Introduction to Time-Delay Systems

lecture no. 2

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Outline

Nyquist criterion for time-delay systems

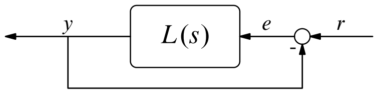
Roots of quasi-polynomials: general observations

Delay sweeping (direct method)

Commensurate delays

Lyapunov's methods

Nyquist stability criterion



The idea is to

- use plot of $L(\mathrm{j}\omega)$ to count the number of closed-loop poles in $\bar{\mathbb{C}}_0$.

Namely, assume that $L(s)$ has no pole/zero cancellations in $\bar{\mathbb{C}}_0$ and denote:

n_{ol} number of poles of $L(s)$ in $\bar{\mathbb{C}}_0$

n_{cl} number of poles of $\frac{1}{1+L(s)}$ in $\bar{\mathbb{C}}_0$

$\varkappa$ number of clockwise encirclements of $-1+\mathrm{j}0$ by Nyquist plot of $L(\mathrm{j}\omega)$ as ω runs from $-\infty$ to ∞

Then

$$n_{\mathrm{cl}} = n_{\mathrm{ol}} + \varkappa$$

What is changed for delay systems ?

Nothing, except that

- we might no longer be interested in $\bar{\mathbb{C}}_0$ as stability region.

Formal workaround: shift Nyquist contour a bit left or, equivalently,

- plot Nyquist plot of $L(\alpha + \mathrm{j}\omega)$ for some $\alpha < 0$,

yet by this intuition we have about frequency response gets lost.

We still can use $L(\mathrm{j}\omega)$ if we

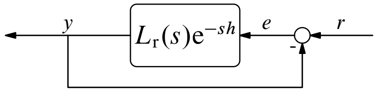
- rule out situation with pole chain around $\mathrm{j}\omega$-axis.

If $L(s) = \frac{b_0(s)+b_h(s)\mathrm{e}^{-sh}}{a_0(s)+a_h(s)\mathrm{e}^{-sh}}$, then closed-loop system is

$$\frac{1}{1+L(s)} = \frac{a_0(s)+a_h(s)\mathrm{e}^{-sh}}{(a_0(s)+b_0(s))+(a_h(s)+b_h(s))\mathrm{e}^{-sh}}$$

and we should rule out $\left|\frac{a_h(\infty)+b_h(\infty)}{a_0(\infty)+b_0(\infty)}\right| = 1$ by considering it unstable.

Dead-time systems



Particularly simple analysis because of simple rules of plotting $L_r(j\omega)e^{-j\omega h}$.

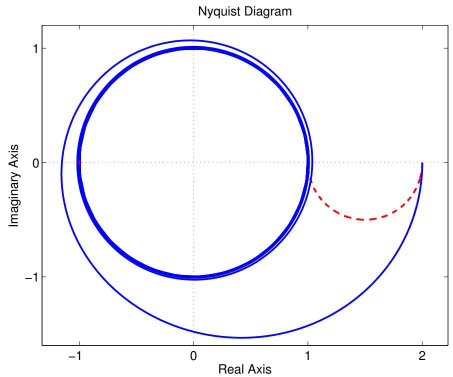
Just remember that

- if $|L_r(\infty)| = 1$, closed-loop systems unstable

no matter how many times the critical point is encircled.

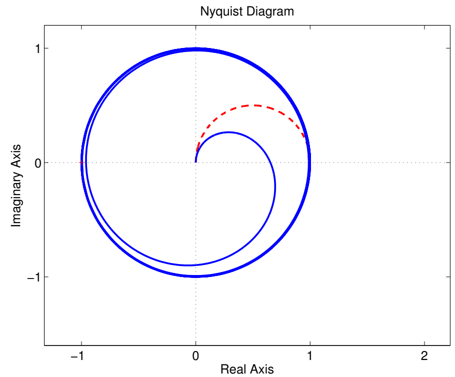
Examples of $|L_r(\infty)| = 1$

$L(s) = \frac{s+2}{s+1}\,e^{-s}$:



infinite number of closed-loop poles in $\bar{\mathbb{C}}_0$

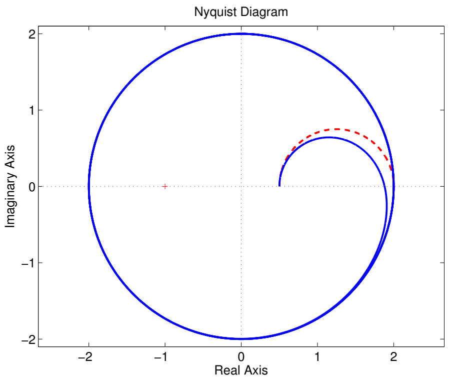
$L(s) = \frac{s}{s+1}\,e^{-s}$:



no closed-loop poles in $\bar{\mathbb{C}}_0$

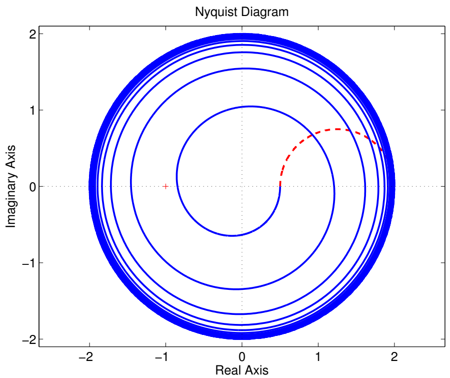
Examples of $|L_r(\infty)| > 1$

$L(s) = \frac{2s+1}{s+2}\,e^{-0.05s}$:



infinite number of closed-loop poles in $\bar{\mathbb{C}}_0$

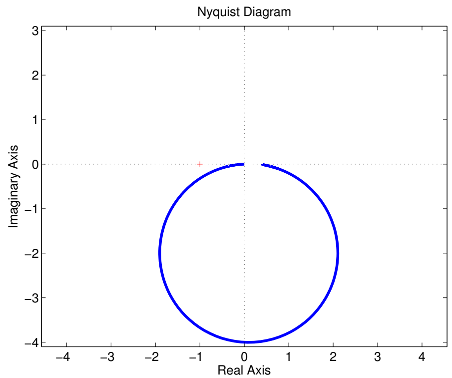
$L(s) = \frac{2s+1}{s+2}\,e^{-5s}$:

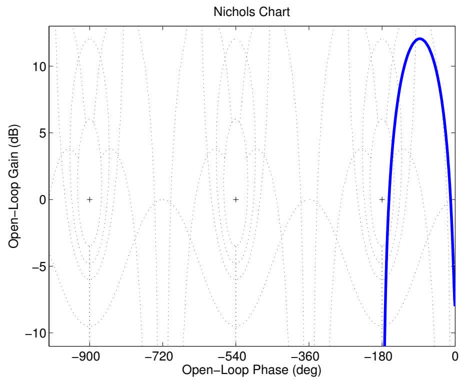


infinite number of closed-loop poles in $\bar{\mathbb{C}}_0$

Examples of $|L_r(\infty)| < 1$

$L(s) = \frac{0.4}{s^2+0.1s+1}$:

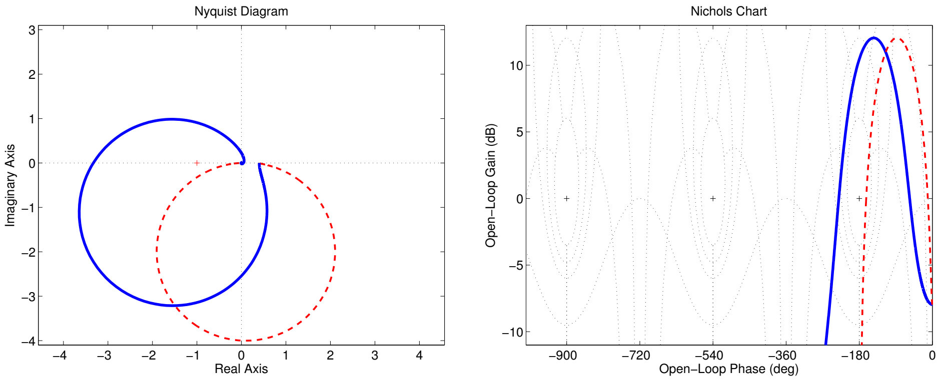




no closed-loop poles in $\bar{\mathbb{C}}_0$

Examples of $|L_{\mathrm{r}}(\infty)| < 1$

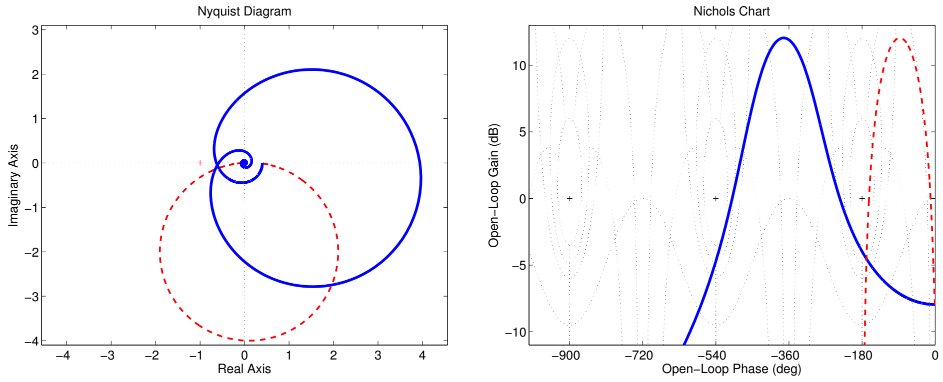
$L(s) = \frac{0.4}{s^2+0.1s+1}\,\mathrm{e}^{-s}$:



two closed-loop poles in $\bar{\mathbb{C}}_0$

Examples of $|L_{\mathrm{r}}(\infty)| < 1$

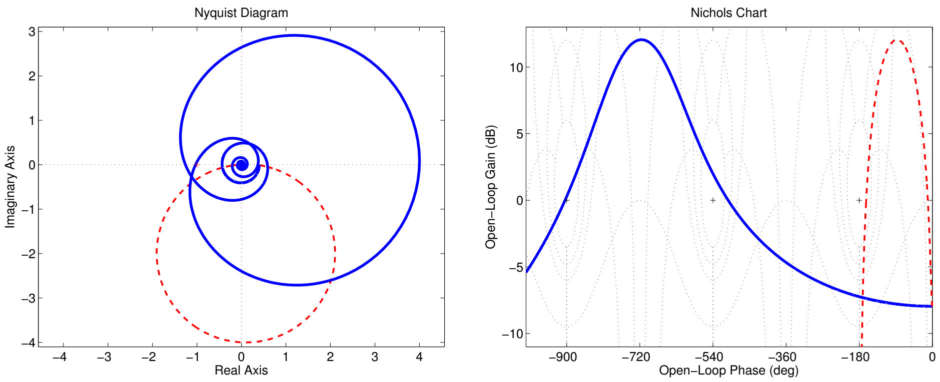
$L(s) = \frac{0.4}{s^2+0.1s+1}\,\mathrm{e}^{-5s}$:



no closed-loop poles in $\bar{\mathbb{C}}_0$

Examples of $|L_{\mathrm{r}}(\infty)| < 1$

$L(s) = \frac{0.4}{s^2+0.1s+1}\,\mathrm{e}^{-11s}$:



four closed-loop poles in $\bar{\mathbb{C}}_0$

Outline

Nyquist criterion for time-delay systems

Roots of quasi-polynomials: general observations

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Commensurate delays

Lyapunov's methods

Problem formulation

Consider (characteristic) quasi-polynomial

$$\chi_h(s) = P(s) + Q(s)\mathrm{e}^{-sh}$$

with

$$P(s) = s^n + p_{n-1}s^{n-1} + \cdots + p_1 s + p_0,$$
$$Q(s) = q_m s^m + q_{m-1}s^{m-1} + \cdots + q_1 s + q_0, \quad q_m \neq 0.$$

The problem is to

- check whether $\chi_h(s)$ has all its roots in $\mathbb{C} \setminus \bar{\mathbb{C}}_\alpha$

for some $\alpha < 0$.

Continuity of roots

Define

$$\lambda_{\mathrm{r},h} := \sup\{\mathrm{Re}\, s : \chi_h(s) = 0\}$$

Then, the following result holds:

Theorem

1. *$\lambda_{\mathrm{r},h}$ is continuous as a function of h for all $h > 0$*
2. *if, in addition, $\chi_h(s)$ is retarded, then $\lambda_{\mathrm{r},h}$ is continuous at $h = 0$ as well*

Important consequence of this is that

- as h changes, roots of $\chi_h(s)$ may transit LHP-2-RHP (or vise versa) by crossing $\mathrm{j}\omega$-axis only.

Assumptions

For

$$P(s) = s^n + p_{n-1}s^{n-1} + \cdots + p_1 s + p_0,$$
$$Q(s) = q_m s^m + q_{m-1}s^{m-1} + \cdots + q_1 s + q_0, \quad q_m \neq 0.$$

we assume that

$\mathcal{A}_1$: $m \leq n$,

$\mathcal{A}_2$: if $m = n$, then $|q_m| < 1$,

$\mathcal{A}_3$: $P(s)$ and $Q(s)$ have no common roots in $\bar{\mathbb{C}}_0$ (i.e., coprime in $\bar{\mathbb{C}}_0$),

$\mathcal{A}_4$: $p_0 + q_0 \neq 0$.

because

- $\mathcal{A}_{1,2}$ guarantee $\left|\frac{Q(\infty)}{P(\infty)}\right| < 1$ (otherwise unstable for all $h > 0$).
- if $\mathcal{A}_3$ does not hold, $\chi_h(s)$ unstable for all $h \geq 0$.
- if $\mathcal{A}_4$ does not hold, $\chi_h(0) = 0$ for all $h \geq 0$.

Outline

Idea

To analyze stability $\chi_h(s)$ by

- continuous increase of h starting from $h = 0$.

Namely, the analysis steps are as follows:

1. locate roots[1] of $\chi_0(s) = P(s) + Q(s)$;
2. increase h and check for jω-axis crossings[2] of roots of $\chi_h(s)$:
    - LHP to RHP crossings called are switches
    - RHP to LHP crossings called are reversals

Stability of $\chi_h(s)$ can then be verified by counting switches and reversals.

[1] As polynomial $\chi_0(s)$ is finite dimensional, this step is trivial.
[2] We'll see below that this step can be efficiently performed.

jω crossings

If at some h roots of $\chi_h(s)$ cross jω-axis, we have (mind $\mathcal{A}_3$):

$$P(\mathrm{j}\omega) + Q(\mathrm{j}\omega)\mathrm{e}^{-\mathrm{j}\omega h} = 0 \iff -\frac{Q(\mathrm{j}\omega)}{P(\mathrm{j}\omega)} = \mathrm{e}^{\mathrm{j}\omega h}.$$

This, in turn, is equivalent to:

1. $\left|\frac{Q(\mathrm{j}\omega)}{P(\mathrm{j}\omega)}\right| = 1$ or $|P(\mathrm{j}\omega)| = |Q(\mathrm{j}\omega)|$ (magnitude relation),
2. $\omega h = \arg\left[-\frac{Q(\mathrm{j}\omega)}{P(\mathrm{j}\omega)}\right] + 2\pi k$ for some $k \in \mathbb{Z}$ (phase relation).

Note that:

- for any $\omega > 0$ satisfying 1, equality 2 is always solvable for h;
- if $\omega > 0$ is solution of 1, then so is $-\omega$;
- if $\omega = 0$ is solution of 1, equality 2 cannot hold because of $\mathcal{A}_4$.

Conclusion:

- existence of jω roots of characteristic equation completely determined by magnitude equation and does not depend on delay.

Positive solutions of $|A(\mathrm{j}\omega)| = |B(\mathrm{j}\omega)|$

This equation can be rewritten as

$$P(\mathrm{j}\omega)P(-\mathrm{j}\omega) - Q(\mathrm{j}\omega)Q(-\mathrm{j}\omega) =: \phi(\omega) = 0,$$

which is polynomial equation in ω^2. Thus, all frequencies ω_i at which rots of $\chi_h(s)$ cross jω-axis can be found from positive real roots of $\phi(s)$.

Example

Let $P(s) = s^2 + 0.1s + 1$ and $Q(s) = q_0 > 0$. Then

$$\begin{aligned}\phi(\omega) &= (-\omega^2 + \mathrm{j}0.1\omega + 1)(-\omega^2 - \mathrm{j}0.1\omega + 1) - q_0^2\\ &= (1-\omega^2)^2 + 0.1^2\omega^2 - q_0^2\\ &= \omega^4 - 2\cdot 0.995\,\omega^2 + 1 - q_0^2 = 0\end{aligned}$$

Three situations possible:

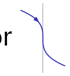
1. if $0 < q_0 < \sqrt{1 - 0.995^2} \approx 0.099875$, there are no real solutions;
2. if $\sqrt{1-0.995^2} \le q_0 < 1$, there are two positive real solutions
$$\omega_1^2 = 0.995 + \sqrt{0.995^2 - 1 + q_0^2}, \quad \omega_2^2 = 0.995 - \sqrt{0.995^2 - 1 + q_0^2};$$
3. if $q_0 \ge 1$, there is one positive real solution
$$\omega_1^2 = 0.995 + \sqrt{0.995^2 + q_0^2 - 1}.$$

Crossing directions

Depend on $\sigma(\omega) := \operatorname{sgn}\operatorname{Re}\frac{ds}{dh}\big|_{s=j\omega}$ at (positive) crossing frequencies:

$\sigma(\omega_i) > 0$ roots migrate from LHP to RHP at $\omega_i > 0$ (switch) 

$\sigma(\omega_i) < 0$ roots migrate from RHP to LHP at $\omega_i > 0$ (reversal) 

$\sigma(\omega_i) = 0$ roots migration depends on higher derivatives  or 

The question now is

- how to compute $\operatorname{sgn}\operatorname{Re}\frac{ds}{dh}$ at $j\mathbb{R}^+$ solutions of $\chi_h(s) = 0$?

Some differential calculus

Note that

$$0 = \frac{d}{dh}\chi_h(s) = \frac{dP(s)}{ds}\frac{ds}{dh} + \frac{dQ(s)}{ds}\frac{ds}{dh}e^{-sh} - Q(s)e^{-sh}\left(h\frac{ds}{dh} + s\right).$$

Thus, denoting by $(\cdot)'$ differentiation with respect to s, we have:

$$\frac{ds}{dh} = \frac{sQ(s)e^{-sh}}{P'(s) + Q'(s)e^{-sh} - hQ(s)e^{-sh}} = -s\left(\frac{P'(s)}{P(s)} - \frac{Q'(s)}{Q(s)} + h\right)^{-1},$$

where equality $Q(s)e^{-sh} = -P(s)$ was used. Since $\operatorname{sgn}\operatorname{Re} z^{-1} = \operatorname{sgn}\operatorname{Re} z$,

$$\begin{aligned}\sigma(\omega) &= \operatorname{sgn}\operatorname{Re}\left[-\frac{1}{j\omega}\left(\frac{P'(j\omega)}{P(j\omega)} - \frac{Q'(j\omega)}{Q(j\omega)} + h\right)\right] \\ &= \operatorname{sgn}\operatorname{Re}\left[\frac{j}{\omega}\left(\frac{P'(j\omega)}{P(j\omega)} - \frac{Q'(j\omega)}{Q(j\omega)}\right)\right] && \left(\text{as } \operatorname{Re}\frac{h}{j\omega} = 0\right) \\ &= \operatorname{sgn}\operatorname{Re}\left[j\left(\frac{P'(j\omega)}{P(j\omega)} - \frac{Q'(j\omega)}{Q(j\omega)}\right)\right], && (\text{as } \omega \in \mathbb{R})\end{aligned}$$

which does not depend on h.

Some differential calculus (contd)

Multiplying expression under "sgn" by $P(j\omega)P(-j\omega) = Q(j\omega)Q(-j\omega) > 0$,

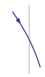
$$\begin{aligned}\sigma(\omega) &= \operatorname{sgn}\operatorname{Re}\left[j\left(\frac{dP(j\omega)}{d(j\omega)}P(-j\omega) - \frac{dQ(j\omega)}{d(j\omega)}Q(-j\omega)\right)\right] \\ &= \operatorname{sgn}\operatorname{Re}\left[\frac{dP(j\omega)}{d\omega}P(-j\omega) - \frac{dQ(j\omega)}{d\omega}Q(-j\omega)\right]\end{aligned}$$

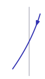
Then, since $\operatorname{sgn}\operatorname{Re} z = \operatorname{sgn}(z + \bar{z})$,

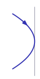
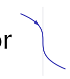
$$\begin{aligned}\sigma(\omega) &= \operatorname{sgn}\operatorname{Re}\left[\frac{dP(j\omega)}{d\omega}P(-j\omega) - \frac{dQ(j\omega)}{d\omega}Q(-j\omega)\right. \\ &\qquad\qquad \left. + \frac{dP(-j\omega)}{d\omega}P(j\omega) - \frac{dQ(-j\omega)}{d\omega}Q(j\omega)\right] \\ &= \operatorname{sgn}\frac{d\phi(\omega)}{d\omega}\end{aligned}$$

Crossing directions (contd)

Thus, we have:

$\operatorname{sgn}\dot{\phi}(\omega_i) > 0$ roots migrate LHP-2-RHP at $\omega_i > 0$ (switch) 

$\operatorname{sgn}\dot{\phi}(\omega_i) < 0$ roots migrate RHP-2-LHP at $\omega_i > 0$ (reversal) 

$\operatorname{sgn}\dot{\phi}(\omega_i) = 0$ roots migration depends on higher derivatives[3]  or 

[3] Still, Walton and Marshall (1987) showed that if $\phi(\omega)$ changes its sign from "$-$" to "$+$" at $\omega = \omega_i$, we have LHP-2-RHP migration, if from "$+$" to "$-$"—RHP-2-LHP migration, and if it doesn't change sign—no migrations take place (touch point).

Example (contd)

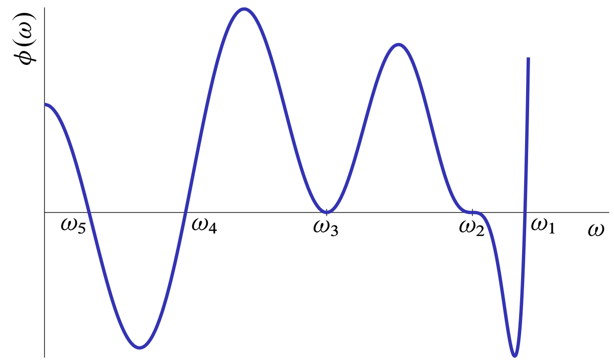
Return to example with $P(s) = s^2 + 0.1s + 1$ and $Q(s) = q_0 > 0$. Here

$$\sigma(\omega) = \tfrac{\mathrm{d}}{\mathrm{d}\omega}(\omega^4 - 2\cdot 0.995\,\omega^2 + 1 - q_0^2) = 4\omega(\omega^2 - 0.995).$$

1. if $q_0 = \sqrt{1-0.995^2}$, then $\omega_1 = \omega_2 = 0.995$ and
$$\sigma(\omega_1) = \sigma(\omega_2) = 0$$
(in fact, in this case no jω-axis crossings take place);
2. if $\sqrt{1-0.995^2} < q_0 < 1$ (two different crossing frequencies), then
$$\sigma(\omega_1) = 4\omega_1\sqrt{0.995^2 - 1 + q_0^2} > 0,$$
$$\sigma(\omega_2) = -4\omega_2\sqrt{0.995^2 - 1 + q_0^2} < 0$$
(so that ω_1 is always switch and ω_2 is always reversal);
3. if $q_0 \geq 1$ (one crossing frequency), then
$$\sigma(\omega_1) = 4\omega_1\sqrt{0.995^2 - 1 + q_0^2} > 0$$
(so that ω_1 is always switch).

Some qualitative observations about crossing frequencies

$\phi(\omega)$ is even real polynomial in ω with positive leading coeff. (mind $\boldsymbol{\mathcal{A}}_{1,2}$):



Crossing frequencies solve $\phi(\omega) = 0$ and crossing directions determined by directions of $\phi(\omega)$ at zero crossings (for *increasing* ω). This means that

- the largest crossing frequency is always a switch,
- switch and reversal frequencies always interlace
(with possible tangential points, at which no crossings occur, between them)

Crossing delays

It's time to make use of the phase relation at jω-crossing:

$$\omega h = \arg\Big[-\tfrac{Q(\mathrm{j}\omega)}{P(\mathrm{j}\omega)}\Big] + 2\pi k, \quad k \in \mathbb{Z} \text{ and such that } h \geq 0. \qquad (1)$$

For each crossing frequency ω_i this equation yields sequence of delays

$$h_{i,k} = h_{i,0} + \tfrac{2\pi}{\omega_i}\,k, \quad k \in \mathbb{N},$$

where $h_{i,0} > 0$ is the smallest solution of (1). Then:

- if ω_i is switch ($\sigma(\omega_i) > 0$), two poles move LHP-2-RHP at each $h_{i,k}$;
- if ω_i is reversal ($\sigma(\omega_i) < 0$), two poles move RHP-2-LHP at each $h_{i,k}$.

Thus, if we know poles of $\chi_0(s)$, stability analysis of $\chi_h(s)$ needs

1. sorting all $h_{i,k}$ in increasing order[3],
2. counting all crossings up to h.

[3] Smallest delay at which first crossing occurs need not correspond to largest frequency.

Example (contd)

Return to example with $P(s) = s^2 + 0.1s + 1$ and $Q(s) = q_0$ and let $q_0 = 0.4$
In this case we have $\omega_1 \approx 1.176$ (switch) and $\omega_2 \approx 0.78$ (reversal) and

$$h_{1,k} \approx 0.254 + 5.344\,k = \{0.254, 5.598, 10.942, 16.286, \ldots\},$$
$$h_{2,k} \approx 3.778 + 8.06\,k = \{3.778, 11.838, 19.898, 27.958, \ldots\}.$$

This yields the following ordered list of crossing (switch and reversal) delays:

$$\{h_{1,0}, h_{2,0}, h_{1,1}, h_{1,2}, h_{2,1}, h_{1,3}, \ldots\}.$$

Since $\chi_0(s) = s^2 + 0.1s + 1.4$ is stable, $\chi_h(s)$ is

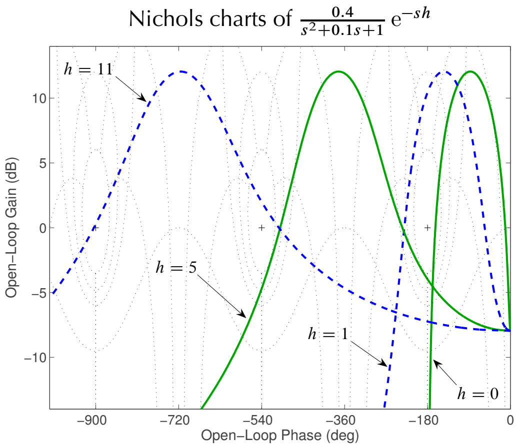
- stable in $h \in [0, 0.254)$,
- unstable in $h \in [0.254, 3.778]$ (two poles went to RHP at $h = 0.254$),
- stable in $h \in (3.778, 5.598)$ (two poles returned to LHP at $h = 3.778$),
- unstable in $h \in [5.598, \infty)$ (two poles went to RHP at $h = 5.598$, then another two—at $h = 10.942$, before first two returned at $h = 11.838$).

Example (contd)

Thus, quasi-polynomial $\chi_h(s) = s^2 + 0.1s + 1 + 0.4e^{-sh}$ is stable in

$$h \in [0, 0.254) \cup (3.778, 5.598).$$

And now compare with Nichols charts (doesn't it ring a bell ?):



Some qualitative observations about crossings

The following facts are worth remembering:

- Delay distance between two crossings at the same frequency ω_i is $\frac{2\pi}{\omega_i}$. Hence, shortest distance is between two switches, which means that at some point there is always ≥ 2 RHP poles. Consequently, there
  - always exists a delay, say $h_{\max}$, such that $\chi_h(s)$ unstable $\forall h \geq h_{\max}$.

  This $h_{\max}$ is calculable.

- If there are no crossing frequencies, then stability / instability properties are delay independent.

Frequency domain vs. modal analysis

There is some nice interplay between these methods. For example:

- roots crossing frequencies are crossover frequencies of loop frequency response,
- root crossing delays correspond to delay margins,
- ...

Arguably (I'm rather opinionated on this:-),

- frequency-response analysis provides more insight (cleaner thinking),
- modal analysis easier leads to reliable computations.

What is certain is that one must

- not confine oneself to either one of these approaches,

interplay between them yields better understanding (and is also fun).

Other modal analysis methods

- bilinear (Rekasius) transformation
  replace e^{-sh} with $\frac{1-\tau s}{1+\tau s}$, which covers the same area in $\mathbb{C}$ as τ grows from $-\infty$ to ∞
- 2D representation
  replace e^{-sh} with z and analyze stability with respect to both $s \in j\mathbb{R}$ and $z \in \mathbb{T}$

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Example

Let
$$\chi_{h,2}(s) = s + \mathrm{e}^{-sh} + \mathrm{e}^{-s2h}.$$
Like in single-delay case, we look for jω-crossings of roots. As $\chi_{h,2}(s)$ real, its jω-axis roots coincide with those of $\chi_{h,2}(-s)$, i.e., they satisfy both
$$s + \mathrm{e}^{-sh} + \mathrm{e}^{-s2h} = 0 \qquad \text{and} \qquad -s + \mathrm{e}^{sh} + \mathrm{e}^{s2h} = 0$$
From the first equation, $\mathrm{e}^{-s2h} = -s - \mathrm{e}^{-sh}$, then from the second equation:
$$0 = 1 + \mathrm{e}^{-sh} - s\mathrm{e}^{-s2h} = 1 + \mathrm{e}^{-sh} + s(s + \mathrm{e}^{-sh}) = 1 + s^2 + (1+s)\mathrm{e}^{-sh}$$
This is a single-delay quasi-polynomial with
$$\phi(\omega) = (1-\omega^2)^2 - (1+\omega^2) = \omega^2(\omega^2 - 3)$$
from which crossing $\omega = \sqrt{3}$ (switch) and crossing $h_{1,0} = \frac{\arg \frac{1+\mathrm{j}\sqrt{3}}{2}}{\sqrt{3}} = \frac{\pi}{3\sqrt{3}}$.
As $\chi_{0,2}(s)$ stable, $\chi_{h,2}(s)$ stable iff $h \in [0, \frac{\pi}{3\sqrt{3}})$.

More general observations

Let
$$\chi_{h,2}(s) = P(s) + Q_1(s)\mathrm{e}^{-sh} + Q_2(s)\mathrm{e}^{-s2h}$$
Clearly, $s_c = \mathrm{j}\omega_c$ is root of $\chi_{h,2}(s)$ iff it is root of
$$\mathrm{e}^{-s2h}\chi_{h,2}(-s) = Q_2(-s) + Q_1(-s)\mathrm{e}^{-sh} + P(-s)\mathrm{e}^{-s2h}$$
Hence, this $s_c = \mathrm{j}\omega_c$ is also root of
$$\begin{aligned}\chi_{h,1}(s) &:= P(-s)\chi_{h,2}(s) - Q_2(s)\mathrm{e}^{-s2h}\chi_{h,2}(-s)\\ &= P(s)P(-s) - Q_2(s)Q_2(-s) + \big(Q_1(s)P(-s) - Q_2(s)Q_1(-s)\big)\mathrm{e}^{-sh}\end{aligned}$$
which is single-delay quasi-polynomial. Yet converse not necessarily true:

- $\chi_{h,1}(s)$ might have more jω-roots[4] than $\chi_{h,2}(s)$,

which complicates the analysis.

[4] In fact, jω-roots of $\chi_{h,1}(s)$ are roots of either $\chi_{h,2}(s)$ or $P(s)P(-s) - Q_2(s)Q_2(-s)$.

More general observations (contd)

If $s_c = \mathrm{j}\omega_c$ is crossing point of $\chi_{h,1}(s)$ such that
$$|P(\mathrm{j}\omega_c)| \neq |Q_2(\mathrm{j}\omega_c)|,$$
then it also crossing point of $\chi_{h,2}(s)$.

Moreover, crosing directions at $\chi_{h,1}(\mathrm{j}\omega_c)$ and $\chi_{h,2}(\mathrm{j}\omega_c)$

- coincide iff $|P(\mathrm{j}\omega_c)| > |Q_2(\mathrm{j}\omega_c)|$
- opposite iff $|P(\mathrm{j}\omega_c)| < |Q_2(\mathrm{j}\omega_c)|$

Outline

Nyquist criterion for time-delay systems

Roots of quasi-polynomials: general observations

Delay sweeping (direct method)

Commensurate delays

Lyapunov's methods

Lyapunov's method for finite-dimensional systems

Consider LTI system

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0$$

and assume there is (differentiable) $V(x) : \mathbb{R}^n \mapsto \mathbb{R}^+$ such that[5]

1. $V(0) = 0$
2. $V(x) > 0$ for all $x \neq 0$
3. derivative along trajectory $\dot{V}(x) := \frac{\partial V(x)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} \leq 0$

called Lyapunov function. Then system is stable (in the sense of Lyapunov).

If 3 replaced with

3′. $\dot{V}(x) < 0$

system is asymptotically stable. Yet another way to end up with asymptotic stability is via LaSalle Invariance Principle

3″. $\dot{V}(x) \leq 0$ and $\dot{V}(x) \equiv 0$ implies $x(t) \equiv 0$.

[5] In principle, statements like $V(x) > 0$ should be more specific, yet we proceed with this sloppiness to simplify exposition.

Lyapunov's method for finite-dimensional systems (contd)

Let's choose

$$V(x) = x'(t)Px(t)$$

for some $P > 0$. Then

$$\begin{aligned}\dot{V}(x) &= \dot{x}'(t)Px(t) + x'(t)P\dot{x}(t) = x'(t)A'Px(t) + x'(t)PAx(t)\\ &= x'(t)(A'P + PA)x(t)\end{aligned}$$

If we can choose $P > 0$ satisfying $A'P + PA = -C'C$ for some C,

$$\dot{V}(x) = -x'(t)C'Cx(t) \leq 0,$$

which implies stability. For asymptotic stability we then need observability of (C, A) as in that case $Cx(t) \equiv 0$ implies $x(t) \equiv 0$. In fact

- $\exists P > 0$ such that $A'P + PA < 0 \iff A$ Hurwitz

and $P = \int_0^\infty \mathrm{e}^{A'\theta}C'C\mathrm{e}^{A\theta}\mathrm{d}\theta > 0$ is observability Gramian of (C, A).

Adding delays: developing intuition via discrete systems

Consider

$$\bar{x}[k+1] = A_0\bar{x}[k] + A_1\bar{x}[k-h]$$

As state vector here $\bar{x}_a = \begin{bmatrix}\bar{x}'[k] & \bar{x}'[k-1] & \cdots & \bar{x}'[k-h]\end{bmatrix}'$, quadratic Lyapunov function should look like

$$\begin{aligned}\bar{V}(\bar{x}_a) &= \begin{bmatrix}\bar{x}'[k] & \bar{x}'[k-1] & \cdots & \bar{x}'[k-h]\end{bmatrix} \overbrace{\begin{bmatrix}\bar{P}_{00} & \bar{P}_{01} & \cdots & \bar{P}_{0h}\\ \bar{P}_{10} & \bar{P}_{11} & \cdots & \bar{P}_{1h}\\ \vdots & \vdots & \ddots & \vdots\\ \bar{P}_{h0} & \bar{P}_{h1} & \cdots & \bar{P}_{hh}\end{bmatrix}}^{>0} \begin{bmatrix}\bar{x}[k]\\ \bar{x}[k-1]\\ \vdots\\ \bar{x}[k-h]\end{bmatrix}\\ &= \sum_{i=0}^{h}\sum_{j=0}^{h}\bar{x}'[k-i]\bar{P}_{ij}\bar{x}[k-j]\end{aligned}$$

Adding delays: Lyapunov-Krasovskiĭ functional

Consider

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h)$$

Quadratic Lyapunov function (actually, functional $\{[-h,0] \mapsto \mathbb{R}^n\} \mapsto \mathbb{R}^+$) for this system could be of the form

$$V(x_\tau) = \int_0^h \int_0^h x'(t-\sigma) P(\sigma,\theta) x(t-\theta) \mathrm{d}\theta \mathrm{d}\sigma$$

for some function $P(\sigma,\theta)$, where $0 \le \sigma, \theta \le h$, such that $P(\sigma,\theta) = P'(\theta,\sigma)$ and $\int_0^h \int_0^h \eta'(\sigma) P(\sigma,\theta) \eta(\theta) \mathrm{d}\theta \mathrm{d}\sigma > 0$ for all $\eta(\cdot) \not\equiv 0$. This functional called Lyapunov-Krasovskiĭ functional.

Alternative expression:

$$V(x_\tau) = \int_{t-h}^t \int_{t-h}^t x'(\sigma) P(t-\sigma, t-\theta) x(\theta) \mathrm{d}\theta \mathrm{d}\sigma$$

Adding delays: Lyapunov-Krasovskiĭ functional (contd)

Choice regarded sufficiently general is

$$P(\sigma,\theta) = P_0 \delta(\sigma)\delta(\theta) + P_1'(\sigma)\delta(\theta) + P_1(\theta)\delta(\sigma) + P_2(\sigma)\delta(\sigma-\theta) + P_3(\sigma-\theta)$$

for some matrix P_0, matrix functions $P_1(\theta)$ and $P_2(\theta)$ defined in $\theta \in [0,h]$, and matrix function $P_3(t)$ defined in $t \in [-h,h]$. In this case

$$\begin{aligned} V(x_\tau) = x'(t) P_0 x(t) &+ \int_0^h \int_0^h x'(t-\sigma) P_3(\sigma-\theta) x(t-\theta) \mathrm{d}\theta \mathrm{d}\sigma \\ &+ 2x'(t) \int_0^h P_1(\theta) x(t-\theta) \mathrm{d}\theta + \int_0^h x'(t-\theta) P_2(\theta) x(t-\theta) \mathrm{d}\theta \end{aligned}$$

the derivative of which is a mess[6]...

[6] It is possible to choose $P(\sigma,\theta)$ of this form by reverse engineering: choose observable "measurement operator" for state vector and construct $P(\sigma,\theta)$ as observability Gramian.

Delay-independent conditions via LK approach

Consider $\dot{x}(t) = A_0 x(t) + A_h x(t-h)$ and choose

$$V(x_\tau) = x'(t) P_0 x(t) + \int_{t-h}^t x'(\theta) P_2 x(\theta) \mathrm{d}\theta$$

for some $P_0 > 0$ and $P_h > 0$. Then, using Leibniz integral rule,

$$\begin{aligned} \dot{V}(x_\tau) &= \dot{x}'(t) P_0 x(t) + x'(t) P_0 \dot{x}(t) + x'(t) P_2 x(t) - x'(t-h) P_2 x(t-h) \\ &= \begin{bmatrix} x'(t) & x'(t-h) \end{bmatrix} \begin{bmatrix} A_0' P_0 + P_0 A_0 + P_2 & P_0 A_h \\ A_h' P_0 & -P_2 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \end{aligned}$$

Thus, if there are $P_0 > 0$ and $P_2 > 0$ such that

$$\begin{bmatrix} A_0' P_0 + P_0 A_0 + P_2 & P_0 A_h \\ A_h' P_0 & -P_2 \end{bmatrix} < 0,$$

$\dot{V}(x_\tau) \le 0$ and system asymptotically stable (mind LaSalle). This is

- Linear Matrix Inequality (LMI), which can be efficiently solved.

Adding delays: Razumikhin approach

Consider again

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h)$$

Lyapunov function for it, $V(x_\tau)$, doesn't have to be quadratic. We may take

$$V(x_\tau) = \max_{\tau \in [-h,0]} \tilde{V}(x(t+\tau))$$

for some "Lyapunov function" $\tilde{V}(x)$. In this case $\dot{\tilde{V}}(x)$ may be even positive at points where $\tilde{V}(x) < V(x_\tau)$. This, relaxed, condition reads as follows:

Theorem (Razumikhin)

System is asymptotically stable if there is $V(x) : \mathbb{R}^n \mapsto \mathbb{R}^+$ such that

1. $V(0) = 0$
2. $V(x) > 0$ *for all* $x \ne 0$
3. $\dot{V}(x) < 0$ *whenever* $\rho V(x(t)) \ge V(x(t+\tau))$, $\tau \in [-h,0]$, *for some* $\rho > 1$

Delay-independent conditions via Razumikhin approach

Consider $\dot{x}(t) = A_0 x(t) + A_h x(t-h)$ and choose

$$V(x) = x'(t) P x(t)$$

for some $P > 0$. For every $\rho > 1$ and $\alpha > 0$ we can define function

$$\begin{aligned}\psi(t) &:= \dot{V}(x) + \alpha\big(\rho V(x(t)) - V(x(t-h))\big) \\ &= \dot{x}'(t) P x(t) + x'(t) P \dot{x}(t) + \alpha\big(\rho x'(t) P x(t) - x'(t-h) P x(t-h)\big) \\ &= \begin{bmatrix} x'(t) & x'(t-h) \end{bmatrix} \begin{bmatrix} A_0' P + P A_0 + \alpha\rho P & P A_h \\ A_h' P & -\alpha P \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}\end{aligned}$$

At $\{x(t) : V(x(t+\tau)) \le \rho V(x(t))\}$ we have that $\psi \ge \dot{V}$. Thus, if $\psi < 0$, the system is stable by Razumikhin arguments. Hence, if there is matrix $P > 0$ and scalar $\alpha > 0$ such that

$$\begin{bmatrix} A_0' P + P A_0 + \alpha P & P A_h \\ A_h' P & -\alpha P \end{bmatrix} < 0,$$

then the system asymptotically stable.

Lyapunov-Krasovskiĭ vs. Razumikhin (not generic)

If there are $P > 0$ and $\alpha > 0$ such that

$$\begin{bmatrix} A_0' P + P A_0 + \alpha P & P A_h \\ A_h' P & -\alpha P \end{bmatrix} < 0,$$

then

$$\begin{bmatrix} A_0' P_0 + P_0 A_0 + P_2 & P_0 A_h \\ A_h' P_0 & -P_2 \end{bmatrix} < 0 \quad \text{for } P_0 = P \text{ and } P_2 = \alpha P.$$

Thus, the LK condition holds whenever so does the Razumikhin condition, but not necessarily vise versa. Hence,

- in this case Razumikhin approach is potentially more conservative

(solvability of LK LMI does not necessarily mean that $P_h = \alpha P_0$).

Delay-independent stability in scalar case

Consider $\dot{x}(t) = a_0 x(t) + a_h x(t-h)$ for $x \in \mathbb{R}$. Delay-independent stability would require stability at $h = 0$ and no positive crossing frequencies. These read

$$a_0 + a_h < 0 \quad \text{and} \quad \omega^2 + a_0^2 = a_h^2 \text{ has no positive real solutions}$$

or, equivalently,

$$a_0^2 \ge a_h^2 \quad \text{and} \quad a_0 < 0$$

(the latter can be interpreted as stability under $h \to \infty$).

Scalar case: Lyapunov-Krasovskiĭ and Razumikhin

LK solvability LMI becomes

$$\exists p_0 > 0, p_2 > 0 \quad \text{such that} \begin{bmatrix} 2a_0 p_0 + p_2 & a_h p_0 \\ a_h p_0 & -p_2 \end{bmatrix} < 0.$$

Taking Schur complement of the $(2,2)$ term, the last condition equivalent to

$$2a_0 p_0 + p_2 + a_h^2 p_0^2 / p_2 < 0 \iff p_2^2 + 2a_0 p_0 p_2 + a_h^2 p_0^2 < 0.$$

The latter reads

$$a_0 < 0 \quad \text{and} \quad a_0^2 p_0^2 - a_h^2 p_0^2 \ge 0 \iff a_0^2 \ge a_h^2,$$

i.e., we recover exact conditions (LK conservative in general). In scalar case

$$\begin{bmatrix} 2a_0 p_0 + p_2 & a_h p_0 \\ a_h p_0 & -p_2 \end{bmatrix} < 0 \iff \begin{bmatrix} 2a_0 p + \alpha p & a_h p \\ a_h p & -\alpha p \end{bmatrix} < 0$$

and Razumikhin result coincides with Lyapunov-Krasovskiĭ one.