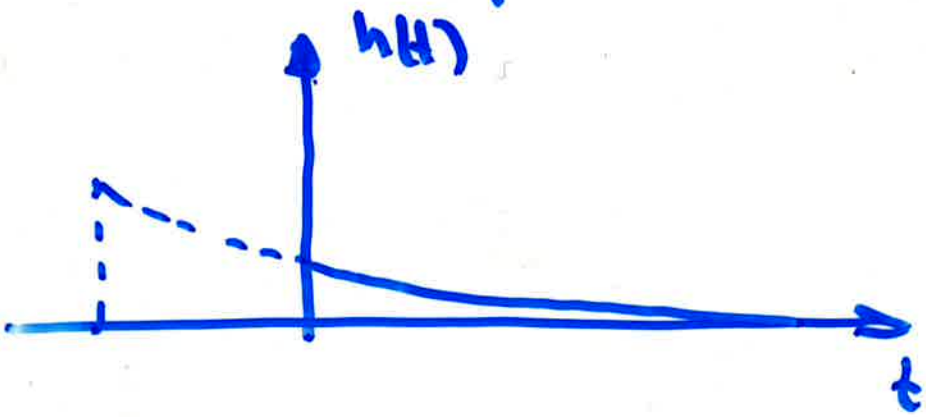
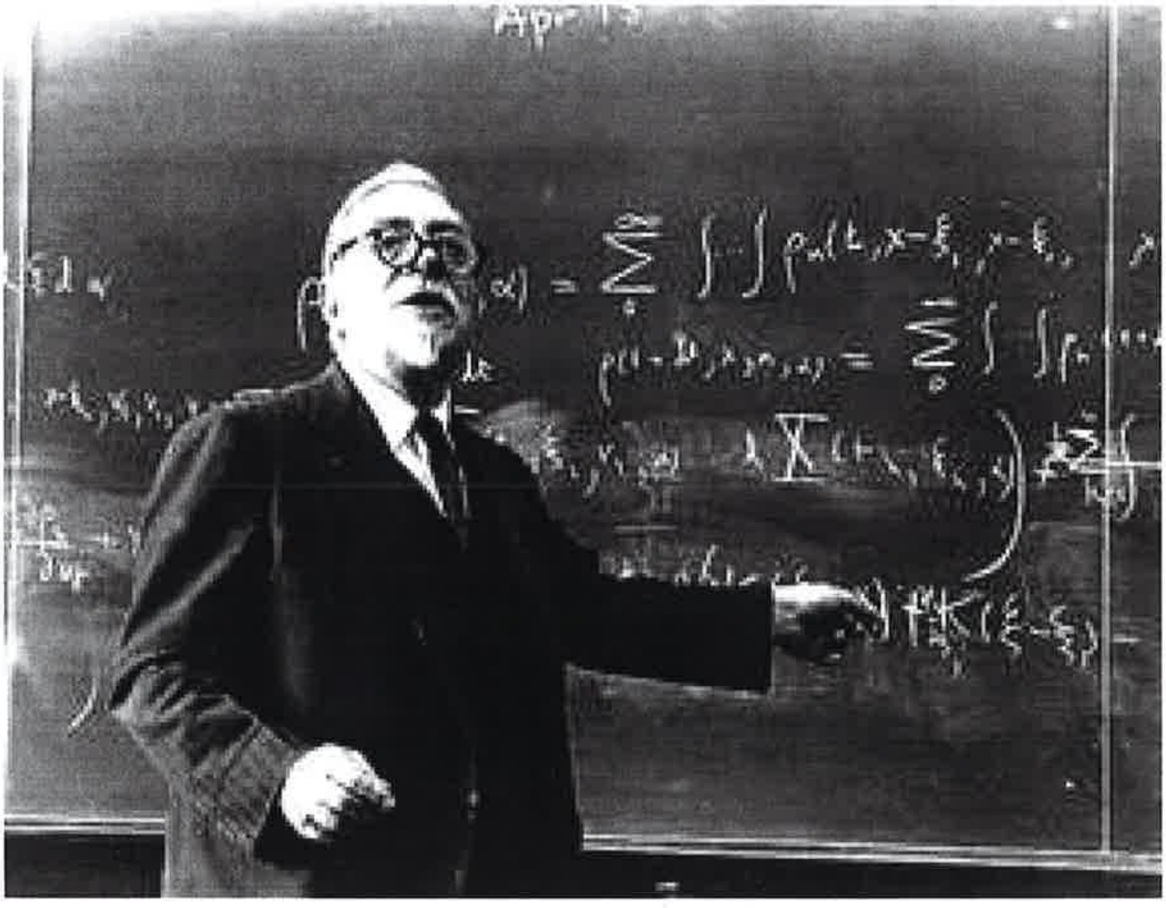
Dagens idé:

Hur fås ett optimalt kausalt system?



Norbert Wiener

1894 - 1964



Wienerteori

1. Inledning
2. Problemformulering
3. Optimala filter
4. Svårigheter
5. Jämförelse med Kalmanfilter
6. Polynommetoder
7. Sammanfattning

Optimala filter

Wiener, N. 1942 MIT

"The Yellow Peril"

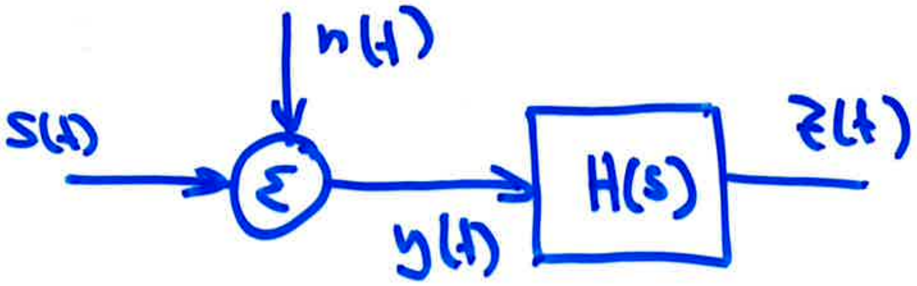
1949

Tidskontinuerligt

Kolmogorov, A.N. 1941

Tidsdiskret

PROBLEM



Bestäm H(s) så att

$$\varepsilon = E\left[z(t) - s(t+\eta) \right]^2 \quad \text{minimeras}$$

$\eta = 0$ Filtrering

$\eta > 0$ Prediktion

$\eta < 0$ Utjämning

OBS Stationäritet

Använd $y(\tau) \quad -\infty < \tau < t$

Förutsättningar

- Tidsinvarianta system
- Kända spektraltätheter och kovariansfunktioner
- Rationella spektraltätheter
- Ergodicitet
- Använd $y \; (-\infty, t]$

Ansätser:

1) Variationskalkyl ⇐

2) Polynomberäkningar

För- och nackdelar

"Lösning"

$$\mathcal{E} = E\left[s(t+\eta) - \int_0^\infty h(\tau)\, y(t-\tau)\, d\tau \right]^2$$

$$= R_s(0) - 2\int_0^\infty h(\tau) R_{sy}(\eta+\tau)\, d\tau$$

$$+ \iint_0^\infty h(\tau) h(\mu) R_y(\tau-\mu)\, d\tau\, d\mu$$

Vilket $h(t)$ sådant att

$h(t) = 0 \quad t < 0$ ger minst

förlust?

Variationskalkyl

- Antag $h(t)$ optimala filtret
- Ersätt $h(t)$ med

  $$h(t) + \varepsilon g(t)$$

- Titta på förändringen i ε

$$\mathcal{E} + \Delta\mathcal{E} = \mathcal{E}$$

$$- 2\varepsilon \left[\int_0^\infty g(\tau) \left[R_{sy}(\eta + \tau) - \int_0^\infty h(\mu) R_y(\tau - \mu)\, d\mu \right] d\tau \right]$$

$$+ \varepsilon^2 \iint_0^\infty g(\tau) g(\mu) R_y(\tau - \mu)\, d\tau\, d\mu$$

$$= \mathcal{E} - 2\varepsilon\, \Delta\mathcal{E}_1 + \varepsilon^2 \underbrace{\Delta\mathcal{E}_2}_{\geq 0}$$

$$\Rightarrow \Delta\mathcal{E} < 0 \quad \text{om} \quad \Delta\mathcal{E}_1 \neq 0$$

$$\boxed{R_{sy}(\tau + \eta) = \int_0^\infty h(\mu) R_y(\tau - \mu)\, d\mu \qquad \tau \geq 0}$$

Wiener-Hopf's integralekv

Nödv. och tillr. villkor

Specialfall: Bortse från realiserbarheten

Fourietransformera W-H

$\Rightarrow$

$$e^{s\eta}\,\phi_{sy}(\omega) = H(s)\,\phi_y(\omega)$$

$$H(s) = \frac{\phi_{sy}(\omega)}{\phi_y(\omega)}\,e^{s\eta}$$

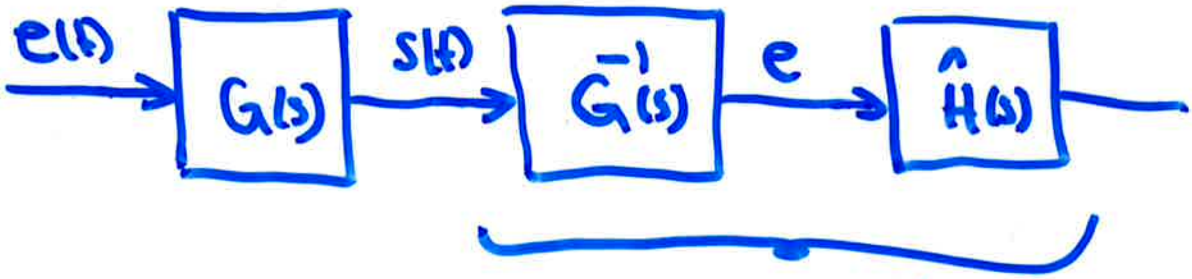
<u>Två problem</u>

- Poler i HHP
- $e^{s\eta}$ då $\eta > 0$ (prediktion)

Specialfall:
Ren prediktion $n = 0$

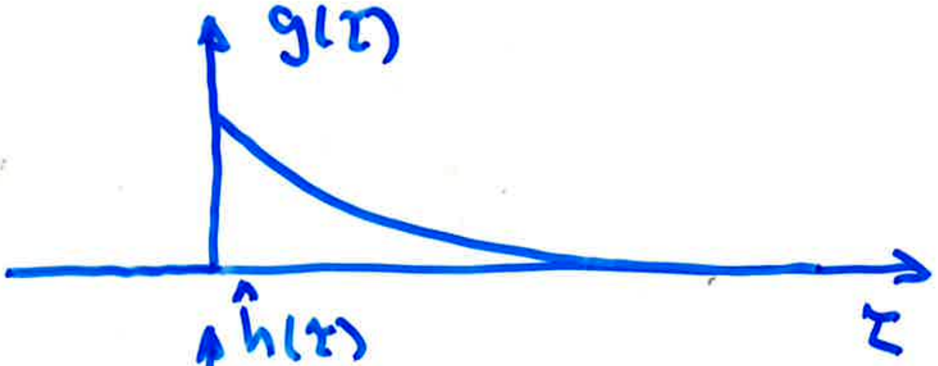
$$y(t) = s(t)$$

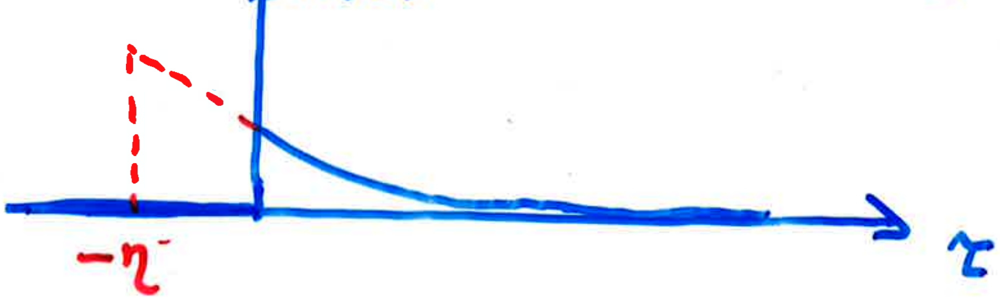
* Innovationsbeskr. av $s(t)$



$$H(s) = G^{-1}(s)\,\hat{H}(s)$$

$$\hat{h}(\tau) = \begin{cases} 0 & \tau < 0 \\ g(\tau + \eta) & \tau \geq 0 \end{cases}$$





TVÅ TRICK

1) Spektralfaktorera

$$\phi_y(\omega) = G(i\omega)G(-i\omega)$$
$$= \underset{g(t)}{G(i\omega)}\,\underset{g'(t)}{G^*(i\omega)}$$

2)

$$\phi_{sy}(\omega) = A(\omega)G^*(i\omega)$$

$$1) \Rightarrow \quad R_y(\tau) = \int_{-\infty}^{0} g(\tau-\mu)g'(\mu)\,d\mu$$

$$2) \Rightarrow \quad R_{sy}(\tau) = \int_{-\infty}^{0} a(\tau-\mu)g'(\mu)\,d\mu$$

OBS Övre gräns 0 för

$g'(\tau) = 0 \quad \tau > 0$

Sätt in i W-H

⋮

$H(i\omega) =$

$$= \frac{1}{G(i\omega)} \cdot \underbrace{\frac{1}{2\pi} \int_{0}^{\infty} e^{-j\omega\tau} \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}\, \Phi_{sy}(\omega)}{G^{*}(i\omega)} \, d\omega \, d\tau}_{\left[\frac{\Phi_{sy}(\omega)}{G^{*}(i\omega)}\right]_{pr}}$$

Realiserbara delen

eller +

Jfr

$$H(i\omega) = \frac{1}{G(i\omega)} \cdot \frac{\Phi_{sy}(\omega)}{G^{*}(i\omega)}$$

HUR BESTÄMS $[\ \ ]_+$?

$$\frac{\Phi_{sy}(\omega)}{G^*(i\omega)} = \underbrace{\sum_{stab} \frac{\beta_i}{i\omega+\alpha_i}}_{\left[\frac{\Phi_{sy}(\omega)}{G^*(i\omega)}\right]_+} + \sum_{instab} \frac{\gamma_i}{i\omega-\alpha_i}$$

EXEMPEL - FILTRERING

$$\Phi_s = \frac{1}{1+\omega^2} \qquad \Phi_n = a^2$$

s, n okorr

$y = s + n$

$$\Phi_y = \Phi_s + \Phi_n$$

$$= \frac{i\omega a + \sqrt{1+a^2}}{i\omega + 1} \cdot \frac{-i\omega a + \sqrt{1+a^2}}{-i\omega + 1} = GG^*$$

$$\Phi_{ys} = \Phi_s$$

$$\frac{\Phi_{ys}}{G^*} = \frac{1}{1+\omega^2} \cdot \frac{-i\omega + 1}{(-ia\omega + \sqrt{1+a^2})}$$

$$= \frac{1}{(i\omega + 1)(-ia\omega + \sqrt{1+a^2})}$$

$$= \underbrace{\frac{1}{a+\sqrt{1+a^2}} \cdot \frac{1}{i\omega+1}}_{[\ ]_+} + \frac{\sim}{-ia\omega + \sqrt{1+a^2}}$$

$$H(s) = \frac{s+1}{sa + \sqrt{1+a^2}} \cdot \frac{1}{s+1} \cdot \frac{1}{a+\sqrt{1+a^2}}$$

$$\boxed{H(s) = \frac{1}{a+\sqrt{1+a^2}} \cdot \frac{1}{a} \frac{1}{s + \sqrt{1+1/a^2}}}$$

KALMAN FILTER

$$\dot{x} = -x + e \qquad e \in N(0,1)$$
$$y = x + v \qquad v \in N(0,a)$$

$$\dot{p} = -p - p - \frac{p^2}{a^2} + 1$$

$$\dot{p} = 0 \Rightarrow \quad p = a\left(\sqrt{1+a^2} - a\right)$$

$$K = \frac{\sqrt{1+a^2}}{a} - 1$$

$$(s+1+K)\hat{X}(s) = K\,Y(s)$$

$$H(s) = \frac{\sqrt{1+a^2} - a}{a} \cdot \frac{1}{s + \sqrt{1+1/a^2}}$$

$$\boxed{H(s) = \frac{1}{\sqrt{1+a^2} + a} \cdot \frac{1}{a} \cdot \frac{1}{s + \sqrt{1+1/a^2}}}$$

EXEMPEL – PREDIKTION $\begin{cases} n = 0 \\ s(t+\eta) \end{cases}$

$$\Phi_s(\omega) = \frac{1}{(1+\omega^2)^2} = \underbrace{\frac{1}{(1+i\omega)^2}}_{G} \cdot \underbrace{\frac{1}{(1-i\omega)^2}}_{G^*}$$

$$\Phi_{\hat{s}y} = \frac{e^{i\omega\eta}}{(1+\omega^2)^2}$$

$$H(i\omega) = \frac{1}{G(i\omega)} \int_0^\infty e^{-i\omega\tau} \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(\tau+\eta)} \cdot \frac{1}{(1+i\omega)^2}\, d\omega}_{(\tau+\eta)\, e^{-(\tau+\eta)}}\, d\tau$$

$$H(i\omega) = (1+i\omega)^2\, e^{-\eta} \left[\frac{1}{(1+i\omega)^2} + \eta \frac{1}{1+i\omega} \right]$$

$$= e^{-\eta} \left[1 + \eta + \eta i\omega \right]$$

$$H(s) = \eta e^{-\eta} \left[s + \frac{1+\eta}{\eta} \right]$$

OBS Derivering

POLYNOMFORMULERING

- Bra sätt att beräkna $[\ ]_+$?
- Polynomidentiteter bra för I/O formuleringar

Kučera ~1978

Ahlén - Sternad 1990

Ref: T. Söderström (2002): Discrete time stochastic systems, Springer Verlag

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Diskret tid

$$G(z) = \left[\frac{\Phi_{sy}(z)}{\lambda^2 H^*(z)} \right]_+ \cdot \frac{1}{H(z)}$$

$y = He$

Söderströms beteckningar

$$A(q)y(t) = C(q)e(t) \quad e \in N(0,\lambda)$$

$$\phi_y(z) = G(z)\phi_u(z)G^*(z^{-*})$$

$$= G(z)G(z^{-1})\,\phi_u(z)$$

(reellt, skalärt)

$$C(z) = z^n + c_1 z^{n-1} + \dots + c_n$$

$$C^*(z^{-*}) = C(z^{-1})$$

$$= z^{-n} + c_1 z^{-n+1} + \dots + c_n$$

$$C^*(z) = [C(z)]^*$$

Exempel

$$A(q)y(t) = C(q)e(t) \qquad Ee^2 = \lambda^2$$

$$s = y(t-j) \qquad j \geq 0$$

Optimala $G(q) = q^{-j}$!

$$G(z) = \left[\frac{\Phi_{sy}(z)}{H^*(z)\lambda^2}\right]_+ \frac{1}{H(z)}$$

$$\Phi_{sy}(z) = z^{-j}\Phi_y(z) \qquad \Phi_y = \lambda^2 H(z)H^*(z)$$

$$G(z) = \left[z^{-j}\frac{\lambda^2 C(z)}{A(z)}\frac{C^*(z)}{A^*(z)}\cdot\frac{A^*(z)}{C^*(z)}\cdot\frac{1}{\lambda^2}\right]_+ \frac{A(z)}{C(z)}$$

$$= \left[z^{-j}\frac{C(z)}{A(z)}\right]_+ \frac{A(z)}{C(z)} = z^{-j}\frac{C(z)A(z)}{A(z)C(z)}$$

$$= z^{-j}$$

$$\hat{s}(t) = q^{-j}y(t) = y(t-j)$$

Exempel - Prediktion

$s(t) = y(t+k) \qquad k > 0$

Icke realiserbara filtret

$$G(z) = z^k \phi_y(z) \phi_y^{-1}(z) = z^k$$

Realiserbara filtret

$$G(z) = \left[z^k \lambda^2 \frac{C C^*}{A A^*} \cdot \frac{A^*}{C^*} \cdot \frac{1}{\lambda^2} \right]_+ \frac{A}{C}$$

$$= \left[z^k \frac{C(z)}{A(z)} \right]_+ \frac{A(z)}{C(z)}$$

Inför

$$z^{k-1} C(z) = A(z) F(z) + L(z)$$

$\deg C = \deg A = n$

$\deg F = k-1$

$\deg L = n-1$

$$G(z) = \left[\frac{zAF + zL}{A} \right]_+ \frac{A}{C} = \frac{zL}{A} \frac{A}{C} = \frac{zL}{C}$$

$$\hat{s}(t) = \hat{y}(t+k|t) = \frac{qL(q)}{C(q)} y(t)$$

$$\underbrace{zF}_{\text{framtid}} + \underbrace{\frac{zL}{A}}_{\text{kausal}}$$

Kausala filtret

$$f(z) = \left[\frac{G(z, z^{-1})}{D(z)\, F(z^{-1})}\right]_+ \qquad \begin{array}{l} \deg D = \nu \\ \deg F = \mu \end{array}$$

D alla nollst innanför enh. c.

$z^{\mu} F(z^{-1})$ — " — utanför — " —

$$G(z, z^{-1}) = \sum_{j=-p}^{l} g_j z^j \qquad p \geq 0 \quad l \geq 0$$

$$G(z, z^{-1}) = z^{\alpha} F(z^{-1}) R(z) + z^{\beta} D(z) L(z^{-1})$$

$\alpha = \min(0, \mu - p)$

$\beta = \max(0, l - \nu)$

$\deg L = \mu + \beta - 1 \geq 0$

$\deg R = \nu - \alpha \geq 0$

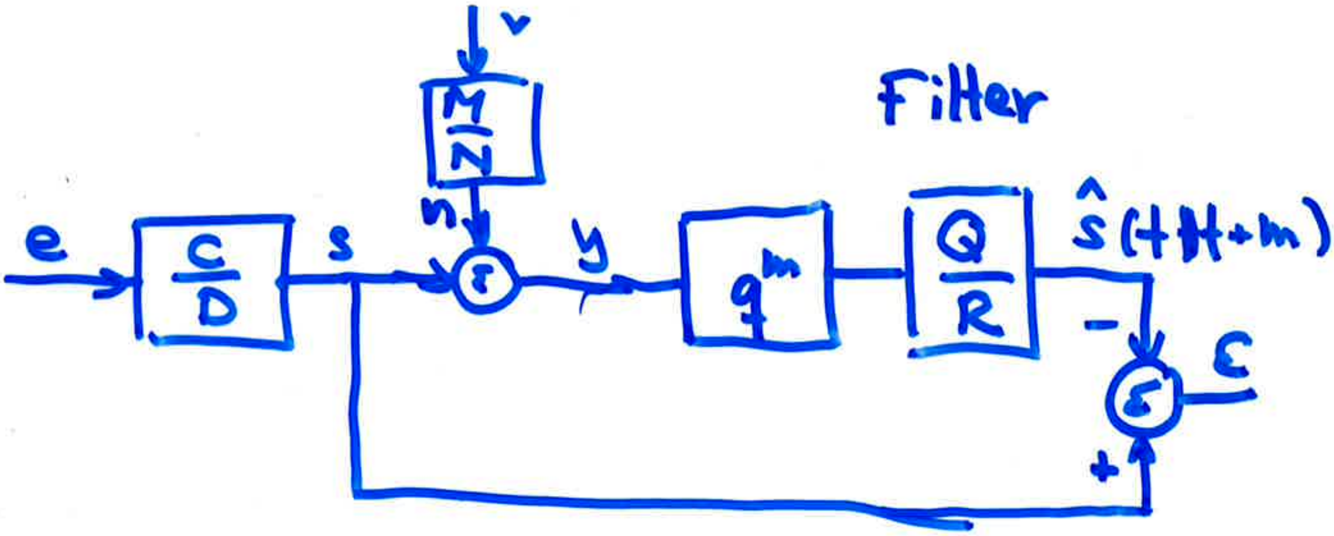
$$\Rightarrow \left[\frac{G(z, z^{-1})}{D(z)\, F(z^{-1})}\right]_+ = \left[\frac{z^{\alpha} R(z)}{D(z)}\right]_+ + \left[\frac{z^{\beta} L(z^{-1})}{F(z^{-1})}\right]_+$$

$$= \frac{z^{\alpha} R(z)}{D(z)}$$

POLYNOMFORMULERING

Kučera ~78

Ahlén – Sternad -90



e, v oberoende

$$\hat{s}(t|t+m) = \frac{Q(q^{-1})}{R(q^{-1})} y(t+m)$$

W-H

$$E\{\varepsilon(t) n^*(t)\} = 0$$

$$= \frac{\lambda_e}{2\pi i} \oint \frac{z^{-m} R C C_* N N_* - Q r \beta \beta_*}{R D D_* N N_*} G_* \frac{dz}{z}$$

$$= \frac{1}{2\pi i} \oint \phi_{\varepsilon n^*} \frac{dz}{z}$$

Alla poler innanför $|z|=1$ förkortas!

1. Gör innovations representation av y

$$y(t) = \frac{\rho}{DN} \sqrt{\lambda_2}\, \eta(t)$$

$$r\rho\rho_* = CC_*NN_* + \varrho MM_*DD_*$$

2. Från W-H

$$z^{-m} RCC_*NN_* - Qr\rho\rho_* = zRDNL_*$$

R, N faktorer $\Rightarrow$

$$z^{-m} CC_*N_* = r\rho_* Q_1 + zDL_*$$

$$Q = Q_1 N \quad R = \rho$$

$$\hat{s}(t|t+m) = \frac{Q_1 N}{\rho}\, y(t+m)$$

Gradtalsproblem för att få unika lösningen