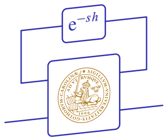
Introduction to Time-Delay Systems



lecture no. 1

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Outline

Course info

Time-delay systems in control applications

System-theoretic preliminaries

Basic properties

Delay systems in the frequency domain

Rational approximations of time delays

State space of delay systems

Modal properties of delay systems

Stability of transfer functions and roots of characteristic equations

General info

- ECTS credits: 7.5
- Prerequisite: any advanced linear control course
- Grading policy: homework 100% (solutions are graded on a scale of 0–100, each must be at least $e^\pi \approx 23.1407$, average grade of 5 best out of 6 assignments must be at least 67)
  Homework solutions are to be submitted electronically to *mirkin@control.lth.se*
- Literature:
  1. My slides.
  2. J. E. Marshall, H. Górecki, A. Korytowski, and K. Walton, *Time-Delay Systems: Stability and Performance Criteria with Applications*. London: Ellis Horwood, 1992.
  3. K. Gu, V. L. Kharitonov, and J. Chen, *Stability of Time-Delay Systems*. Boston: Birkhäuser, 2003.
  4. R. F. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*. New York: Springer-Verlag, 1995.

Syllabus

1. Introduction
   - time delays in engineering applications; system-theoretic preliminaries
2. Mathematical modeling of time-delay systems
   - frequency domain & modal analyses; state space; rational approximations
3. Stability analysis
   - stability notions; frequency sweeping; Lyapunov's method
4. Stabilization methods
   - fixed-structure controllers; finite spectrum assignment; coprime factorization
5. Dead-time compensation
   - Smith predictor and its modifications; implementation issues
6. Handling uncertain delays
   - Lyapunov-based methods; unstructured uncertainty embedding
7. Optimal control and estimation (??)
   - H^2 optimizations

On a less formal side

What this course is about…

- system-theoretic and control aspects of delays in dynamical systems
- exploiting the structure of the delay element
- giving a flavor of ideas in the field
- showing that things are (relatively) simple if the viewpoint is right

…and what it isn't

- digging up most general and mathematically intriguing cases
- answering the very problem that motivated you to take this course

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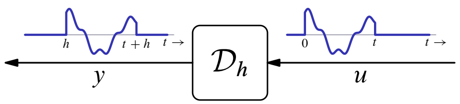
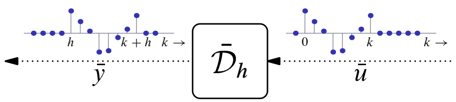
Time delay element

$$y = \mathcal{D}_h u: \quad y(t) = u(t-h)$$



or

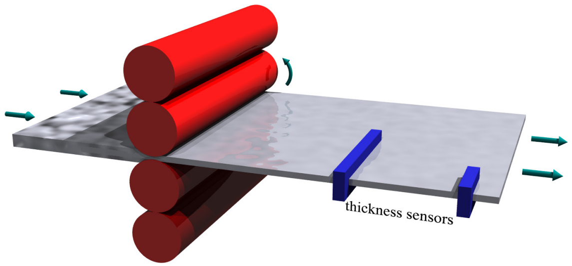
$$\bar{y} = \bar{\mathcal{D}}_h \bar{u}: \quad y[k] = u[k-h]$$



Why delays?

- Ubiquitous in physical processes
  - loop delays
  - process delays
  - …
- Compact/economical approximations of complex dynamics
- Exploiting delays to improve performance

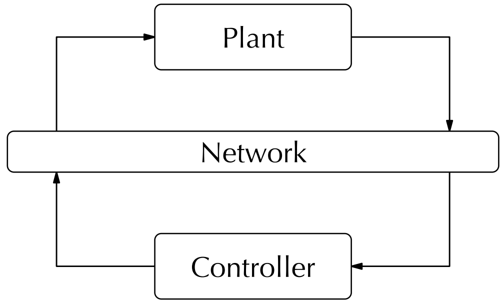
Loop delays: steel rolling



Thickness can only be measured at some distance from rolls, leading to

- measurement delays

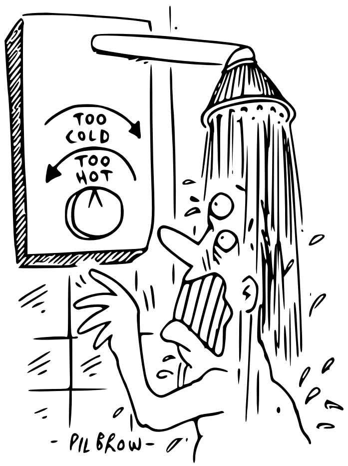
Loop delays: networked control



Sampling, encoding, transmission, decoding need time. This gives rise to

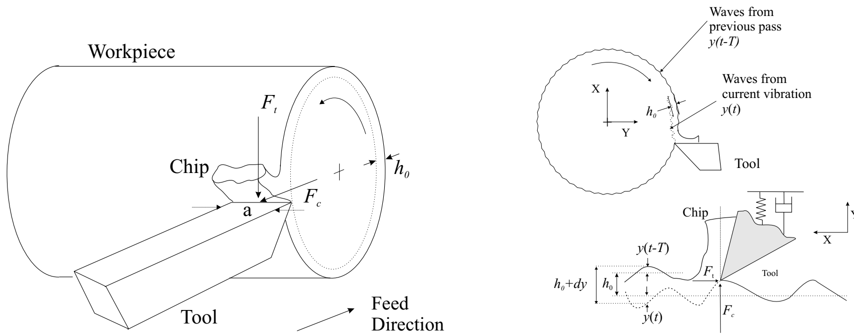
- measurement delays
- actuation delays

Loop delays: temperature control



Everybody experienced this, I guess. . .

Internal delays: regenerative chatter in metal cutting



Deviation from the ideal cut satisfies

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = K_f a\big(h_0 + y(t-T) - y(t)\big),$$

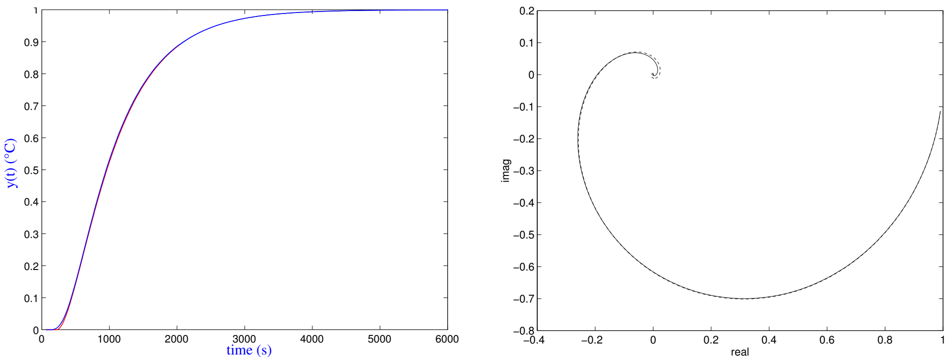
where T is time of one full rotation of workpiece.

Delays as modeling tool: heating a can

Transfer function of a heated can (derived from PDE model):

$$G(s) = \frac{1}{J_0\left(\sqrt{\frac{-s}{\alpha}}R\right)} + \sum_{m=1}^{\infty} \frac{2}{\lambda_m R\, J_1(\lambda_m R)} \frac{s}{s+\alpha\lambda_m^2} \frac{1}{\cosh\left(\sqrt{\frac{s}{\alpha}+\lambda_m^2}\cdot\frac{L}{2}\right)}.$$

Its approximation by $G_2(s) = \frac{1}{(\tau_1 s+1)(\tau_2 s+1)}\mathrm{e}^{-sh}$ is reasonably accurate:

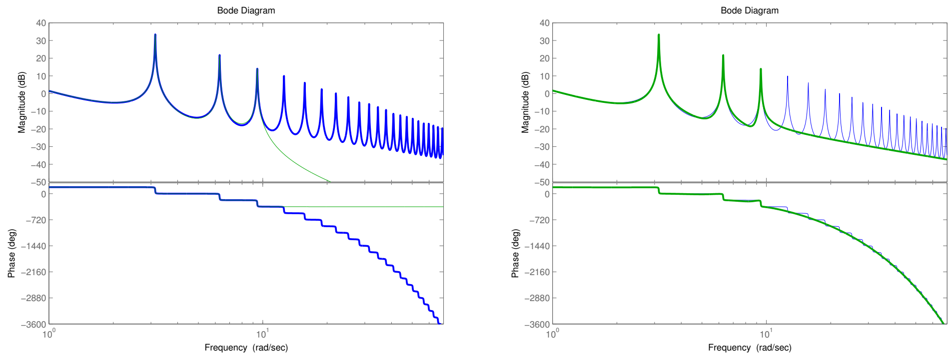


Delays as modeling tool: torsion of a rod

Transfer function of a free-free uniform rod at distance x from actuator:

$$G(s) = \frac{k}{s}\,\frac{\mathrm{e}^{-xhs} + \mathrm{e}^{-(2-x)hs}}{1-\mathrm{e}^{-2hs}}, \quad 0 \le x \le 1.$$

Its approximation by $G_{\mathrm{r}}(s)\mathrm{e}^{-xhs}$ does capture high-frequency phase lag:

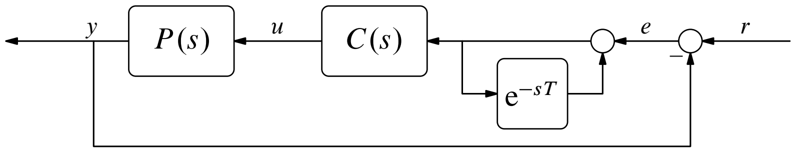


Exploiting delays: repetitive control

Any T-periodic signal $f(t)$ satisfies (with suitable initial conditions)

$$f(t) - f(t-T) = 0 \quad \text{or, in frequency domain,} \quad f(s) = \frac{1}{1-\mathrm{e}^{-sT}}$$

This motivates the configuration, called repetitive control:



which is

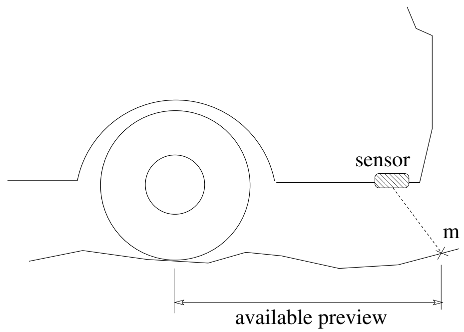
- generalization of the internal model controllers (including I)

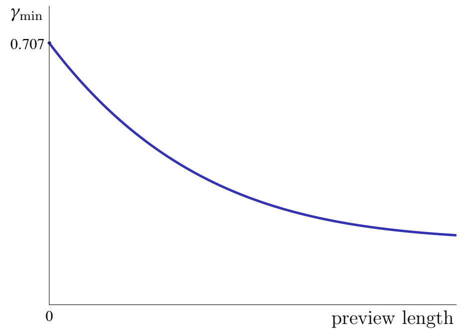
and guarantees (if stable!)

- asymptotically perfect tracking of any T-periodic reference r

Exploiting delays: preview control

If we know reference in advance, we can exploit it to improve performance:





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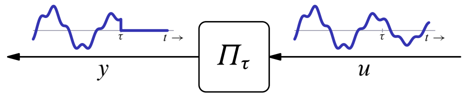
Linear systems

We think of systems as linear operators between input and output signals:

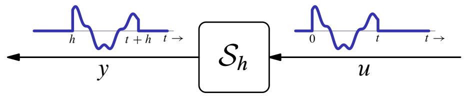
$$y = \mathcal{G} u.$$

$\mathcal{G}$ is said to be

- causal if $\mathit{\Pi}_\tau \mathcal{G}(I - \mathit{\Pi}_\tau) = 0$, $\forall \tau$, where $\mathit{\Pi}_\tau$ is truncation at time τ



- time-invariant if it commutes with shift operator: $\mathcal{G}\mathcal{S}_h = \mathcal{S}_h\mathcal{G}$, $\forall h$



- stable if $\exists \gamma \geq 0$ (independent of u) such that $\|\mathcal{G}u\| \leq \gamma \|u\|$, $\forall u$

LTI systems

Let $g(t)$ be impulse response of an LTI system $\mathcal{G}$. Then

$$y(t) = \int_{-\infty}^{\infty} g(t-\tau)u(\tau)\mathrm{d}\tau \qquad \text{(convolution integral)}$$

For causal systems $g(t) = 0$ whenever $t < 0$. Then

$$y(t) = \int_{-\infty}^{t} g(t-\tau)u(\tau)\mathrm{d}\tau$$

or even

$$y(t) = \int_{0}^{t} g(t-\tau)u(\tau)\mathrm{d}\tau$$

if we assume that $u(t) = 0$ whenever $t < 0$.

Transfer functions

In the s-domain (Laplace transform domain) convolution becomes product:

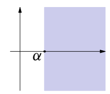
$$y(t) = \int_0^t g(t-\tau)u(\tau)\mathrm{d}\tau \iff y(s) = G(s)u(s),$$

where $G(s) = \mathcal{L}\{g(t)\} := \int_0^\infty g(t)\mathrm{e}^{-st}\mathrm{d}t$ is called the transfer function of $\mathcal{G}$.

Some basic definitions:

- Static gain: $G(0)$
- Frequency response: $G(\mathrm{j}\omega) := G(s)|_{s=\mathrm{j}\omega}$
- High-frequency gain: $\limsup_{\omega\to\infty} \|G(\mathrm{j}\omega)\|$
- $G(s)$ is said to be proper if $\exists \alpha > 0$ such that $\sup_{s\in\mathbb{C}_\alpha} G(s) < \infty$, where

$$\mathbb{C}_\alpha := \{s \in \mathbb{C} : \operatorname{Re} s > \alpha\} =$$



- $G(s)$ is said to be strictly proper if $\lim_{\alpha\to\infty} \sup_{s\in\mathbb{C}_\alpha} G(s) = 0$

Transfer functions: causality & L^2-stability

Theorem
LTI $\mathcal{G}$ is causal iff its transfer function $G(s)$ is proper.

Some definitions:

- L^2-norm of $f(t)$: $\|f\|_2 := \bigl(\int_0^\infty \|f(t)\|^2 \mathrm{d}t\bigr)^{1/2}$ ($\|f\|_2^2$ is energy of $f(t)$)
- $\mathcal{G}$ is said to be L^2 stable if $\|\mathcal{G}\|_{L^2\mapsto L^2} := \sup_{\|u\|_2=1}\|\mathcal{G}u\|_2 < \infty$
- $H^\infty := \bigl\{G(s) : G(s) \text{ analytic in } \mathbb{C}_0 \text{ and } \sup_{s\in\mathbb{C}_0}\|G(s)\| < \infty\bigr\}$

Theorem
LTI $\mathcal{G}$ is causal and L^2-stable iff $G \in H^\infty$. Moreover, in this case

$$\|\mathcal{G}\|_{L^2\mapsto L^2} = \|G\|_\infty := \sup_{s\in\mathbb{C}_0}\|G(s)\| = \sup_{\omega\in\mathbb{R}}\|G(\mathrm{j}\omega)\|.$$

Note that $G \in H^\infty$ implies that $G(s)$ is proper.

Rational transfer functions

A $p \times q$ transfer function $G(s)$ is said to be rational if $\forall i = \overline{1,p}$ and $j = \overline{1,q}$

$$G_{ij}(s) = \frac{b^{m_{ij}} s^{m_{ij}} + \cdots + b_1 s + b_0}{a^{n_{ij}} s^{n_{ij}} + \cdots + a_1 s + a_0}$$

for some $n_{ij}, m_{ij} \in \mathbb{Z}^+$.

System $\mathcal{G}$ is finite dimensional iff its transfer function $G(s)$ is rational.

Some properties greatly simplified when $G(s)$ is rational:

- rational $G(s)$ is proper (strictly proper) iff $n_{ij} \geq m_{ij}$ ($n_{ij} > m_{ij}$) $\forall i, j$
- rational $G \in H^\infty$ iff $G(s)$ proper and has no poles in $\overline{\mathbb{C}}_0 := \mathrm{j}\mathbb{R} \cup \mathbb{C}_0$
- high-frequency gain of rational $G(s)$ is $\|G(\infty)\|$

State-space realizations

Any causal LTI finite-dimensional system $\mathcal{G}$ admits state space realization in the sense that there are $n \in \mathbb{Z}^+$ and $x(t) \in \mathbb{R}^n$ (called state vector) such that

$$y = \mathcal{G}u \quad \Rightarrow \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

for some $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, and initial condition x_0. In this case

$$x(t+\theta) = \mathrm{e}^{A\theta}x(t) + \int_0^\theta \mathrm{e}^{A(\theta-\tau)}Bu(t+\tau)\mathrm{d}\tau,$$

implying that

- if we know $x(t)$ and future inputs, we can calculate future outputs

(i.e., no knowledge of past inputs required). This means that

- state vector is history accumulator,

which is the defining property of state vector.

State-space realizations (contd)

Assume that $x_0 = 0$ (no history to accumulate). Then:

- impulse response of $\mathcal{G}$ is $g(t) = D\delta(t) + C\mathrm{e}^{At}B$,
- transfer function of $\mathcal{G}$ is $G(s) = D + C(sI - A)^{-1}B =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$

State-space realization is not unique. There even are realizations of the very same system with different state dimensions. A realization called

- minimal if no other realizations of lower dimension exist

and (A, B, C, D) is minimal iff (A, B) controllable and (C, A) observable.

Any two minimal realizations connected via similarity transformation:

$$x(t) \mapsto Tx(t) \quad \Rightarrow \quad \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \mapsto \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right],$$

which changes neither impulse response nor transfer function, obviously.

System properties via state-space realizations

An LTI causal finite-dimensional system is

- L^2 stable iff its minimal realization has Hurwitz "A" matrix[1].

This is independent of the realization chosen.

The "D" matrix is informative too:

- $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is strictly proper iff $D = 0$
- High-frequency gain of $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is exactly $\|D\|$

[1]A matrix is said to be Hurwitz if it has no eigenvalues in $\overline{\mathbb{C}}_0$ (closed right-half plane).

State-space calculus

These can be verified via simple flow-tracing:

- Addition:

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_2 & D_1 \end{array}\right] + \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

- Multiplication:

$$\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2 C_1 & A_2 & B_2 D_1 \\ \hline D_2 C_1 & C_2 & D_2 D_1 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & B_2 C_1 & B_2 D_1 \\ 0 & A_1 & B_1 \\ \hline C_2 & D_2 C_1 & D_2 D_1 \end{array}\right]$$

- Inverse (exists iff $\det D \neq 0$):

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right]$$

Schur complement

If $\det M_{11} \neq 0$,

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ M_{21} M_{11}^{-1} & I \end{bmatrix}\begin{bmatrix} M_{11} & 0 \\ 0 & \Delta_{11} \end{bmatrix}\begin{bmatrix} I & M_{11}^{-1} M_{12} \\ 0 & I \end{bmatrix},$$

while if $\det M_{22} \neq 0$,

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} I & M_{12} M_{22}^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} \Delta_{22} & 0 \\ 0 & M_{22} \end{bmatrix}\begin{bmatrix} I & 0 \\ M_{22}^{-1} M_{21} & I \end{bmatrix},$$

where

$$\Delta_{11} := M_{22} - M_{21} M_{11}^{-1} M_{12} \quad \text{and} \quad \Delta_{22} := M_{11} - M_{12} M_{22}^{-1} M_{21}$$

are Schur complements of M_{11} and M_{22}, respectively. Consequently,

$$\det M = \det M_{11} \det \Delta_{11} = \det M_{22} \det \Delta_{22},$$

provided corresponding invertibility holds true.

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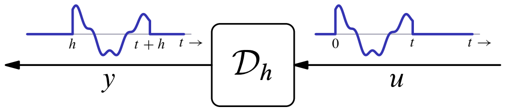
Time delay element: continuous time

$$y = \mathcal{D}_h u: \quad y(t) = u(t-h)$$



Causality follows by

$$\mathcal{D}_h \Pi_\tau = \Pi_{\tau+h} \mathcal{D}_h \quad \Rightarrow \quad \Pi_\tau \mathcal{D}_h (I - \Pi_\tau) = \Pi_\tau (I - \Pi_{\tau+h}) \mathcal{D}_h.$$

Since $\Pi_\tau \Pi_{\tau+h} = \Pi_\tau$, we have that

$$\Pi_\tau \mathcal{D}_h (I - \Pi_\tau) = (\Pi_\tau - \Pi_\tau) \mathcal{D}_h = 0$$

Time-invariance follows by

$$\mathcal{D}_h \mathcal{S}_\tau = \mathcal{D}_{h+\tau} = \mathcal{S}_\tau \mathcal{D}_h$$

Impulse response is clearly $\delta(t-h)$

Transfer function of $\mathcal{D}_h$

By the time shifting property of the Laplace transform:

$$y(t) = u(t-h) \iff y(s) = \mathrm{e}^{-sh} u(s)$$

Thus

$$D_h(s) = \mathrm{e}^{-sh},$$

which is irrational.

Transfer function of $\mathcal{D}_h$ is

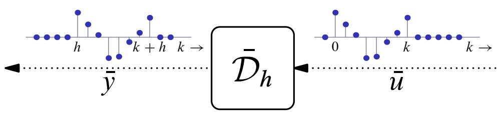
- entire (i.e., analytic in whole $\mathbb{C}$)
- bounded in $\mathbb{C}_\alpha$ for every $\alpha \in \mathbb{R}$

Hence,

$$\mathrm{e}^{-sh} \in H^\infty$$

Time delay element: discrete time

$$\bar{y} = \bar{\mathcal{D}}_h \bar{u}: \quad \bar{y}[k] = \bar{u}[k-h]$$



By the time shifting property of the z-transform:

$$\bar{y}[k] = \bar{u}[k-h] \iff \bar{y}(z) = z^{-h} \bar{u}(z)$$

Thus

$$\bar{D}_h(z) = \frac{1}{z^h},$$

which is rational (of order h, all h poles are at the origin, no finite zeros).

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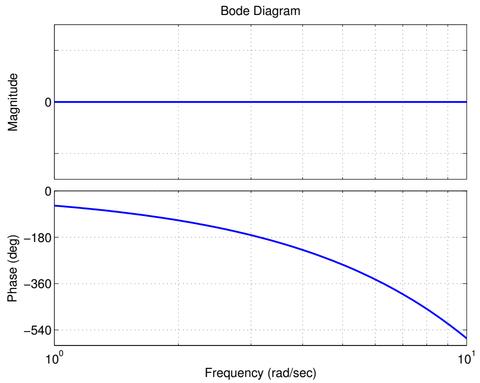
Frequency response

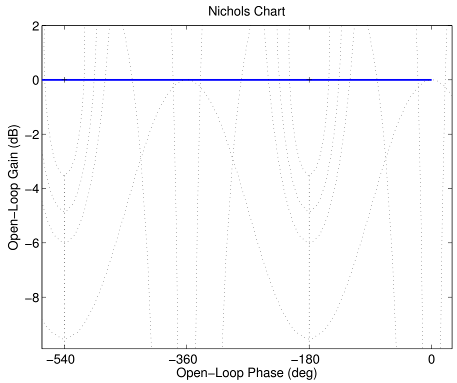
Obviously,

$$e^{-sh}|_{s=j\omega} = e^{-j\omega h} = \cos(\omega h) - j\sin(\omega h)$$

Has

- unit magnitude ($|e^{j\omega h}| \equiv 1$) and
- linearly decaying phase ($\arg e^{j\omega h} = -\omega h$, in radians if ω—in rad/sec)





Effects of I/O delay on rational transfer functions

Consider the simplest interconnection:

$$L(s) = L_r(s)e^{-sh} \qquad \text{for some rational } L_r(s).$$

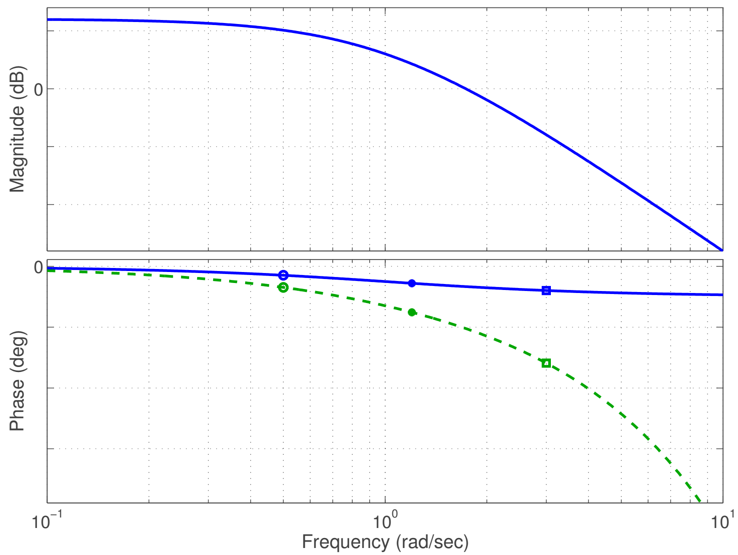
In this case $L(j\omega) = L_r(j\omega)e^{-j\omega h}$, meaning that

$$|L(j\omega)| = |L_r(j\omega)| \quad \text{and} \quad \arg L(j\omega) = \arg L_r(j\omega) - \omega h.$$

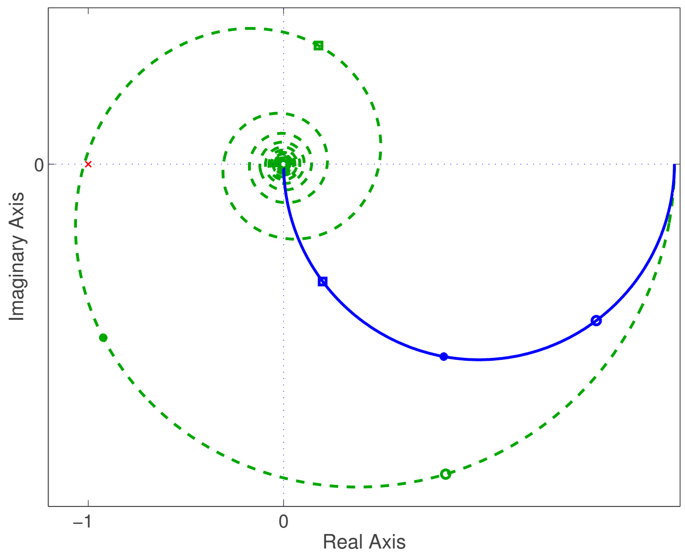
In other words, delay in this case

- does not change the magnitude of $L_r(j\omega)$ and
- adds phase lag proportional to ω.

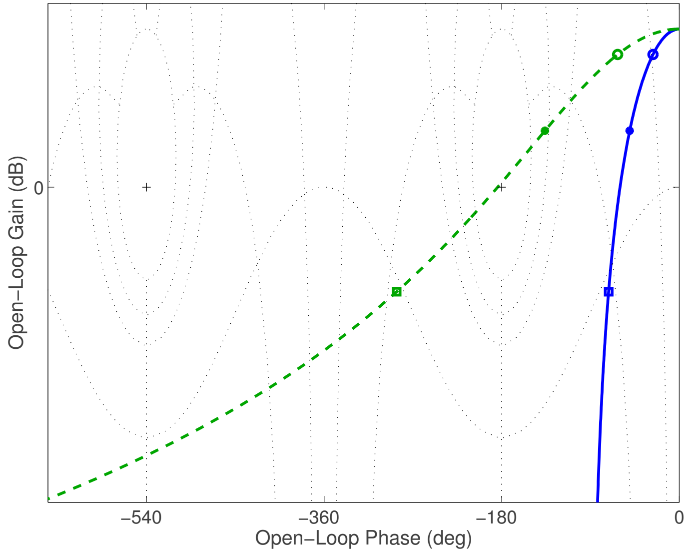
Effects of I/O delay: Bode diagram



Effects of I/O delay: Nyquist diagram



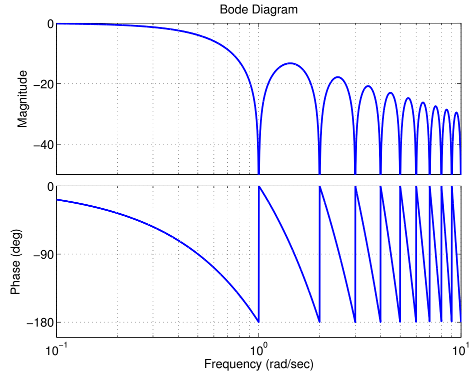
Effects of I/O delay: Nichols chart

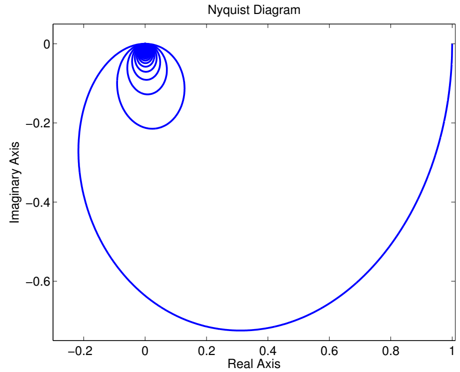


It's not always so easy

If delay not I/O, frequency response plots might be much more complicated. Consider for example the (stable) system:

$$G(s) = \frac{1 - \mathrm{e}^{-2\pi s}}{2\pi s}, \quad \text{with impulse response} \quad g(t) = \frac{1(t) - 1(t - 2\pi)}{2\pi},$$





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Why to approximate

Delay element is infinite dimensional, which complicates its treatment. It is not a surprise then that we want to approximate delay by finite-dimensional (rational) elements to

- use standard methods in analysis and design,
- use standard software for simulations,
- ~~avoid learning new methods,~~
- . . .

What to approximate: bad news

On the one hand,

- phase lag of the delay element is not bounded (and continuous in ω).

On the other hand,

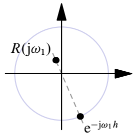
- rational systems can only provide finite phase lag.

Therefore, phase error between e^{-sh} and any rational transfer function $R(s)$ is arbitrarily large. Moreover, for every $R(s)$ there always is ω_0 such that

- $\arg \mathrm{e}^{-\mathrm{j}\omega h} - \arg R(\mathrm{j}\omega)$ continuously decreasing function of ω, $\forall \omega \geq \omega_0$.

Hence there always is frequency ω_1 such that

$$\arg \mathrm{e}^{-\mathrm{j}\omega_1 h} - \arg R(\mathrm{j}\omega_1) = -\pi - 2\pi k \qquad \text{i.e.,}$$



This, together with the fact that $|\mathrm{e}^{-\mathrm{j}\omega h}| \equiv 1$, means that

- rational approximation of pure delay, e^{-sh}, is pretty senseless

as there always will be frequencies at which error[2] is ≥ 1 (i.e., $\geq 100\%$).

[2] Thus, we never can get better approximation than with $R(s) = 0 \ldots$

What to approximate: good news

Yet we never work over infinite bandwidth. Hence, we

- need to approximate e^{-sh} in finite frequency range

or, equivalently,

- approximate $F(s)\mathrm{e}^{-sh}$ for low-pass (strictly proper) $F(s)$.

This can be done, since

- phase lag of delay over finite bandwidth is finite

and

- magnitude of $F(\mathrm{j}\omega)\mathrm{e}^{-\mathrm{j}\omega h}$ decreases as ω increases,

which implies that at frequencies where the phase lag of $F(\mathrm{j}\omega)\mathrm{e}^{-\mathrm{j}\omega h}$ large, the function effectively vanishes.

Also, we may consider $h = 1$ w.l.o.g., otherwise $s \to s/h$ makes the trick.

Truncation-based methods

General idea is to

- truncate some power series,

which could give accurate results in a (sufficiently large) neighborhood of 0.

Truncation-based methods: naïve approach

Note that

$$\mathrm{e}^{-s} = \frac{\mathrm{e}^{-s/2}}{\mathrm{e}^{s/2}}$$

and truncate Taylor series of numerator and denominator. We could get:

$$\mathrm{e}^{-s} \approx \frac{\sum_{i=0}^{n} \frac{1}{i!2^i} (-s)^i}{\sum_{i=0}^{n} \frac{1}{i!2^i} s^i}$$

This yields:

n	1	2	3	4
$\mathrm{e}^{-sh} \approx$	$\frac{1-\frac{sh}{2}}{1+\frac{sh}{2}}$	$\frac{1-\frac{sh}{2}-\frac{s^2h^2}{8}}{1+\frac{sh}{2}+\frac{s^2h^2}{8}}$	$\frac{1-\frac{sh}{2}+\frac{s^2h^2}{8}-\frac{s^3h^3}{48}}{1+\frac{sh}{2}+\frac{s^2h^2}{8}+\frac{s^3h^3}{48}}$	$\frac{1-\frac{sh}{2}+\frac{s^2h^2}{8}-\frac{s^3h^3}{48}+\frac{s^4h^4}{384}}{1+\frac{sh}{2}+\frac{s^2h^2}{8}+\frac{s^3h^3}{48}+\frac{s^4h^4}{384}}$

For $n = 2$ called Kautz formula. But

- becomes unstable for $n > 4$!

Truncation-based methods: Padé approximation

Consider approximation

$$e^{-s} \approx \frac{P_m(s)}{Q_n(s)} =: R_{[m,n]}(s),$$

where $P_m(s)$ and $Q_n(s)$ are polynomials of degrees m and n, respectively. Taylor expansions at $s = 0$ of each side are

$$e^{-s} = 1 - \frac{s}{1!} + \frac{s^2}{2!} - \frac{s^3}{3!} + \cdots$$

$$R_{[m,n]}(s) = R_{[m,n]}(0) + \frac{R'_{[m,n]}(0)s}{1!} + \frac{R''_{[m,n]}(0)s^2}{2!} + \frac{R'''_{[m,n]}(0)s^3}{3!} + \cdots$$

The idea of $[m,n]$-Padé approximation is to find coefficients of $R_{[m,n]}(s)$ via

- matching first $n + m + 1$ Taylor coefficients

of these two series. If $n = m$, it can be shown that $P_n(s) = Q_n(-s)$.

Example: $[2,2]$-Padé approximation

In this case $R_{[2,2]}(s) = \frac{s^2 - q_1 s + q_0}{s^2 + q_1 s + q_0}$ and Taylor expansions are

$$e^{-s} = 1 - s + \frac{s^2}{2} - \frac{s^3}{6} + \frac{s^4}{24} - \cdots$$

$$R_{[2,2]}(s) = 1 - \frac{2q_1}{q_0}s + \frac{2q_1^2}{q_0^2}s^2 - \frac{2(q_1^3 - q_1 q_0)}{q_0^3}s^3 + \frac{2(q_1^4 - 2q_1^2 q_0)}{q_0^4}s^4 - \cdots$$

from which

$$q_0 = 2q_1 \quad \text{and} \quad \frac{q_1 - 2}{4q_1} = \frac{1}{6}$$

and then $q_1 = 6$ and $q_0 = 12$, matching 5 coefficients.

Thus, $[2,2]$-Padé approximation is

$$e^{-s} \approx \frac{s^2 - 6s + 12}{s^2 + 6s + 12} = \frac{1 - \frac{s}{2} + \frac{s^2}{12}}{1 + \frac{s}{2} + \frac{s^2}{12}}.$$

Truncation-based methods: Padé approximation (contd)

General formula for $[n,n]$-Padé approximation is

$$e^{-s} \approx \frac{\sum_{i=0}^{n} \binom{n}{i} \frac{(2n-i)!}{(2n)!} (-s)^i}{\sum_{i=0}^{n} \binom{n}{i} \frac{(2n-i)!}{(2n)!} s^i} = \frac{\sum_{i=0}^{n} \frac{(2n-i)!n!}{(2n)!(n-i)!i!} (-s)^i}{\sum_{i=0}^{n} \frac{(2n-i)!n!}{(2n)!(n-i)!i!} s^i}$$

This yields:

n	1	2	3	4
$e^{-sh} \approx$	$\frac{1-\frac{sh}{2}}{1+\frac{sh}{2}}$	$\frac{1-\frac{sh}{2}+\frac{s^2h^2}{12}}{1+\frac{sh}{2}+\frac{s^2h^2}{12}}$	$\frac{1-\frac{sh}{2}+\frac{s^2h^2}{10}-\frac{s^3h^3}{120}}{1+\frac{sh}{2}+\frac{s^2h^2}{10}+\frac{s^3h^3}{120}}$	$\frac{1-\frac{sh}{2}+\frac{s^2h^2}{28}-\frac{s^3h^3}{84}+\frac{s^4h^4}{1680}}{1+\frac{sh}{2}+\frac{s^2h^2}{28}+\frac{s^3h^3}{84}+\frac{s^4h^4}{1680}}$

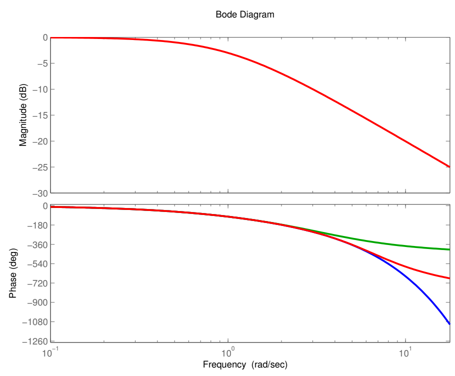
Using Routh-Hurtitz test one can prove that

- $[n,n]$-Padé approximation stable for all n.

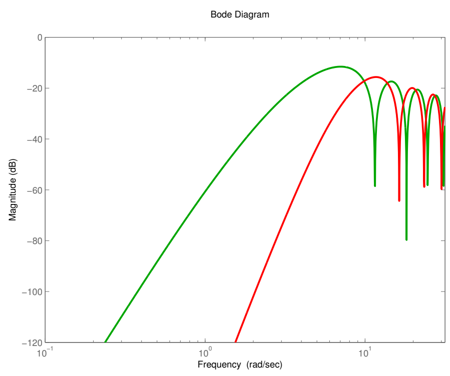
Padé approximation: example

Consider Padé approximation of $\frac{1}{s+1}e^{-s}$. This can be calculated by Matlab function `pade(tf(1,[1 1],'InputDelay',1),N)`.

$\frac{e^{-s}}{s+1}$ and its 2nd and 4th order approximations



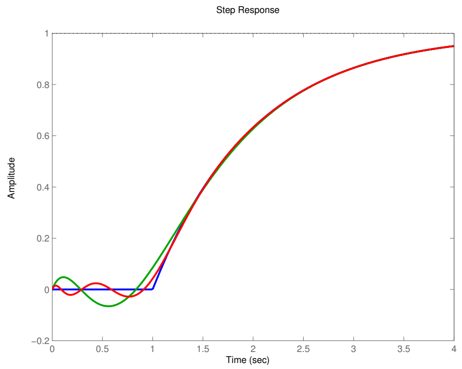
Corresponding approximation errors

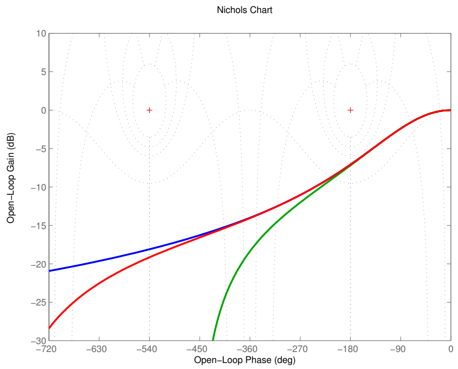


Padé approximation: example (contd)

We may also compare step responses and Nichols charts

$\frac{e^{-s}}{s+1}$ and its 2nd and 4th order approximations





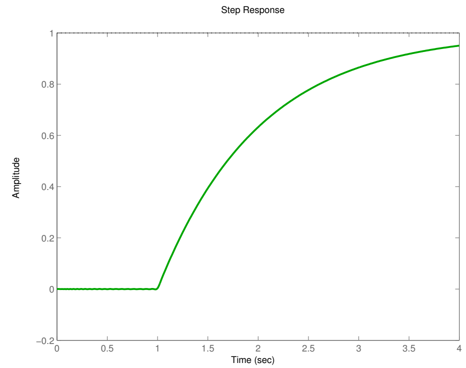
From loop shaping perspectives,

- approximation performance depends on crossover requirements.

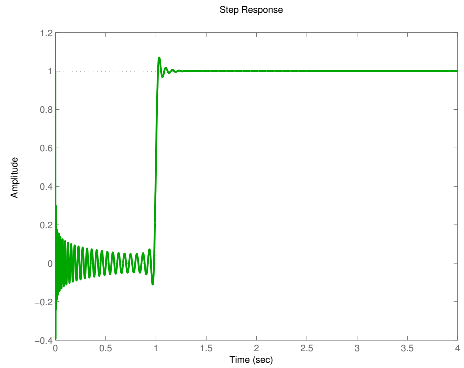
Padé approximation: example (contd)

Increasing approximation order improves the match between step responses of $\frac{1}{s+1}e^{-s}$ and its Padé approximation:

$\frac{e^{-s}}{s+1}$ and its 50th order approximation



e^{-s} and its 50th order approximation



Not true for the approximation of the pure delay e^{-s} !

Outline

Course info

Time-delay systems in control applications

System-theoretic preliminaries

Basic properties

Delay systems in the frequency domain

Rational approximations of time delays

State space of delay systems

Modal properties of delay systems

Stability of transfer functions and roots of characteristic equations

State equation with input delay

$$\dot{x}(t) = Ax(t) + Bu(t) \xrightarrow{\text{if input delayed by } h} \dot{x}(t) = Ax(t) + Bu(t-h)$$

Solution then becomes:

$$\begin{aligned}
x(t+\theta) &= e^{A\theta}x(t) + \int_t^{t+\theta} e^{A(t+\theta-\tau)}Bu(\tau-h)d\tau \\
&= e^{A\theta}\left(x(t) + \int_t^{t+\theta} e^{A(t-\tau)}Bu(\tau-h)d\tau\right) \\
&= e^{A\theta}\left(x(t) + \int_{t-h}^{t-h+\theta} e^{A(t-h-\tau)}Bu(\tau)d\tau\right) \\
&= e^{A\theta}\left(x(t) + \int_{t-h}^{t} e^{A(t-h-\tau)}Bu(\tau)d\tau + \int_t^{t-h+\theta} e^{A(t-h-\tau)}Bu(\tau)d\tau\right)
\end{aligned}$$

It depends on initial "state" $x(t)$, future inputs over $[t, t-h+\theta]$ (if $\theta > h$), and past inputs over $[t-h, t]$ (or $[t-h, t-h+\theta]$ if $\theta < h$).

"In my country there is problem…" (B. Sagdiyev)

…and that problem is:

- $x(t)$ no longer accumulates the history.

This, in turn, implies that $x(t)$ can no longer be regarded as the "state".

Intuitively, the "true" state vector at every time instance t should contain

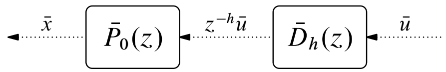
- $x(t)$
- $u(t+\tau)$ for all $\tau \in [-h, 0]$—denoted $u_\tau(t)$

Checking intuition on discrete-time case

Consider

$$\bar{x}[k+1] = \bar{A}\bar{x}[k] + \bar{B}\bar{u}[k-h].$$

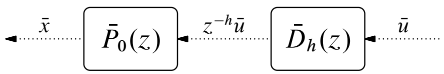
This system can be thought of as serial interconnection



where

$$\bar{P}_0(z) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline I & 0 \end{array}\right] \quad \text{and} \quad \bar{D}_h(z) = \left[\begin{array}{ccccc|c} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \\ 0 & 0 & 0 & \cdots & 0 & I \\ \hline I & 0 & 0 & \cdots & 0 & 0 \end{array}\right]$$

Checking intuition on discrete-time case (contd)



Thus, for $\bar{P}_h(z) := \bar{P}_0(z)\bar{D}_h(z)$ we have:

$$\bar{P}_h(z) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline I & 0 \end{array}\right] \left[\begin{array}{ccccc|c} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \\ 0 & 0 & 0 & \cdots & 0 & I \\ \hline I & 0 & 0 & \cdots & 0 & 0 \end{array}\right] = \left[\begin{array}{c:ccccc|c} \bar{A} & \bar{B} & 0 & 0 & \cdots & 0 & 0 \\ \hdashline 0 & 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & I \\ \hline I & 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right]$$

Checking intuition on discrete-time case (contd)

To recover the state vector, write state equation:

$$\underbrace{\begin{bmatrix} \bar{x}[k+1] \\ \bar{u}[k-h+1] \\ \bar{u}[k-h+2] \\ \vdots \\ \bar{u}[k-1] \\ \bar{u}[k] \end{bmatrix}}_{\bar{x}_a[k+1]} = \begin{bmatrix} \bar{A} & \bar{B} & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \underbrace{\begin{bmatrix} \bar{x}[k] \\ \bar{u}[k-h] \\ \bar{u}[k-h+1] \\ \vdots \\ \bar{u}[k-2] \\ \bar{u}[k-1] \end{bmatrix}}_{\bar{x}_a[k]} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix} \bar{u}[k]$$

Thus, the state vector at time k, $\bar{x}_a[k]$, indeed

- includes both $\bar{x}[k]$ and whole input history $\bar{u}[i]$ in $k-h \le i \le k-1$

State equation with input delay (contd)

Thus, the state vector of

$$\dot{x}(t) = Ax(t) + Bu(t-h)$$

at time t is $(x(t), u_\tau(t)) \in (\mathbb{R}^n, \{[-h,0] \mapsto \mathbb{R}^m\})$. This implies (among many other things) that

- initial conditions for this system are $(x(0), u_\tau(0))$,

which is also a function. For example,

- zero initial conditions should read

$$x(0) = 0 \quad \text{and} \quad u(\tau) = 0, \quad \forall \tau[-h, 0].$$

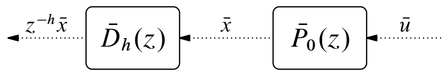
There is more consistent (and elegant) way to reflect all this via state-space description, using semigroup formalism. See (Curtain and Zwart, 1995) for details.

State equation with output delay

Consider

$$\dot{x}(t) = Ax(t) + Bu(t)$$

and assume that that we measure delayed x, i.e., $y(t) = x(t-h)$. Discrete counterpart looks like this:



with

$$\bar{P}_h(z) = \left[\begin{array}{cccccc|c} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \\ 0 & 0 & 0 & \cdots & 0 & I \\ \hline I & 0 & 0 & \cdots & 0 & 0 \end{array}\right] \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline I & 0 \end{array}\right] = \left[\begin{array}{ccccc:c|c} 0 & I & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & I & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 0 & \bar{A} & \bar{B} \\ \hline I & 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right]$$

State equation with output delay (contd)

$$\underbrace{\begin{bmatrix} \bar{x}[k-h+1] \\ \bar{x}[k-h+2] \\ \vdots \\ \bar{x}[k-1] \\ \bar{x}[k] \\ \bar{x}[k+1] \end{bmatrix}}_{\bar{x}_a[k+1]} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \\ 0 & 0 & 0 & \cdots & 0 & I \\ 0 & 0 & 0 & \cdots & 0 & \bar{A} \end{bmatrix} \underbrace{\begin{bmatrix} \bar{x}[k-h] \\ \bar{x}[k-h+1] \\ \vdots \\ \bar{x}[k-2] \\ \bar{x}[k-1] \\ \bar{x}[k] \end{bmatrix}}_{\bar{x}_a[k]} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \bar{B} \end{bmatrix} \bar{u}[k]$$

Thus, the state vector at time k, $\bar{x}_a[k]$,

- includes whole history of $\bar{x}[i]$ in $k-h \leq i \leq k-1$

State equation with output delay (contd)

Returning to continuous-time case, state vector of

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = x(t-h) \end{cases}$$

at time t is $x_\tau(t) \in \{[-h,0] \mapsto \mathbb{R}^n\}$ (may be convenient to write $(x(t), x_\tau(t))$).

The

- initial condition is then the function $x_\tau(0)$

and zero initial conditions would mean

$$x(\tau) = 0, \quad \forall \tau \in [-h, 0].$$

State delay equations

If we use a P controller $u(t) = Ky(t)$, the closed loop system becomes

$$\dot{x}(t) = Ax(t) + BKx(t-h).$$

This kind of equations called retarded functional differential equation.

If we use a D controller $u(t) = K\dot{y}(t)$, the closed loop system becomes

$$\dot{x}(t) = Ax(t) + BK\dot{x}(t-h) \quad \text{or} \quad \dot{x}(t) - BK\dot{x}(t-h) = Ax(t)$$

This kind of equations called neutral functional differential equation.

Homogeneous LTI state equations: classification

(Lumped-delay) retarded equation:

$$\dot{x}(t) = \sum_{i=0}^{r} A_i x(t-h_i), \quad 0 = h_0 < h_1 < \cdots < h_r = h$$

(Lumped-delay) neutral equation:

$$\sum_{i=0}^{r} E_i \dot{x}(t-h_i) = \sum_{i=0}^{r} A_i x(t-h_i), \quad 0 = h_0 < h_1 < \cdots < h_r = h$$

Distributed-delay retarded equation:

$$\dot{x}(t) = \int_{-h}^{0} \alpha(\tau) x(t+\tau) \mathrm{d}\tau$$

If $\alpha(\tau) = \sum_i A_i \delta(t+h_i)$, we have the lumped-delay equation above.

- In all cases the "true" state is $x_\tau(t) \in \{[-h,0] \mapsto \mathbb{R}^n\}$.

Adding inputs and outputs

(Still not the most) general form:

$$\begin{cases} \dot{x}(t) + \int_{-h_x}^{0} \epsilon(\tau)\dot{x}(t+\tau)\mathrm{d}\tau = \int_{-h_x}^{0} \alpha(\tau)x(t+\tau)\mathrm{d}\tau + \int_{-h_u}^{0} \beta(\tau)u(t+\tau)\mathrm{d}\tau \\ y(t) = \int_{-h_x}^{0} \gamma(\tau)x(t+\tau)\mathrm{d}\tau + \int_{-h_u}^{0} \delta(\tau)u(t+\tau)\mathrm{d}\tau \end{cases}$$

with the state "vector" $(x_\tau(t), u_\tau(t)) \in \big(\{[-h_x,0] \mapsto \mathbb{R}^n\}, \{[-h_u,0] \mapsto \mathbb{R}^m\}\big)$.

Important special (lumped-delay) case:

$$\begin{cases} \sum_{i=0}^{r_x} E_i \dot{x}(t-h_i) = \sum_{i=0}^{r_x} A_i x(t-h_i) + \sum_{i=0}^{r_u} B_i u(t-h_i) \\ y(t) = \sum_{i=0}^{r_x} C_i x(t-h_i) + \sum_{i=0}^{r_u} D_i u(t-h_i) \end{cases}$$

with $E_0 = I$ and $0 = h_0 < h_1 < \cdots < h_{\max\{r_x,r_u\}} = h$.

The same in s domain (with zero initial conditions)

(Still not the most) general form:

$$\begin{cases} s\left(I + \int_{-h_x}^{0} \epsilon(\tau)\mathrm{e}^{\tau s}\mathrm{d}\tau\right) X(s) = \int_{-h_x}^{0} \alpha(\tau)\mathrm{e}^{\tau s}\mathrm{d}\tau X(s) + \int_{-h_u}^{0} \beta(\tau)\mathrm{e}^{\tau s}\mathrm{d}\tau U(s) \\ Y(s) = \int_{-h_x}^{0} \gamma(\tau)\mathrm{e}^{\tau s}\mathrm{d}\tau X(s) + \int_{-h_u}^{0} \delta(\tau)\mathrm{e}^{\tau s}\mathrm{d}\tau U(s) \end{cases}$$

with zero initial conditions $x(\tau) = 0$ ($\tau \in [-h_x, 0]$), $u(\tau) = 0$ ($\tau \in [-h_u, 0]$).

Important special (lumped-delay) case:

$$\begin{cases} s\sum_{i=0}^{r_x} E_i \mathrm{e}^{-sh_i} X(s) = \sum_{i=0}^{r_x} A_i \mathrm{e}^{-sh_i} X(s) + \sum_{i=0}^{r_u} B_i \mathrm{e}^{-sh_i} U(s) \\ Y(s) = \sum_{i=0}^{r_x} C_i \mathrm{e}^{-sh_i} X(s) + \sum_{i=0}^{r_u} D_i \mathrm{e}^{-sh_i} U(s) \end{cases}$$

with $E_0 = I$ and $0 = h_0 < h_1 < \cdots < h_{\max\{r_x,r_u\}} = h$.

Outline

Characteristic equation

Distributed-delay equation

$$s\left(I+\int_{-h_x}^{0}\epsilon(\tau)e^{\tau s}d\tau\right)X(s)=\int_{-h_x}^{0}\alpha(\tau)e^{\tau s}d\tau X(s)+\int_{-h_u}^{0}\beta(\tau)e^{\tau s}d\tau U(s)$$

(or its lumped-delay counterpart) can be rewritten as

$$X(s)=\Delta^{-1}(s)\int_{-h_u}^{0}\beta(\tau)e^{\tau s}d\tau U(s)\quad\text{or}\quad X(s)=\Delta^{-1}(s)\sum_{i=0}^{r_u}B_i e^{-sh_i}U(s),$$

where

$$\Delta(s):=\int_{-h_x}^{0}\bigl(s(I+\epsilon(\tau))-\alpha(\tau)\bigr)e^{\tau s}d\tau\quad\text{or}\quad\Delta(s):=\sum_{i=0}^{r_x}(sE_i-A_i)e^{-sh_i}$$

called the characteristic matrix. Then equation

$$\det[\Delta(s)]=:\chi(s)=0$$

called characteristic equation.

Example 1

Consider

$$\dot{x}(t)=\begin{bmatrix}0&1\\0&-1\end{bmatrix}x(t)-\begin{bmatrix}0&0\\k_1&k_2\end{bmatrix}x(t-h)$$

Then

$$\Delta(s)=s\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}0&1\\0&-1\end{bmatrix}+\begin{bmatrix}0&0\\k_1&k_2\end{bmatrix}e^{-sh}=\begin{bmatrix}s&-1\\k_1e^{-sh}&s+1+k_2e^{-sh}\end{bmatrix}$$

and

$$\chi(s)=s^2+s+(k_2s+k_1)e^{-sh}$$

Example 2

Consider

$$\dot{x}(t)+\begin{bmatrix}0&0\\k_1&k_2\end{bmatrix}\dot{x}(t-h)=\begin{bmatrix}0&1\\0&-1\end{bmatrix}x(t)$$

Then

$$\Delta(s)=s\begin{bmatrix}1&0\\0&1\end{bmatrix}+s\begin{bmatrix}0&0\\k_1&k_2\end{bmatrix}e^{-sh}-\begin{bmatrix}0&1\\0&-1\end{bmatrix}=\begin{bmatrix}s&-1\\sk_1e^{-sh}&s+1+sk_2e^{-sh}\end{bmatrix}$$

and

$$\chi(s)=s^2+s+(k_2s^2+k_1s)e^{-sh}$$

Example 3

Consider

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) - \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} x(t-h)$$

Then

$$\Delta(s) = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \mathrm{e}^{-sh} = \begin{bmatrix} s + k_1\mathrm{e}^{-sh} & -1 \\ 0 & s + 1 + k_2\mathrm{e}^{-sh} \end{bmatrix}$$

and

$$\chi(s) = s^2 + s + \big((k_1 + k_2)s + k_1\big)\mathrm{e}^{-sh} + k_1k_2\mathrm{e}^{-2sh}$$

Quasi-polynomials

General form:

$$\det\left(\sum_{i=0}^{r_x}(sE_i - A_i)\mathrm{e}^{-s\tilde{h}_i}\right) = P(s) + \sum_i Q_i(s)\mathrm{e}^{-sh_i}$$

for polynomials $P(s) \not\equiv 0$ and $Q_i(s)$ ($\exists j,\ Q_j(s) \not\equiv 0$) and delays $h_i > 0$.

Classification by delay pattern:

1. single-delay: $P(s) + Q(s)\mathrm{e}^{-sh}$
2. commensurate-delay: $P(s) + \sum_i Q_i(s)\mathrm{e}^{-sih}$
3. incommensurate-delay: if at least one of $\frac{h_i}{h_j}$ is irrational

Classification by principal degrees of s:

1. retarded: $\deg P(s) > \deg Q_i(s)$, $\forall i$
2. neutral: $\deg P(s) \geq \deg Q_i(s)$, $\forall i$, and $\exists j$ s.t. $\deg P(s) = \deg Q_j(s)$
3. advanced: $\exists j$ s.t. $\deg P(s) < \deg Q_j(s)$

Roots of quasi-polynomials

Apparently, the simplest example is

$$1 + k\mathrm{e}^{-sh}, \quad k > 0.$$

It is readily seen that it has

- infinite number of roots,

those at

$$s = \frac{\ln k}{h} + \mathrm{j}\frac{(1+2i)\pi}{h}, \quad i \in \mathbb{Z}$$

(all these roots are in $\bar{\mathbb{C}}_0$ iff $k \geq 1$ and in $\mathbb{C} \setminus \bar{\mathbb{C}}_0$ iff $0 < k < 1$).

This is generic property, i.e.,

- quasi-polynomials have infinite number of roots.

Roots of quasi-polynomials: where are they located

Some fundamental properties:

1. there is a finite number of roots within any finite region of $\mathbb{C}$
   (meaning there are no accumulation points for roots of quasi-polynomials)
2. roots for large values of $|s|$ belong to a finite number of areas

   $$\big|\mathrm{Re}\, s + \beta_i \ln|s|\big| < \gamma$$

   for some $\gamma > 0$ and $\beta_i \in \mathbb{R}$. For
   - retarded quasi-polynomials $\beta_i > 0$
   - neutral quasi-polynomials $\beta_i = 0$
   - advanced quasi-polynomials $\beta_i < 0$
3. retarded quasi-polynomials have a finite number of roots in $\mathbb{C}_\alpha$ for all α

Rightmost root: retarded case

Consider

$$\dot{x}(t) = \sum_{i=0}^{r_x} A_i x(t - h_i), \quad x_\tau(0) = \phi(\tau)$$

and let $\chi(s)$ be its characteristic quasi-polynomial. Define

$$\lambda_{\mathrm{r}} := \max\{\operatorname{Re} s : \chi(s) = 0\}$$

Theorem

For any $\lambda > \lambda_r$ there is a $\mu > 0$ such that

$$\|x(t)\| \le \mu \, \mathrm{e}^{\lambda t} \max_{\tau \in [-h,0]} \|\phi(\tau)\|, \quad \forall t \in \mathbb{R}^+$$

for all continuous initial conditions ϕ.

Rightmost root: neutral case

Consider

$$\sum_{i=0}^{r_x} E_i \dot{x}(t - h_i) = \sum_{i=0}^{r_x} A_i x(t - h_i), \quad x_\tau(0) = \phi(\tau)$$

and let $\chi(s)$ be its characteristic quasi-polynomial. Define

$$\lambda_{\mathrm{r}} := \sup\{\operatorname{Re} s : \chi(s) = 0\}$$

Theorem

For any $\lambda > \lambda_r$ there is a $\mu > 0$ such that

$$\|x(t)\| \le \mu \, \mathrm{e}^{\lambda t} \max_{\tau \in [-h,0]} (\|\phi(\tau)\| + \|\dot{\phi}(\tau)\|), \quad \forall t \in \mathbb{R}^+$$

for all continuous and differential initial conditions ϕ.

Outline

Course info

Time-delay systems in control applications

System-theoretic preliminaries

Basic properties

Delay systems in the frequency domain

Rational approximations of time delays

State space of delay systems

Modal properties of delay systems

Stability of transfer functions and roots of characteristic equations

Simple special case

Special (SISO single-delay) case:

$$G(s) = \frac{N_0(s)}{M_0(s) + M_h(s)\mathrm{e}^{-sh}},$$

for some real polynomials $M_0(s)$, $M_h(s)$, $N_0(s)$ such that $\deg M_0 \ge \deg M_h$ and $\deg M_0 \ge \deg N_0$. This transfer function

- is proper
- is analytic on $\mathbb{C} \setminus \Lambda$, where Λ is set of roots of $M_0(s) + M_h(s)\mathrm{e}^{-sh} = 0$
- has only poles as its singularities

To further simplify matters, assume also that there is

- no pole / zero cancellations in $G(s)$

Asymptotic poles location

Let $\gamma_a := P_a(\infty)$, where $P_a := \frac{M_h}{M_0}$ ($|\gamma_a|$ is the high-frequency gain of P_a).

Theorem
As $|s| \to \infty$, poles of $G(s)$ are asymptotic to points with $\operatorname{Re} s = \frac{\ln|\gamma_a|}{h}$.

Proof.
Poles are solutions of the characteristic equation $M_0(s) + M_h(s)e^{-sh} = 0$ or

$$e^{sh} = -P_a(s) = -\gamma_a + O\left(\tfrac{1}{s}\right)$$

By Rouché's arguments, as $|s| \to \infty$ roots approach solutions of $e^{sh} = -\gamma_a$, i.e., $sh = \ln(-\gamma_a) + j2\pi k$, $k \in \mathbb{Z}$. Thus,

$$sh \to \begin{cases} \ln|\gamma_a| + j2k\pi & \text{if } \gamma_a < 0 \\ \ln|\gamma_a| + j(2k+1)\pi & \text{if } \gamma_a > 0 \end{cases}$$

which for $\gamma_a \neq 0$ approaches vertical line with $\operatorname{Re} s = \ln|\gamma_a|$ and for $\gamma_a = 0$ approaches $\operatorname{Re} s = -\infty$. □

$|\gamma_a| < 1$

In this case, $G(s)$ has at most finitely many unstable poles. Moreover, the following result can be formulated:

Lemma
Let $|\gamma_a| < 1$. Then $G \in H^\infty$ iff $G(s)$ has no poles in $\bar{\mathbb{C}}_0$.

Proof (outline).
If $G(s)$ has a pole in $\bar{\mathbb{C}}_0$, it does not belong to H^∞.
If $G(s)$ has no poles in $\bar{\mathbb{C}}_0$, there is a bounded $\mathbb{S} \subset \bar{\mathbb{C}}_0$ such that in $\bar{\mathbb{C}}_0 \setminus \mathbb{S}$

1. $M_0(s)$ has no roots
2. $|P_a(s)| < \gamma_b$ and $\left|\frac{N_0(s)}{M_0(s)}\right| < \gamma_c$ for some $|\gamma_a| \le \gamma_b < 1$ and $\gamma_c > 0$.

Thus, $|G(s)|$ is bounded in $\mathbb{S}$ (no poles and the set is bounded) and

$$|G(s)| = \frac{|N_0(s)/M_0(s)|}{|1 + P_a(s)e^{-sh}|} < \frac{\gamma_c}{1-\gamma_b} < \infty, \quad \forall s \in \bar{\mathbb{C}}_0 \setminus \mathbb{S}.$$

Hence, $G(s)$ is analytic and bounded in $\mathbb{C}_0$. □

$|\gamma_a| > 1$

In this case, $G(s)$ has infinitely many unstable poles. Hence, the following result can be formulated:

Lemma
Let $|\gamma_a| > 1$. Then $G \notin H^\infty$.

Proof.
Obvious. □

$|\gamma_a| = 1$: example 1

Let $G(s) = \frac{1}{s+1+(s+2)e^{-s}}$. This transfer function

- has infinitely many poles in $\mathbb{C}_0$, hence unstable.

To see this, consider its characteristic equation at $s = \sigma + j\omega$:

$$e^{\sigma}e^{j\omega} = -\frac{\sigma + j\omega + 2}{\sigma + j\omega + 1} \quad \Longrightarrow \quad e^{\sigma} = \sqrt{\frac{(\sigma+2)^2+\omega^2}{(\sigma+1)^2+\omega^2}}$$

Right-hand side here > 1 iff $\sigma > -\frac{3}{2}$. Hence, whenever $\sigma > -\frac{3}{2}$, $e^{\sigma} > 1$ or, equivalently, $\sigma > 0$. Since roots accumulate around $\sigma = 0$, there might be only a finite number of roots in $\sigma \le -\frac{3}{2}$.

$|\gamma_a| = 1$: example 2

Let $G(s) = \frac{1}{s+1+s\mathrm{e}^{-s}}$. This transfer function

- has no poles in $\bar{\mathbb{C}}_0$.

To see this, consider its characteristic equation at $s = \sigma + \mathrm{j}\omega$:

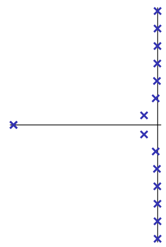
$$\mathrm{e}^{-\sigma}\mathrm{e}^{-\mathrm{j}\omega} = -\frac{\sigma + \mathrm{j}\omega + 1}{\sigma + \mathrm{j}\omega} \quad\Longrightarrow\quad \mathrm{e}^{-\sigma} = \left|1 + \frac{1}{\sigma + \mathrm{j}\omega}\right| \geq \left|1 + \frac{\sigma}{\sigma^2 + \omega^2}\right|$$

If $\sigma = 0$, $1 = 1 + 1/|\omega|$: unsolvable. If $\sigma > 0$, $\mathrm{e}^{-\sigma} \geq 1$: contradiction.

$|\gamma_a| = 1$: example 2 (contd)

We know that there is a sequence $\{s_k\} \in \mathbb{C} \setminus \bar{\mathbb{C}}_0$ of poles of $G(s)$ satisfying

$$s_k + 1 + s_k\mathrm{e}^{-s_k} = 0, \quad \text{with } \lim_{k\to\pm\infty}|s_k| = \infty :$$



Then

$$G(-s_k) = \frac{1}{1 - s_k - s_k\mathrm{e}^{s_k}} = \frac{1}{1 - s_k + s_k^2/(1 + s_k)} = 1 + s_k,$$

so that $|G(-s_k)| \to 1 + |2k + 1|\pi$. Thus, $G(s)$ unbounded on $\{-s_k\} \in \mathbb{C}_0$, so

- $G \notin H^\infty$ and hence unstable.

$|\gamma_a| = 1$: example 3

Let $G(s) = \frac{1}{(s+1)(s+1+s\mathrm{e}^{-s})}$. In this case

- $G \in H^\infty$

(in fact, $\|G(s)\|_\infty = 2$). Hence,

- $G(s)$ is stable.

In general, if $|\gamma_a| = 1$, the transfer function $\frac{N_0(s)}{M_0(s)+M_1(s)\mathrm{e}^{-sh}} \in H^\infty$ only if[3]

$$\deg M_0(s) \geq \deg N_0(s) + 2.$$

[3] For proof and further details see "H^∞ and BIBO stabilization of delay systems of neutral type," by Partington and Bonnet, *Systems & Control Letters*, **52**, pp. 283–288, 2004.

$|\gamma_a| = 1$: on the safe side

Thus, we saw that in this case $G(s)$ might be stable. Yet this

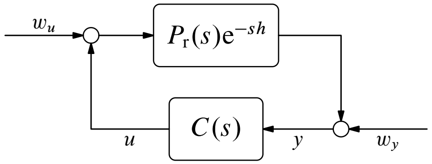
- stability is fragile (extremely non-robust).

Indeed,

- infinitesimal increase of $|P_a(\infty)|$ leads to instability.

It is thus safe to regard such systems as practically unstable.

Internal stability and high-frequency gain



This feedback system called internally stable if transfer matrix $\left[\begin{smallmatrix} w_y \\ w_u \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$,

$$\frac{1}{1-P_r(s)C(s)e^{-sh}}\begin{bmatrix} 1 & P_r(s)e^{-sh} \\ C(s) & P_r(s)C(s)e^{-sh} \end{bmatrix} \in H^\infty$$

Its $(1,1)$ entry, the sensitivity function, is of the form

$$\frac{1}{1-P_r(s)C(s)e^{-sh}} =: \frac{1}{1-L_r(s)e^{-sh}} = \frac{M_L(s)}{M_L(s)-N_L(s)e^{-s}} =: \frac{N_0(s)}{M_0(s)-M_1(s)e^{-s}},$$

for which $\deg M_0(s) = \deg N_0(s)$ ($\not\geq \deg N_0(s) + 2$). Thus, if $|L_r(\infty)| = 1$,

- this system cannot be internally stable.

Summary

So far we learned that

- if $|\gamma_a| < 1$, $G \in H^\infty$ iff it has no poles in $\bar{\mathbb{C}}_0$,
- if $|\gamma_a| > 1$, $G \notin H^\infty$ because it has infinitely many poles in $\mathbb{C}_0$
- if $|\gamma_a| = 1$, $G(s)$ practically unstable

Important point:

- classical "no poles in $\bar{\mathbb{C}}_0$" stability criterion might fail

Slight modification doing the trick

Theorem

Transfer function

$$G(s) = \frac{N_0(s)}{M_0(s) + M_h(s)e^{-sh}}, \quad \deg M_0 \geq \max\{\deg M_h, \deg N_0\},$$

is (practically) stable iff $\exists \alpha < 0$ such that $G(s)$ has no poles in $\bar{\mathbb{C}}_\alpha$.

This result says that in time-delay systems (both retarded and neutral)

- poles play essentially the same role as in the rational case,

we just should slightly redefine stability region ($\bar{\mathbb{C}}_0 \to \bar{\mathbb{C}}_\alpha$, for some $\alpha < 0$).
This, in turn, makes it possible to

- extend classical stability analysis methods to time-delay systems.

We still may use $\bar{\mathbb{C}}_0$, yet make sure that

- no pole chain is asymptotic to $\operatorname{Re} s = 0$.