Convex Optimization: from Real-Time Embedded to Large-Scale Distributed

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Outline

Convex Optimization

Real-Time Embedded Optimization

Large-Scale Distributed Optimization

Summary

Outline

Convex optimization — Classical form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- variable $x \in \mathbf{R}^n$
- $f_0, \ldots, f_m$ are **convex**: for $\theta \in [0, 1]$,

$$f_i(\theta x + (1-\theta)y) \leq \theta f_i(x) + (1-\theta) f_i(y)$$

i.e., f_i have nonnegative (upward) curvature

Convex optimization — Cone form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in K \\ & Ax = b \end{array}$$

- variable $x \in \mathbf{R}^n$
- $K \subset \mathbf{R}^n$ is a proper cone
  - K nonnegative orthant $\longrightarrow$ LP
  - K Lorentz cone $\longrightarrow$ SOCP
  - K positive semidefinite matrices $\longrightarrow$ SDP
- the ‘modern’ canonical form

Why

- beautiful, nearly complete theory
  - duality, optimality conditions, ...

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  - get **global solution** (and optimality certificate)
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Why

- beautiful, nearly complete theory
  - duality, optimality conditions, ...
- effective algorithms, methods (in theory and practice)
  - get **global solution** (and optimality certificate)
  - polynomial complexity
- conceptual unification of many methods
- **lots of applications** (many more than previously thought)

Application areas

- machine learning, statistics
- finance
- supply chain, revenue management, advertising
- control
- signal and image processing, vision
- networking
- circuit design
- combinatorial optimization
- quantum mechanics

Applications — Machine learning

- parameter estimation for regression and classification
  - least squares, lasso regression
  - logistic, SVM classifiers
  - ML and MAP estimation for exponential families
- modern ℓ_1 and other sparsifying regularizers
  - compressed sensing, total variation reconstruction
- *k*-means, EM (bi-convex)

Example — Support vector machine

- data (a_i, b_i), $i = 1, \ldots, m$
  - $a_i \in \mathbf{R}^n$ feature vectors; $b_i \in \{-1, 1\}$ Boolean outcomes
- prediction: $\hat{b} = \mathrm{sign}(w^T a - v)$
  - $w \in \mathbf{R}^n$ is weight vector; $v \in \mathbf{R}$ is offset

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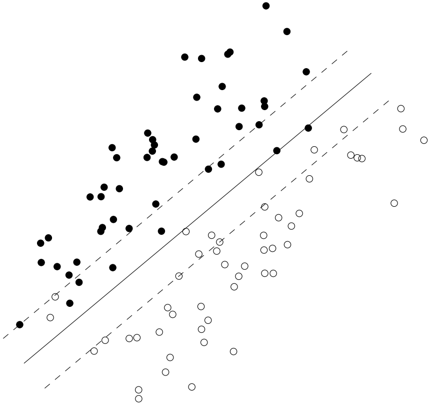
- SVM: choose w, v via (convex) optimization problem

$$\text{minimize} \quad L + (\lambda/2)\|w\|_2^2$$

$L = (1/m)\sum_{i=1}^m \left(1 - b_i(w^T a_i - v)\right)_+$ is avg. loss

SVM

$$w^T z - v = 0 \text{ (solid)}; \quad |w^T z - v| = 1 \text{ (dashed)}$$



Sparsity via ℓ_1 regularization

- adding ℓ_1-norm regularization

$$\lambda\|x\|_1 = \lambda(|x_1| + |x_2| + \cdots + |x_n|)$$

  to objective results in **sparse** x

- $\lambda > 0$ controls trade-off of sparsity versus main objective
- **preserves convexity, hence tractability**
- used for many years, in many fields
  - sparse design
  - feature selection in machine learning (lasso, SVM, ...)
  - total variation reconstruction in signal processing
  - compressed sensing

Example — Lasso

- regression problem with ℓ_1 regularization:

$$\text{minimize} \quad (1/2)\|Ax - b\|_2^2 + \lambda\|x\|_1$$

  with $A \in \mathbf{R}^{m \times n}$
- useful even when $n \gg m$ (!!); does **feature selection**

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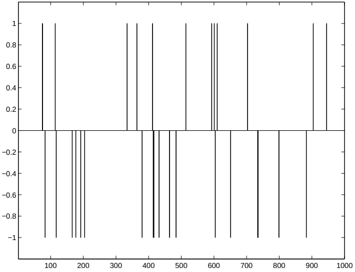
- lasso, ridge regression have **same computational cost**

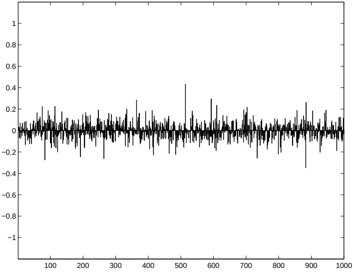
Example — Lasso

- $m = 200$ examples, $n = 1000$ features
- examples are noisy linear measurements of true x
- true x is sparse (30 nonzeros)

true x

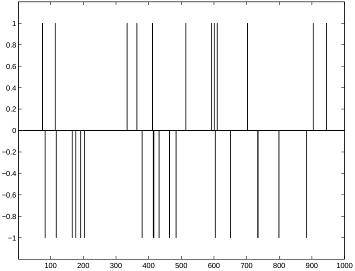


ℓ_2 reconstruction

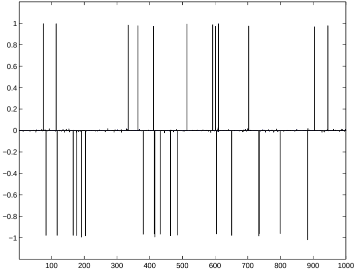


Example — Lasso

true x



ℓ_1 (lasso) reconstruction



State of the art — Medium scale solvers

- 1000s–10000s variables, constraints
- reliably solved by interior-point methods on single machine
- exploit problem sparsity
- not quite a technology, but getting there

State of the art — Modeling languages

- (new) high level language support for convex optimization
  - describe problem in high level language
  - description is automatically transformed to cone problem
  - solved by standard solver, transformed back to original form

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- (new) high level language support for convex optimization
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- enables rapid prototyping (for small and medium problems)
- ideal for teaching (can do a lot with short scripts)

CVX

- parser/solver written in Matlab (M. Grant, 2005)
- SVM:

$$\text{minimize} \quad L + (\lambda/2)\|w\|_2^2$$

$L = (1/m)\sum_{i=1}^{m} \left(1 - b_i(w^T a_i - v)\right)_+$ is avg. loss

- CVX specification:

```
cvx_begin
    variables w(n) v   % weight, offset
    L=(1/m)*sum(pos(1-b.*(A*w-v)));   % avg. loss
    minimize (L+(lambda/2)*sum_square(w))
cvx_end
```

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Motivation

- in many applications, need to solve the same problem repeatedly with different data
  - control: update actions as sensor signals, goals change
  - finance: rebalance portfolio as prices, predictions change
- used now when solve times are measured in minutes, hours
  - supply chain, chemical process control, trading

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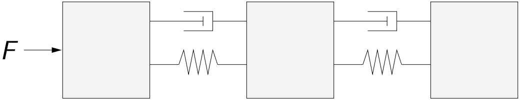
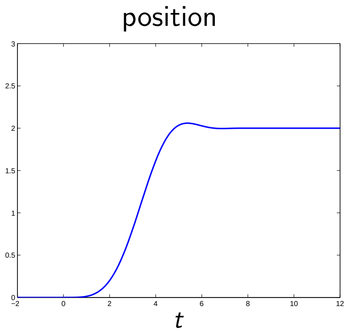
- (using new techniques) can be used for applications with solve times measured in **milliseconds** or **microseconds**

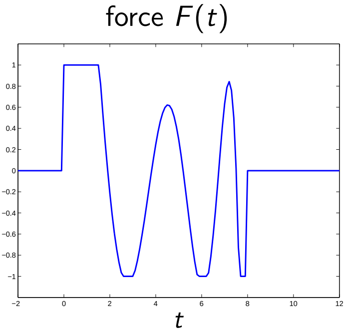
Example — Disk head positioning



- force $F(t)$ moves disk head/arm modeled as 3 masses (2 vibration modes)
- goal: move head to commanded position as quickly as possible, with $|F(t)| \leq 1$
- reduces to a (quasi-) convex problem

Optimal force profile





Embedded solvers — Requirements

- high speed
  - hard real-time execution limits
- extreme reliability and robustness
  - no floating point exceptions
  - must handle poor quality data
- small footprint
  - no complex libraries

Embedded solvers

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  - by hand
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- otherwise, develop custom code
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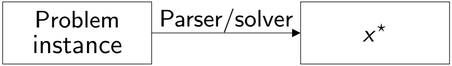
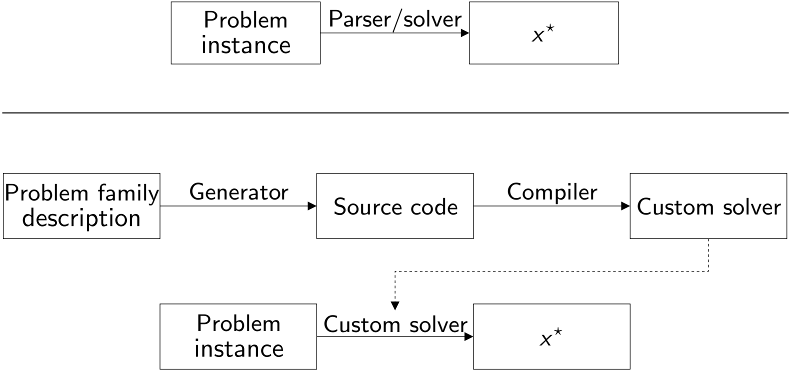
  - automatically via code generation
- can exploit known sparsity pattern, data ranges, required tolerance at solver code development time
- typical speed-up over general solver: **100–10000**$\times$

Parser/solver vs. code generator



Parser/solver vs. code generator



CVXGEN code generator

- handles small, medium size problems transformable to QP (J. Mattingley, 2010)
- uses primal-dual interior-point method
- generates flat library-free C source

CVXGEN example specification — SVM

```
dimensions
  m = 50   % training examples
  n = 10   % dimensions
end
parameters
  a[i] (n), i = 1..m   % features
  b[i], i = 1..m   % outcomes
  lambda positive
end
variables
  w (n)   % weights
  v     % offset
end
minimize
  (1/m)*sum[i = 1..m](pos(1 - b[i]*(w'*a[i] - v))) +
    (lambda/2)*quad(w)
end
```

CVXGEN sample solve times

problem	SVM	Disk
variables	61	590
constraints	100	742
CVX, Intel i3	270 ms	2100 ms
CVXGEN, Intel i3	230 μs	4.8 ms

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Motivation and goal

motivation:

- want to solve **arbitrary-scale** optimization problems
  - machine learning/statistics with huge datasets
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goal:

- ideally, a system that
  - has CVX-like interface
  - targets modern large-scale computing platforms
  - scales arbitrarily

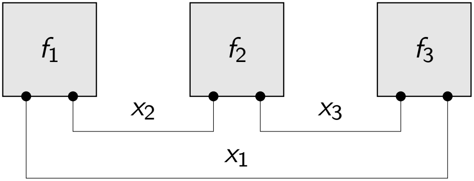
...not there yet, but there's promising progress

Distributed optimization

- devices/processors/agents coordinate to solve large problem, by passing relatively small messages
- can split variables, constraints, objective terms among processors
- variables that appear in more than one processor called 'complicating variables'
  (same for constraints, objective terms)

Example — Distributed optimization

$$\text{minimize} \quad f_1(x_1, x_2) + f_2(x_2, x_3) + f_3(x_1, x_3)$$



Distributed optimization methods

- dual decomposition (Dantzig-Wolfe, 1950s–)
- subgradient consensus
  (Tsitsiklis, Bertsekas, Nedić, Ozdaglar, Jadbabaie, 1980s–)

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  (Tsitsiklis, Bertsekas, Nedić, Ozdaglar, Jadbabaie, 1980s–)

- alternating direction method of multipliers (1980s–)
  - equivalent to many other methods
    (*e.g.*, Douglas-Rachford splitting)
  - **well suited to modern systems and problems**

Consensus optimization

- want to solve problem with N objective terms

$$\text{minimize} \quad \sum_{i=1}^{N} f_i(x)$$

  e.g., f_i is the loss function for ith block of training data

- consensus form:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{N} f_i(x_i) \\ \text{subject to} & x_i - z = 0 \end{array}$$

  - x_i are **local variables**
  - z is the **global variable**
  - $x_i - z = 0$ are **consistency** or **consensus** constraints

Consensus optimization via ADMM

with $\overline{x}^k = (1/N)\sum_{i=1}^N x_i^k$ (average over local variables)

$$\begin{aligned} x_i^{k+1} &:= \operatorname*{argmin}_{x_i} \left(f_i(x_i) + (\rho/2)\|x_i - \overline{x}^k + u_i^k\|_2^2 \right) \\ u_i^{k+1} &:= u_i^k + (x_i^{k+1} - \overline{x}^{k+1}) \end{aligned}$$

- get **global** minimum, under very general conditions
- u^k is running sum of inconsistencies (PI control)
- minimizations carried out independently and in parallel
- coordination is via averaging of local variables x_i

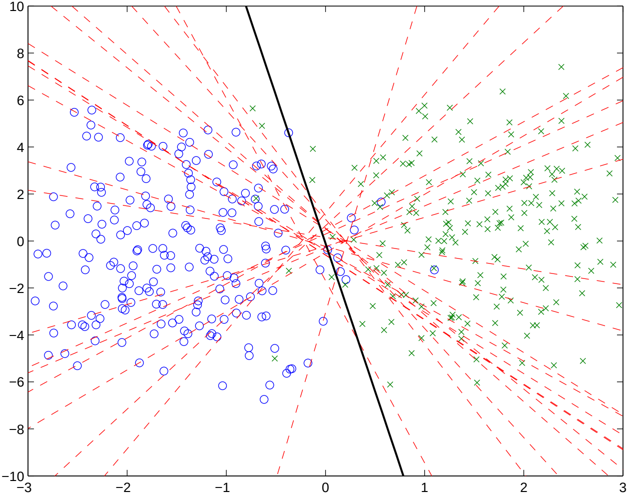
Statistical interpretation

- f_i is negative log-likelihood (loss) for parameter x given ith data block
- x_i^{k+1} is MAP estimate under prior $\mathcal{N}(\overline{x}^k - u_i^k, \rho I)$
- processors only need to support a Gaussian MAP method
  - type or number of data in each block not relevant
  - consensus protocol yields global ML estimate
- **privacy preserving**: agents never reveal data to each other

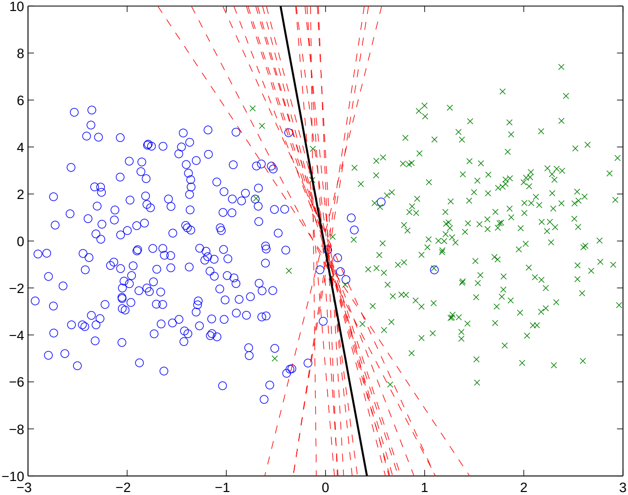
Example — Consensus SVM

- baby problem with $n = 2$, $m = 400$ to illustrate
- examples split into $N = 20$ groups, in worst possible way: each group contains only positive or negative examples

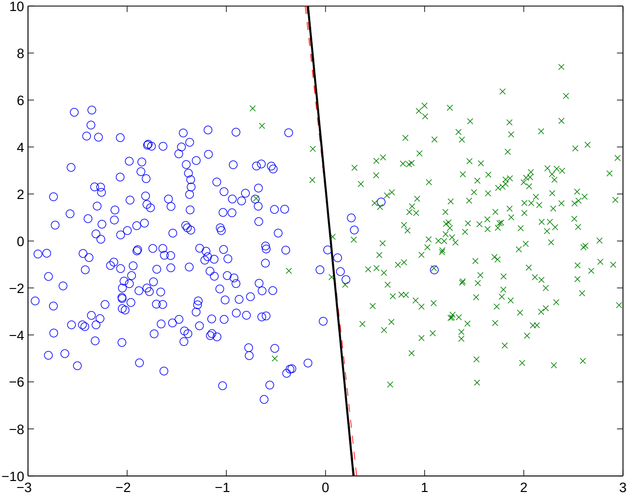
Iteration 1



Iteration 5



Iteration 40



Example — Distributed lasso

- example with **dense** $A \in \mathbf{R}^{400000 \times 8000}$ ($\sim$30 GB of data)
  - distributed solver written in C using MPI and GSL
  - no optimization or tuned libraries (like ATLAS, MKL)
  - split into 80 subsystems across 10 (8-core) machines on Amazon EC2

- computation times

loading data	30s
factorization (5000×8000 matrices)	5m
subsequent ADMM iterations	0.5–2s
total time (about 15 ADMM iterations)	**5–6m**

Outline

Summary

convex optimization problems

- **arise in many applications**
- **can be solved effectively**
  - small problems at microsecond/millisecond time scales
  - medium-scale problems using general purpose methods
  - arbitrary-scale problems using distributed optimization

References

- *Convex Optimization* (Boyd & Vandenberghe)
- *CVX: Matlab software for disciplined convex programming* (Grant & Boyd)
- *CVXGEN: A code generator for embedded convex optimization* (Mattingley & Boyd)
- *Distributed optimization and statistical learning via the alternating direction method of multipliers* (Boyd, Parikh, Chu, Peleato, & Eckstein)

all available (with code) from `stanford.edu/~boyd`