Introduction to Time-Delay Systems

Fall 2012

Homework no. 1

(submission deadline: 31.10.2012, 10:00am)

Problem 1 (25%)**.** Suggest a proper transfer function $G(s)$ whose frequency response $G(\mathrm{j}\omega)$ is unbounded for $\omega \to \infty$. Does it correspond to a causal system ?

Problem 2 (25%)**.** Consider time-varying delay element $\mathcal{D}_{h(t)}$ defined as

$$y(t) = \mathcal{D}_{h(t)} u(t) \iff y(t) = u(t - h(t))$$

for a function $h(t)$ such that $0 \le h(t) \le t$ for all t.

1. Is $\mathcal{D}_{h(t)}$ bounded as an operator $L^2(\mathbb{R}^+) \mapsto L^2(\mathbb{R}^+)$?
2. Is $\mathcal{D}_{h(t)}$ bounded as an operator $L^2(\mathbb{R}^+) \mapsto L^2(\mathbb{R}^+)$ if we assume in addition that $h(t) \le 1$ for all t ?

Problem 3 (25%)**.** The transfer function

$$\Delta_h(s) := \frac{1 - \mathrm{e}^{-sh}}{h}, \qquad h > 0,$$

may be considered as a causal approximation of the derivative element $\Delta(s) = s$.

1. Prove that $\Delta_h \in H^\infty$.
2. What is the high-frequency gain of $\Delta_h(s)$?

Another frequently used approximation of the derivative is the rational approximation $\Delta_{\mathrm{r},\tau}(s) := \frac{s}{\tau s + 1}$.

3. Find τ_h such that the high-frequency gain of $\Delta_{\mathrm{r},\tau_h}(s)$ is the same as that of $\Delta_h(s)$.
4. Consider now two approximations of the derivative: $\Delta_h(s)$ and $\Delta_{\mathrm{r},\tau_h}(s)$. Which of them results in a smaller approximation error in a given bandwidth ω_b? Does the answer depend on ω_b? Does the answer depend on h ? By approximation errors consider the quantities

$$\max_{\omega \in [0,\omega_\mathrm{b}]} \left| 1 - \frac{\Delta_h(\mathrm{j}\omega)}{\Delta(\mathrm{j}\omega)} \right| \quad \text{and} \quad \max_{\omega \in [0,\omega_\mathrm{b}]} \left| 1 - \frac{\Delta_{\mathrm{r},\tau_h}(\mathrm{j}\omega)}{\Delta(\mathrm{j}\omega)} \right|,$$

respectively.

Transcendental equations may be solved numerically, the rest must be done analytically.

Problem 4 (25%)**.** Derive $R_{[2,3]}(s)$, which is the $[2,3]$-Padé approximation of $\frac{\mathrm{e}^{-s}}{s+1}$ and compute $\left\| \frac{\mathrm{e}^{-s}}{s+1} - R_{[2,3]}(s) \right\|_\infty$ (numerically). Compare this quantity with $\left\| \frac{\mathrm{e}^{-s}}{s+1} - \frac{R_{[2,2]}(s)}{s+1} \right\|_\infty$, where $R_{[2,2]}(s)$ is the $[2,2]$-Padé approximation of e^{-s}.