H^\infty.$$

Outline

Smith controller

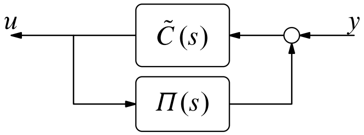
Finite spectrum assignment

Alternative viewpoint on FSA: Kwon-Pearson-Artstein reduction

Fiagbedzi-Pearson reduction

"Just because you can explain it doesn't mean it's not still a miracle."

Controllers with internal infinite-dimensional feedback



Smith controller: $\Pi(s) = -P_{\mathrm{r}}(s)(1 - \mathrm{e}^{-sh})$ and $\tilde{C}(s)$ is designed for $P_{\mathrm{r}}(s)$

FSA: $\Pi(s) = \mathrm{e}^{-Ah}\int_0^h \mathrm{e}^{-(sI-A)\theta}\mathrm{d}\theta B$ and $\tilde{C}(s) = F\mathrm{e}^{Ah}$, with F designed for (rational) $\tilde{P}(s) = (sI-A)^{-1}B$

Reduction: $\Pi(s) = \mathrm{e}^{-Ah}\int_0^h \mathrm{e}^{-(sI-A)\theta}\mathrm{d}\theta B$ and $\tilde{C}(s) = \tilde{F}$, with $\tilde{F}$ designed for (rational) $\tilde{P}(s) = (sI-A)^{-1}\mathrm{e}^{-Ah}B$

Each of them introduced as a clever trick, but there should be a reason for

- ending up with essentially the same structure...
- having essentially the same rationale (prediction) behind $\Pi(s)$...

Static state feedback: discrete-time case

Consider

$$\bar{\Sigma}_h : \bar{x}[k+1] = \bar{A}\bar{x}[k] + \bar{B}\bar{u}[k-h]$$

and assume that we measure whole $\bar{x}$. "True" state-space representation:

$$\begin{bmatrix} \bar{u}[k] \\ \bar{u}[k-1] \\ \vdots \\ \bar{u}[k-h+1] \\ \bar{x}[k+1] \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 \\ I & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & I & 0 & 0 \\ 0 & \cdots & 0 & \bar{B} & \bar{A} \end{bmatrix} \begin{bmatrix} \bar{u}[k-1] \\ \bar{u}[k-2] \\ \vdots \\ \bar{u}[k-h] \\ \bar{x}[k] \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \bar{u}[k]$$

and we measure whole state vector $\bar{x}_{\mathrm{a}}[k] = \begin{bmatrix} \bar{u}'[k-h] & \cdots & \bar{u}'[k-1] & \bar{x}'[k] \end{bmatrix}'$. Static state feedback in this case is

$$\bar{u}[k] = \underbrace{\begin{bmatrix} \bar{F}_{u,1} & \bar{F}_{u,2} & \cdots & \bar{F}_{u,h} & \bar{F}_x \end{bmatrix}}_{\bar{F}_{\mathrm{a}}} \bar{x}_{\mathrm{a}}[k] = \bar{F}_x x[k] + \sum_{i=1}^{h} \bar{F}_{u,i}\bar{u}[k-i],$$

which is dynamic control law in $\bar{x}$: $\bar{U}(z) = \bigl(I - \sum_{i=1}^{h} \bar{F}_{u,i} z^{-i}\bigr)^{-1}\bar{F}_x \bar{X}(z)$.

Choice of $\bar{F}_{\mathrm{a}}$: what can we do ?

If $(\bar{A}, \bar{B})$ is controllable, then so is the realization

$$\begin{bmatrix} \bar{u}[k] \\ \bar{u}[k-1] \\ \vdots \\ \bar{u}[k-h+1] \\ \bar{x}[k+1] \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 \\ I & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & I & 0 & 0 \\ 0 & \cdots & 0 & \bar{B} & \bar{A} \end{bmatrix} \begin{bmatrix} \bar{u}[k-1] \\ \bar{u}[k-2] \\ \vdots \\ \bar{u}[k-h] \\ \bar{x}[k] \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \bar{u}[k]$$

because its controllability matrix

$$\bar{\mathcal{M}}_{\mathrm{c,a}} = \left[\begin{array}{cccc|cccc} I & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & \bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{n-1}\bar{B} \end{array}\right] =: \begin{bmatrix} I & 0 \\ 0 & \bar{\mathcal{M}}_{\mathrm{c}} \end{bmatrix} \in \mathbb{R}^{(mh+n)\times(mh+n)}$$

So, in principle, we can assign closed-loop poles arbitrarily by $\bar{F}_{\mathrm{a}}$.

Choice of $\bar{F}_a$: what makes sense to do?

Handling augmented system $\bar{x}_a[k+1] = \bar{A}_a\bar{x}_a[k] + \bar{B}_a\bar{u}[k]$ as structureless finite-dimensional system is straightforward, yet

- numerically expensive (computational burden grows rapidly with h)
- conceptually wasteful (there is a plenty of structure to exploit)

Poles of system $\bar{P}_h(z) = \bar{P}_0(z)z^{-h} = \bar{P}_0(z)\cdot z^{-h}I_m$ are union of

1. n poles of delay-free system $\bar{P}_0(z)$
2. mh, where $m := \dim \bar{u}$, of delay z^{-h} at the origin

Poles at the origin are perfectly good, so we

- may only concentrate on poles of $\bar{P}_0(z)$

and don't need to waste our efforts on moving mh poles at $z = 0$.

Approach 1: pole placement via Ackermann's formula

Assume $m = 1$ and let required closed-loop characteristic polynomial be

$$\bar{\chi}_{\mathrm{cl,a}}(z) = z^h\bar{\chi}_{\mathrm{cl}}(z) = z^{n+h} + \bar{a}_{n-1}z^{n+h-1} + \cdots + a_1z^{h+1} + a_0z^h$$

By Ackermann's formula,

$$\bar{F}_a = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \bar{\mathcal{M}}_{\mathrm{c,a}}^{-1}\,\bar{\chi}_{\mathrm{cl,a}}(\bar{A}_a).$$

Looks cumbersome, yet good news is that

- structure of $\bar{A}_a$ can be exploited

to obtain tangible results.

Approach 1: exploiting structure of $(\bar{A}_a, \bar{B}_a)$

Straightforward, albeit boring, manipulations yield:

$$\bar{A}_a = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & \bar{B} & \bar{A} \end{bmatrix}, \quad \bar{A}_a^2 = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & \cdots & 0 & \bar{B} & \bar{A}\bar{B} & \bar{A}^2 \end{bmatrix},$$

$$\ldots, \bar{A}_a^h = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{h-1}\bar{B} & \bar{A}^h \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{h-1}\bar{B} & \bar{A}^h \end{bmatrix}$$

Approach 1: exploiting structure of $(\bar{A}_a, \bar{B}_a)$ (contd)

Thus,

$$\bar{A}_a^{h+i} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} \bar{A}^i \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{h-1}\bar{B} & \bar{A}^h \end{bmatrix}, \quad \forall i = 0, 1, \ldots$$

Hence,

$$\bar{\chi}_{\mathrm{cl,a}}(\bar{A}_a) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} \bar{\chi}_{\mathrm{cl}}(\bar{A}) \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{h-1}\bar{B} & \bar{A}^h \end{bmatrix}$$

and

$$\bar{\mathcal{M}}_{\mathrm{c,a}}^{-1}\,\bar{\chi}_{\mathrm{cl,a}}(\bar{A}_a) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} \bar{\mathcal{M}}_{\mathrm{c}}^{-1}\bar{\chi}_{\mathrm{cl}}(\bar{A}) \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{h-1}\bar{B} & \bar{A}^h \end{bmatrix}.$$

Approach 1: state feedback gain

Finally, we have:

$$\begin{aligned}\bar F_{\mathrm a} &= \begin{bmatrix} \bar F_{u,1} & \bar F_{u,2} & \cdots & \bar F_{u,h} & \bar F_x \end{bmatrix} \\ &= \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \bar{\mathcal M}_{\mathrm a}^{-1} \bar\chi_{\mathrm{cl}}(\bar A) \begin{bmatrix} \bar B & \bar A \bar B & \cdots & \bar A^{h-1}\bar B & \bar A^h \end{bmatrix}\end{aligned}$$

Clearly, $\bar F := \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \bar{\mathcal M}_{\mathrm a}^{-1} \bar\chi_{\mathrm{cl}}(\bar A)$ is state feedback gain assigning poles of delay-free system ($h = 0$) to $\bar\chi_{\mathrm{cl}}(z)$. Thus, we end up with

$$\bar F_{\mathrm a} = \bar F \begin{bmatrix} \bar B & \bar A \bar B & \cdots & \bar A^{h-1}\bar B & \bar A^h \end{bmatrix}$$

and corresponding control law

$$\bar u[k] = \bar F \left(\bar A^h \bar x[k] + \sum_{i=1}^{h} \bar A^{i-1} \bar B \bar u[k-i] \right),$$

which assigns closed-loop poles to $\bar\chi_{\mathrm{cl}}(z) = z^h \det\bigl(zI - (\bar A + \bar B \bar F)\bigr)$.

Approach 2: geometric reasonings

Let

$$\bar\Sigma_0 : \bar x[k+1] = \bar A \bar x[k] + \bar B \bar u[k]$$

and assume that we'd like to shift only a part of $\operatorname{spec}(\bar A)$ by state feedback. An easy way to accomplish this is to use any similarity transform T such that

$$T \bar x[k+1] = \begin{bmatrix} \bar A_{\mathrm{shift}} & 0 \\ \bar A_{21} & \bar A_{\mathrm{keep}} \end{bmatrix} T \bar x[k] + \begin{bmatrix} \bar B_{\mathrm{shift}} \\ \bar B_2 \end{bmatrix} \bar u[k]$$

and then, if $(\bar A_{\mathrm{shift}}, \bar B_{\mathrm{shift}})$ controllable, use $\bar u[k] = \bar F_{\mathrm{shift}} \begin{bmatrix} I & 0 \end{bmatrix} T \bar x[k]$. In fact, we do not need T, but only its first block row $T_{\mathrm{shift}} := \begin{bmatrix} I & 0 \end{bmatrix} T$ verifying

$$T_{\mathrm{shift}} \bar A = \bar A_{\mathrm{shift}} T_{\mathrm{shift}}$$

with any $\bar A_{\mathrm{shift}}$ whose spectrum coincides with the part of $\operatorname{spec}(\bar A)$ that we want to shift. Indeed, with $\bar u[k] = \bar F_{\mathrm{shift}} T_{\mathrm{shift}} \bar x[k]$ we have that

$$T_{\mathrm{shift}} \bar x[k+1] = (\bar A_{\mathrm{shift}} + T_{\mathrm{shift}} B \cdot \bar F_{\mathrm{shift}}) T_{\mathrm{shift}} \bar x[k].$$

Approach 2: exploiting structure of $(\bar A_{\mathrm a}, \bar B_{\mathrm a})$

Now, we have

$$\begin{bmatrix} \bar u[k] \\ \bar u[k-1] \\ \vdots \\ \bar u[k-h+1] \\ \bar x[k+1] \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 \\ I & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & I & 0 & 0 \\ 0 & \cdots & 0 & \bar B & \bar A \end{bmatrix} \begin{bmatrix} \bar u[k-1] \\ \bar u[k-2] \\ \vdots \\ \bar u[k-h] \\ \bar x[k] \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \bar u[k]$$

and want to shift only the modes of $\bar A$. Let's see whether

$$\begin{bmatrix} T_1 & T_2 & \cdots & T_h & T_x \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 \\ I & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & I & 0 & 0 \\ 0 & \cdots & 0 & \bar B & \bar A \end{bmatrix} = \bar A \begin{bmatrix} T_1 & T_2 & \cdots & T_h & T_x \end{bmatrix}$$

could be solved for some appropriately dimensioned $T_1, T_2, \ldots, T_h$ and T_x.

Approach 2: exploiting structure of $(\bar A_{\mathrm a}, \bar B_{\mathrm a})$ (contd)

Equivalently, we seek for $T_1, T_2, \ldots, T_h$ and T_x satisfying

$$T_{i+1} = \bar A T_i \;\; (i = 1, \ldots, h-1) \quad \text{and} \quad T_x \begin{bmatrix} \bar B & \bar A \end{bmatrix} = \bar A \begin{bmatrix} T_h & T_x \end{bmatrix}.$$

Hence, we need to find T_1 and T_x such that $T_x \bar B = \bar A^h T_1$ and $T_x \bar A = \bar A T_x$. An *easy* guess is $T_x = \bar A^h$ and $T_1 = \bar B$, so that

$$\begin{bmatrix} T_1 & T_2 & \cdots & T_h & T_x \end{bmatrix} = \begin{bmatrix} \bar B & \bar A \bar B & \cdots & \bar A^{h-1}\bar B & \bar A^h \end{bmatrix}$$

and we again end up with the feedback gain

$$\bar F_{\mathrm a} = \bar F \begin{bmatrix} \bar B & \bar A \bar B & \cdots & \bar A^{h-1}\bar B & \bar A^h \end{bmatrix}$$

and corresponding control law

$$\bar u[k] = \bar F \left(\bar A^h \bar x[k] + \sum_{i=1}^{h} \bar A^{i-1} \bar B \bar u[k-i] \right),$$

which assigns closed-loop poles to $\bar\chi_{\mathrm{cl}}(z) = z^{mh} \det\bigl(zI - (\bar A + \bar B \bar F)\bigr)$.

Discrete-time state feedback: interpretation

State-feedback control law can be presented as $\bar{u}[k] = \bar{F}\bar{x}_h[k]$, where

$$\bar{x}_h[k] := \bar{A}^h \bar{x}[k] + \sum_{i=1}^{h} \bar{A}^{i-1} \bar{B}\bar{u}[k-i].$$

At the same time, we know that solution of $\bar{x}[k+1] = \bar{A}\bar{x}[k] + \bar{B}\bar{u}[k-h]$ is

$$\bar{x}[k+h] = \bar{A}^h \bar{x}[k] + \sum_{i=1}^{h} \bar{A}^{i-1} \bar{B}\bar{u}[k-i].$$

This means that

- $\bar{x}_h[k]$ is h steps ahead prediction of $\bar{x}[k+h]$

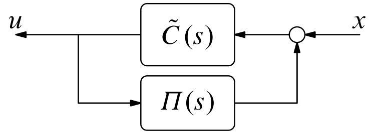
and therefore control law

- $\bar{u}[k] = \bar{F}\bar{x}_h[k]$ may be called predictive feedback

so we end up with exactly the same control rationale as in the FSA case.

Continuous-time FSA feedback: rationale

Thus, predictive control law



with

$$\Pi(s) = \mathrm{e}^{-Ah} \int_0^h \mathrm{e}^{-(sI-A)\theta} \mathrm{d}\theta B \quad \text{and} \quad \tilde{C}(s) = F\mathrm{e}^{Ah}$$

is nothing but

- a static state feedback
- shifting only the finite modes of the plant to $A + BF$